ANNALES ACADEMIAE SCIENTIARUM FENNICAE

Series A

I. MATHEMATICA

587

BIHARMONIC MEASURE

BY

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HELSINKI 1974
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doi:10.5186/aasfm.1975.587

ISSN 0066-1953
ISBN 951-41-0201-0

Communicated 11 March 1974

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Biharmonic measure

The harmonic Green's function g is known to exist on a Riemannian manifold R if and only if the harmonic measure ω of the ideal boundary β does not reduce to a constant. This measure is a harmonic function on the complement $S_0 = R - \bar{R}_0$ of the closure of a regular region R_0, with essentially the boundary values 1 on ∂R_0 and 0 on β. The biharmonic Green's function γ on R, with »boundary values» $\gamma = \Delta\gamma = 0$ on β, exists if and only if $\omega \in L^2(S_0)$ [22]. In the present paper we ask: can a nonharmonic biharmonic function be introduced on S_0 with the property that its nondegeneracy characterizes the existence of γ, in analogy with the nondegeneracy of ω characterizing the existence of g? We shall show that this is possible. The function, which we will call the *biharmonic measure* σ of β, is the limit of biharmonic functions σ_Ω on subregions $S_0 \cap \Omega$ of S_0, with the Ω regular subregions of R containing $\bar{R}_0$. In contrast with approximating harmonic measures ω_Ω with $\omega_\Omega = 1$ on ∂R_0, $\omega_\Omega = 0$ on $\partial\Omega$, the function σ_Ω, with $\Delta\sigma_\Omega = \omega_\Omega$, vanishes on the entire boundary of $S_0 \cap \Omega$, and spans $S_0 \cap \Omega$ like an arc shaped bridge. As Ω increases, the height of this arch σ_Ω increases, and its limit σ as Ω exhausts R is either an arch spanning S_0 or else the constant ∞. We shall show that the finiteness of σ is independent of the choice of S_0, and we can therefore introduce the class O_Σ of Riemannian manifolds with boundaries of infinite biharmonic measure.

We first explore O_Σ in its own right. For radial spaces, which play a central role in biharmonic classification theory, we decompose σ into its biharmonic, harmonic, quasiharmonic, and constant components. The biharmonic type of R can then be easily tested. In particular, $\sigma < \infty$ if both the biharmonic and harmonic components of σ tend to zero as one approaches β. We use this test to determine the type of a number of fundamental manifolds used in biharmonic classification theory. E.g., for the Poincaré N-ball

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The work was sponsored by the U. S. Army Research Office, Grant DA-ARO-31-124-73-G39, University of California, Los Angeles.

MOS Classification 31B30.

$$B_\alpha^N = \{x = (x^1, \ldots, x^N),\ |x| = r,\ r < 1,\ ds = (1 - r^2)^\alpha\ |dx|,\ \alpha \in \mathbf{R}\}$$

we obtain the following complete characterizations:

$$B_\alpha^N \in O_\Sigma \Leftrightarrow \begin{cases} \alpha \leq -3/2, & N = 2, \\ \alpha \notin (-3, 1), & N = 3, \\ \alpha \geq (N-2)^{-1}, & N > 3. \end{cases}$$

After this study of σ we establish its characteristic property in our original problem:

$$O_\Sigma = O_\Gamma,$$

where O_Γ is the class of Riemannian manifolds which do not carry the biharmonic Green's function γ. In particular, the above values of α exclude γ on the Poincaré ball B_α^N.

The role of O_Σ in general biharmonic classification theory will be discussed in another context.

1. The Laplace-Beltrami operator $\Delta = d\delta - \delta d$ gives the class H of harmonic functions h, $\Delta h = 0$, and the class H^2 of nonharmonic biharmonic functions u, $\Delta^2 u = 0$, $\Delta u \neq 0$. Let R_0, Ω be regular subregions of R, with $\bar{R}_0 \subset \Omega$, and set $S_0 = R - \bar{R}_0$, $\alpha = \partial R_0$, $\beta_\Omega = \partial\Omega$. Take $\omega_\Omega \in C(\bar{S}_0 \cap \bar{\Omega})$, $\omega_\Omega \in H(S_0 \cap \Omega)$, $\omega_\Omega|\alpha = 1$, $\omega_\Omega|\beta_\Omega = 0$. The directed limit $\omega = \lim_{\Omega \to R} \omega_\Omega$ is the harmonic measure on $\bar{S}_0$ of the ideal boundary β of R.

We introduce:

Definition. *The directed limit*

$$\sigma = \lim_{\Omega \to R} \sigma_\Omega,$$

where

$$\sigma_\Omega \in C(\bar{S}_0 \cap \bar{\Omega}) \cap H^2(S_0 \cap \Omega),\quad \Delta\sigma_\Omega = \omega_\Omega,\quad \sigma_\Omega|\alpha = \sigma_\Omega|\beta_\Omega = 0,$$

is the biharmonic measure of the ideal boundary β of the Riemannian manifold R.

The limit always exists. In fact, if $g_{S_0 \cap \Omega}(x, y)$ is the harmonic Green's function on $S_0 \cap \Omega$ with pole y, then

$$\sigma_\Omega(x) = \int_{S_0 \cap \Omega} g_{S_0 \cap \Omega}(x, y)\omega_\Omega(y)dy,$$

with dy the Riemannian volume element at y. Since both $g_{S_0 \cap \Omega}$ and ω_Ω increase with Ω, so does σ_Ω, and the limit σ exists, finite or infinite, at every $x \in S_0$. We shall show in No. 7 that the finiteness is independent of x and of S_0.

2. We first study the biharmonic measure σ on radial spaces, e.g., the N-space and the N-ball, each endowed with a radial metric $ds = \lambda(r)|dx|$, $r = |x|$. We choose $R_0 = \{r < r_0\}$, $S_0 = \{r > r_0\}$.

Let Q be the class of quasiharmonic functions q, defined by $\Delta q = 1$. The space of all radial biharmonic functions $u(r)$, $\Delta^2 u(r) = 0$, is generated by four functions: any $u_0(r) \in H^2 - Q$, $\Delta u_0(r)$, any $q_0(r) \in Q$, and the constant 1. Thus every $u(r)$ has a decomposition

$$u(r) = au_0(r) + b\Delta u_0(r) + cq_0(r) + d\,,$$

which depends on the choice of u_0 and q_0. In particular, if the biharmonic measure

$$\sigma(x) = \sigma(r) = \int_{S_0} g_{S_0}(x\,,y)\omega(y)dy$$

is finite, it has such a decomposition, and $\sigma(x) \to 0$ as $x \to \beta$.

Testing of the finiteness of σ is facilitated by the following simple criterion:

Lemma. *If every radial $u \in H^2$ on S_0 is unbounded, then the biharmonic measure σ is infinite. If there exists a function $u_0(r) \in H^2 - Q$ with $u_0(r) \to 0$, $\Delta u_0(r) \to 0$ as $x \to \beta$, then $\sigma < \infty$.*

Proof. The first part of the Lemma is clear, since $\Delta\sigma = \omega$ entails $\sigma \in H^2$. If $u_0(r) \to 0$, $\Delta u_0(r) \to 0$ as $x \to \beta$ for some $u_0 \in H^2 - Q$, then $a = 1/\Delta u_0(r_0)$ gives $\Delta(au_0) = \omega$, and $b = -au_0(r_0)/\Delta u_0(r_0)$ provides us with the function $\sigma = au_0 + b\Delta u_0$ which has all the properties required of the biharmonic measure.

3. For our first radial space we take the Euclidean N-space E^N. The type distinction is here fascinatingly simple:

Theorem 1. *The ideal boundary of E^N has a finite biharmonic measure if and only if $N > 4$.*

Proof. Let us first construct σ for $N > 4$ by means of the definition $\sigma = \lim_{\Omega \to R} \sigma_\Omega$. Choose $r_0 = 1$, $\varrho > 1$, $\Omega = \{r < \varrho\}$, and write σ_ϱ for σ_Ω. For the decomposition

$$\sigma_\varrho = a_\varrho u_0(r) + b_\varrho \Delta u_0(r) + c_\varrho q_0(r) + d_\varrho$$

we take

$$\sigma_\varrho = a_\varrho r^{-N+4} + b_\varrho r^{-N+2} - c_\varrho r^2 + d_\varrho\,,$$

where we have absorbed in b_ϱ the constant $[2(N-4)]^{-1}$ from $\Delta r^{-N+4} = 2(N-4)r^{-N+2}$, and in c_ϱ the constant $-(2N)^{-1}$ from $\Delta r^2 = -2N$. By the definition of σ_ϱ,

$$\begin{cases} \sigma_\varrho(1) = a_\varrho + b_\varrho + c_\varrho + d_\varrho = 0\,, \\ \sigma_\varrho(\varrho) = a_\varrho \varrho^{-N+4} + b_\varrho \varrho^{-N+2} + c_\varrho \varrho^2 + d_\varrho = 0\,, \\ \Delta\sigma_\varrho(1) = 2(N-4)a_\varrho - 2Nc_\varrho = 1\,, \\ \Delta\sigma_\varrho(\varrho) = 2(N-4)a_\varrho \varrho^{-N+2} - 2Nc_\varrho = 0\,. \end{cases}$$

From the third and fourth equations we obtain

$$a_\varrho = [2(N-4)(1-\varrho^{-N+2})]^{-1}\,,$$
$$c_\varrho = \varrho^{-N+2}[2N(1-\varrho^{-N+2})]^{-1}\,.$$

If a, b, c, d are the limits of $a_\varrho, b_\varrho, c_\varrho, d_\varrho$ as $\varrho \to \infty$, then

$$a = [2(N-4)]^{-1}\,, \quad c = 0\,.$$

The first and second equations above then give

$$b = \lim_{\varrho\to\infty} \{-(1-\varrho^{-N+2})^{-1}[a_\varrho(1-\varrho^{-N+4}) + c_\varrho(1-\varrho^2)]\} = -[2(N-4)]^{-1}\,,$$
$$d = \lim_{\varrho\to\infty} [-(a_\varrho + b_\varrho + c_\varrho)] = 0\,.$$

Thus we have the explicit expression

$$\sigma(r) = [2(N-4)]^{-1}(r^{-N+4} - r^{-N+2})$$

for the biharmonic measure of the ideal boundary of E^N for $N > 4$.

4. We now deduce the same result for $N > 4$ using the Lemma and its proof. For $u_0(r) = r^{-N+4}$, $\Delta u_0(r) = 2(N-4)r^{-N+2}$, we have

$$a = [2(N-4)]^{-1} r_0^{N-2}\,, \quad b = -[2(N-4)]^{-2} r_0^N\,,$$

and for $r_0 = 1$ the function $\sigma = au_0 - b\Delta u_0$ is the biharmonic measure $[2(N-4)]^{-1}(r^{-N+4} - r^{-N+2})$.

To prove that $\sigma = \infty$ for $N \le 4$, we use the representation for $\sigma < \infty$

$$\sigma(r) = \begin{cases} ar^2 \log r + b \log r + cr^2 + d\,, & N = 2\,, \\ ar + br^{-1} + cr^2 + d\,, & N = 3\,, \\ a \log r + br^{-2} + cr^2 + d\,, & N = 4\,. \end{cases}$$

For $N = 2$, this is unbounded unless $a = b = c = 0$, and σ is constant. For $N = 3$ or 4, it is unbounded unless $a = c = 0$ and σ is harmonic. However, by $\Delta\sigma = \omega$, σ cannot be harmonic, a contradiction. The proof of Theorem 1 is complete.

5. Can the biharmonic measure of the ideal boundary of E^N be made finite even for $N \le 4$ if we »shrink» or »expand» the boundary by replacing the Euclidean metric $ds = |dx|$ by the metric $(1 + r^2)^\alpha |dx|$, with α a sufficiently small or large constant? Denote the resulting space by R^N_α. The answer is perhaps somewhat unexpected:

Theorem 2. *The biharmonic measure* σ *of the ideal boundary of* R^N_α *is infinite for all* α *if* $N \leq 4$. *For* $N > 4$, σ *is infinite if and only if* $\alpha \leq -\frac{1}{2}$.

Proof. An explicit construction of σ as the limit of σ_Ω, as in No. 3, is now not possible, and we make use of the Lemma. We know that σ has the form

$$\sigma(r) = au_0(r) + b\Delta u_0(r) + cq_0(r) + d\,.$$

First we shall find bounded functions $u_0\,, \Delta u_0$ for $N > 4$. Choose again $S_0 = \{r > 1\}$. For $h(r) \in H(S_0)$,

$$\Delta h(r) = -\,r^{-N+1}(1 + r^2)^{-N\alpha}[r^{N-1}(1 + r^2)^{(N-2)\alpha}h'(r)]' = 0\,,$$

$$h(r) = c \int^r r^{-N+1}(1 + r^2)^{-(N-2)\alpha}dr \sim c \int^r r^{-N+1-2(N-2)\alpha}dr$$

$$= \begin{cases} a + br^{-(N-2)(1+2\alpha)} & \text{if } N > 2 \text{ and } \alpha \neq -\frac{1}{2}\,, \\ a + b\log r & \text{if } N = 2 \text{ or } \alpha = -\frac{1}{2}\,. \end{cases}$$

Thus $h(r)$ belongs to the family B of bounded functions if and only if $N > 2\,, \alpha > -\frac{1}{2}$, an assumption we shall make for the present. Here and later we disregard irrelevant multiplicative and additive constants, and we choose an h_0 with

$$h_0(r) \sim r^{-(N-2)(1+2\alpha)}\,,$$

which $\to 0$ as $r \to \infty$ if $N > 2\,, \alpha > -\frac{1}{2}$.

For $\Delta u(r) = h_0(r)$, we obtain

$$[r^{N-1}(1 + r^2)^{(N-2)\alpha}u'(r)]' \sim r^{1+4\alpha}\,.$$

Accordingly, in view of $\alpha \neq -\frac{1}{2}$, we have $[\,] \sim r^{2+4\alpha}$, and, again by virtue of $\alpha \neq -\frac{1}{2}$, we can take u_0 with

$$u_0(r) \sim r^{-(N-4)(1+2\alpha)}\,.$$

This $\to 0$ as $r \to \infty$ if $-2(N-4)\alpha < N-4$, which in turn, under our assumption $\alpha > -\frac{1}{2}$, holds if and only if $N > 4$. By the Lemma, we conclude that $\sigma < \infty$ if $N > 4\,, \alpha > -\frac{1}{2}$.

In the discussion of the case $\sigma = \infty$, the nonuniqueness of the generators $u_0\,, \Delta u_0\,, q_0$ (and 1) makes it necessary to consider the unboundedness of all four components of $\sigma(r) \in H^2$. For $\Delta q(r) = 1$, we obtain

$$[r^{N-1}(1+r^2)^{(N-2)\alpha}q'(r)]' \sim r^{N-1+2N\alpha} ,$$

and therefore

$$q'(r) \sim \begin{cases} r^{1+4\alpha} , & \alpha \neq -\frac{1}{2} , \\ r^{-1}\log r , & \alpha = -\frac{1}{2} . \end{cases}$$

We choose

$$q_0(r) \sim \begin{cases} r^{2+4\alpha} , & \alpha \neq -\frac{1}{2} ; \\ (\log r)^2 , & \alpha = -\frac{1}{2} . \end{cases}$$

For $N = 2$ and any α , $h_0(r) \sim \log r$, and $u'(r)$ satisfies

$$[ru'(r)]' \sim r^{1+4\alpha}\log r , [] \sim r^{2+4\alpha}\log r , u'(r) \sim r^{1+4\alpha}\log r ,$$

so that we can take

$$u_0(r) \sim \begin{cases} r^{2+4\alpha}\log r , & \alpha \neq -\frac{1}{2} , \\ (\log r)^2 , & \alpha = -\frac{1}{2} . \end{cases}$$

Therefore,

$$\sigma(r) \sim \begin{cases} ar^{2+4\alpha}\log r + b\log r + cr^{2-4\alpha} + d ; & \alpha \neq -\frac{1}{2} \\ a(\log r)^2 + b\log r + d , & \alpha = -\frac{1}{2} . \end{cases}$$

By virtue of $\Delta\sigma = \omega$, we have $a \neq 0$, $b \neq 0$, hence $\sigma \notin B$, and $\sigma = \infty$.

For $N = 3$, $\alpha \neq -\frac{1}{2}$,

$$h_0(r) \sim r^{-1-2\alpha} , \quad u_0(r) \sim r^{1+2\alpha} , \quad q_0(r) \sim r^{2+4\alpha} .$$

Since $a \neq 0$ and hence $b \neq 0$, we have $\sigma \notin B$, hence $\sigma = \infty$.

For $N = 3$, $\alpha = -\frac{1}{2}$, $h_0(r) \sim \log r$ and

$$[r^2(1+r^2)^{-1/2}u'(r)]' \sim r^2(1+r^2)^{-3/2}\log r \sim r^{-1}\log r ,$$

$$[] \sim (\log r)^2 , \quad u'(r) \sim r^{-1}(\log r)^2 , \quad u_0(r) \sim (\log r)^3 .$$

Since $q_0(r) \sim (\log r)^2$, we have $\sigma(r) \sim (\log r)^3 \notin B$, hence $\sigma = \infty$.

For $N = 4$, $\alpha \neq -\frac{1}{2}$, $h_0(r) \sim r^{-2-4\alpha}$,

$$[r^3(1+r^2)^{2\alpha}u'(r)]' \sim r^{1-4\alpha} , \quad u'(r) \sim r^{-1} , \quad u_0(r) \sim \log r .$$

In view of $q_0(r) \sim r^{2+4\alpha}$, $\sigma(r)$ grows at least as rapidly as $\log r$, hence $\sigma = \infty$.

For $N = 4$, $\alpha = -\frac{1}{2}$, $h_0(r) \sim \log r$,

$$[r^3(1+r^2)^{-1}u'(r)]' \sim r^{-1}\log r , \quad u'(r) \sim r^{-1}(\log r)^2 , \quad u_0(r) \sim (\log r)^3 .$$

Since $q_0(r) \sim (\log r)^2$, we have $\sigma(r) \sim (\log r)^3 \notin B$, hence $\sigma = \infty$. We have proved that $\sigma = \infty$ for $N \leq 4$, all α.

For $N > 4$, $\alpha < -\frac{1}{2}$,

$$h_0(r) \sim r^{-(N-2)(1+2\alpha)}, \quad u_0(r) \sim r^{-(N-4)(1+2\alpha)}, \quad q_0(r) \sim r^{2+4\alpha}.$$

Therefore, $\sigma \sim r^\beta$, with

$$\beta \geq \min\left[-(N-2)(1+2\alpha), -(N-4)(1+2\alpha)\right].$$

The two quantities are both positive for $\alpha < -\frac{1}{2}$, so that $\sigma \notin B$, hence $\sigma = \infty$.

For $N > 4$, $\alpha = -\frac{1}{2}$, $h_0(r) \sim \log r$,

$$[r^{N-1}(1+r^2)^{-(N-2)/2}u'(r)]' \sim r^{-1}\log r, \; u'(r) \sim r^{-1}(\log r)^2, \; u_0(r) \sim (\log r)^3,$$

and $q_0(r) \sim (\log r)^2$. Consequently $\sigma(r) \sim (\log r)^3 \notin B$, and $\sigma = \infty$.

We have shown that $\sigma = \infty$ for $N > 4$, $\alpha \leq -\frac{1}{2}$. The proof of Theorem 2 is complete.

6. Next we consider the Poincaré N-ball B_α^N, which plays an important role in general biharmonic classification theory. By definition,

$$B_\alpha^N = \{x = (x^1, \ldots, x^N), |x| = r, r < 1, ds = (1-r^2)^\alpha |dx|, \alpha \in \mathbf{R}\}.$$

We shall give a complete characterization of the finiteness of σ:

Theorem 3. *The biharmonic measure of the ideal boundary of the Poincaré ball B_α^N is finite if and only if*

$$\begin{cases} \alpha > -3/2, & N = 2, \\ \alpha \in (-3, 1), & N = 3, \\ \alpha < (N-2)^{-1}, & N > 3. \end{cases}$$

Proof. For $h(r) \in H$,

$$\Delta h(r) = -r^{-N+1}(1-r^2)^{-N\alpha}[r^{N-1}(1-r^2)^{(N-2)\alpha}h'(r)]' = 0,$$

and we choose

$$h_0(r) \sim \begin{cases} \log r \sim 1 - r, & N = 2, \text{ any } \alpha, \\ (1-r)^{-(N-2)\alpha-1}, & N > 2, \; \alpha \neq (N-2)^{-1}, \\ \log(1-r), & N > 2, \; \alpha = (N-2)^{-1}. \end{cases}$$

For $\Delta u(r) = h_0(r)$,

$$[r^{N-1}(1-r^2)^{(N-2)\alpha}u'(r)]' \sim (1-r)^{N\alpha}h_0(r),$$

$$u'(r) \sim (1-r)^{-(N-2)\alpha}\int^r (1-r)^{N\alpha}h_0(r)dr.$$

Fore $N = 2$, we take

$$u_0(r) \sim \int^r \int^r (1-s)^{2\alpha+1} ds dr \sim \begin{cases} \int^r (1-r)^{2\alpha+2} dr \sim (1-r)^{2\alpha+3}, & \alpha \neq -1, \\ & -3/2, \\ \int^r \log(1-r) dr \sim (1-r)\log(1-r), & \\ & \alpha = -1, \\ \int^r (1-r)^{-1} dr \sim \log(1-r), & \alpha = -3/2. \end{cases}$$

For $N = 3$, we obtain successively

$$u(r) \sim \begin{cases} \int^r (1-r)^{-\alpha} \int^r (1-s)^{2\alpha+1} ds dr, & \alpha \neq 1, \\ \int^r (1-r)^{-\alpha} \int^r (1-s)^3 \log(1-s)\, ds dr, & \alpha = 1, \end{cases}$$

$$u(r) \sim \begin{cases} \int^r (1-r)^{\alpha+2} dr, & \alpha \neq 1, -1, \\ \int^r (1-r)^3 \log(1-r) dr, & \alpha = 1, \\ \int^r (1-r) \log(1-r) dr, & \alpha = -1, \end{cases}$$

$$u_0(r) \sim \begin{cases} (1-r)^{\alpha+3}, & \alpha \neq 1, -1, -3, \\ (1-r)^4 \log(1-r), & \alpha = 1, \\ (1-r)^2 \log(1-r), & \alpha = -1, \\ \log(1-r), & \alpha = -3. \end{cases}$$

For $N = 4$,

$$u(r) \sim \begin{cases} \int^r (1-r)^{-2\alpha} \int^r (1-s)^{2\alpha+1} ds dr, & \alpha \neq \frac{1}{2}, \\ \int^r (1-r)^{-1} \int^r (1-s)^2 \log(1-s) ds dr, & \alpha = \frac{1}{2}, \end{cases}$$

$$u_0(r) \sim \begin{cases} (1-r)^3 , & \alpha \neq \frac{1}{2} , -1 , \\ (1-r)^3 \log (1-r) , & \alpha = \frac{1}{2} , \\ (1-r)^3 \log (1-r) , & \alpha = -1 . \end{cases}$$

For $N > 4$,

$$u(r) \sim \begin{cases} \int^r (1-r)^{-(N-2)\alpha} \int^r (1-s)^{2\alpha+1} ds dr , & \alpha \neq (N-2)^{-1} , \\ \int^r (1-r)^{-1} \int^r (1-s)^{N/(N-2)} \log (1-s) ds dr , & \alpha = (N-2)^{-1} , \end{cases}$$

$$u_0(r) \sim \begin{cases} (1-r)^{-(N-4)\alpha+3} , & \alpha \neq 3(N-4)^{-1} , (N-2)^{-1} , -1 , \\ \log (1-r) , & \alpha = 3(N-4)^{-1} , \\ (1-r)^{(2N-2)/(N-2)} \log (1-r) , & \alpha = (N-2)^{-1} , \\ (1-r)^{N-1} \log (1-r) , & \alpha = -1 . \end{cases}$$

For $\Delta q(r) = 1$,

$$[r^{N-1}(1-r^2)^{(N-2)\alpha} q'(r)]' \sim (1-r)^{N\alpha} ,$$

$$q'(r) \sim \begin{cases} (1-r)^{2\alpha+1} , & \alpha \neq -N^{-1} , \\ (1-r)^{(N-2)/N} \log (1-r) , & \alpha = -N^{-1} . \end{cases}$$

For $N \geq 2$,

$$q_0(r) \sim \begin{cases} (1-r)^{2\alpha+2} , & \alpha \neq -N^{-1} , -1 , \\ (1-r)^{(2N-2)/N} \log (1-r) , & \alpha = -N^{-1} , \\ \log (1-r) , & \alpha = -1 . \end{cases}$$

As $r \to 1$,

$$h_0(r) \to 0 \quad \text{if} \quad \begin{cases} N = 2 , & \text{any } \alpha , \\ N > 2 , & \alpha < (N-2)^{-1} , \end{cases}$$

$$u_0(r) \to 0 \quad \text{if} \quad \begin{cases} N = 2 , & \alpha > -3/2 , \\ N = 3 , & \alpha > -3 , \\ N = 4 , & \text{any } \alpha , \\ N > 4 , & \alpha < 3(N-4)^{-1} . \end{cases}$$

We conclude by the Lemma that

$$\sigma < \infty \quad \text{if} \quad \begin{cases} N = 2 , & \alpha > -3/2 , \\ N = 3 , & \alpha \in (-3 , 1) , \\ N > 3 , & \alpha < (N-2)^{-1} , \end{cases}$$

as claimed.

In preparation for the case $\sigma = \infty$, we observe that

$$h_0 \notin B \Leftrightarrow \begin{cases} N = 2\,, & \text{no } \alpha\,, \\ N > 2\,, & \alpha \geq (N-2)^{-1}\,, \end{cases}$$

$$u_0 \notin B \Leftrightarrow \begin{cases} N = 2\,, & \alpha \leq -3/2\,, \\ N = 3\,, & \alpha \leq -3\,, \\ N = 4\,, & \text{no } \alpha\,, \\ N > 4\,, & \alpha \geq 3(N-4)^{-1}\,, \end{cases}$$

$$q_0 \notin B \Leftrightarrow N \geq 2\,, \quad \alpha \leq -1\,.$$

We have obtained

$$au_0 + b\Delta u_0 \notin B \Leftrightarrow \begin{cases} N = 2\,, & \alpha \leq -3/2\,, \\ N = 3\,, & \alpha \notin (-3\,,1)\,, \\ N > 3\,, & \alpha \geq (N-2)^{-1}\,, \end{cases}$$

except that we shall return later to the case $N > 4$, $\alpha \geq 3(N-4)^{-1}$. Here for $N = 2$, we have $h_0 \in B$, $u_0 \notin B$, with

$$u_0(r) \sim \begin{cases} (1-r)^{2\alpha+3}\,, & \alpha < -3/2\,, \\ \log(1-r)\,, & \alpha = -3/2\,, \end{cases}$$

whereas

$$q_0(r) \sim (1-r)^{2\alpha+2}\,, \quad \alpha \leq -3/2\,.$$

Thus the rates of growth of u_0 and q_0 are different for $\alpha \leq -3/2$, and we have $\sigma \notin B$, hence $\sigma = \infty$ as claimed.

For $N = 3$, $h_0 \notin B$, $u_0 \in B$ if $\alpha \geq 1$, with

$$h_0(r) \sim \begin{cases} (1-r)^{-\alpha+1}\,, & \alpha > 1\,, \\ \log(1-r)\,, & \alpha = 1\,, \end{cases}$$

whereas

$$q_0(r) \sim (1-r)^{2\alpha+2}\,, \quad \alpha \geq 1\,.$$

Thus the rates of growth are different for $\alpha \geq 1$, hence $\sigma = \infty$. Moreover, $h_0 \in B$, $u_0 \notin B$ for $\alpha \leq -3$, with

$$u_0(r) \sim \begin{cases} (1-r)^{\alpha-3}\,, & \alpha < -3\,, \\ \log(1-r)\,, & \alpha = -3\,, \end{cases}$$

whereas

$$q_0(r) \sim (1-r)^{2\alpha+2}\,, \quad \alpha \leq -3\,.$$

The rates of growth are different for $\alpha \leq -3$, hence $\sigma = \infty$.

For $N = 4$, $h_0 \notin B$, $u_0 \in B$ if $\alpha \geq \frac{1}{2}$, with

$$h_0(r) \sim \begin{cases} (1-r)^{-2\alpha+1}, & \alpha > \frac{1}{2}, \\ \log(1-r), & \alpha = \frac{1}{2}, \end{cases}$$

whereas

$$q_0(r) \sim (1-r)^{2\alpha+2}, \quad \alpha \geq \tfrac{1}{2}.$$

The rates of growth are different for $\alpha \geq \frac{1}{2}$, hence $\sigma = \infty$.

For $N > 4$, $h_0 \notin B$, $u_0 \in B$ if $\alpha \in [(N-2)^{-1}, 3(N-4)^{-1}]$, with

$$h_0(r) \sim \begin{cases} (1-r)^{-(N-2)\alpha+1}, & \alpha \in ((N-2)^{-1}, 3(N-4)^{-1}), \\ \log(1-r), & \alpha = (N-2)^{-1}, \end{cases}$$

whereas

$$q_0(r) \sim (1-r)^{2\alpha+2}, \quad \alpha \in [(N-2)^{-1}, 3(N-4)^{-1}).$$

Thus the rates of growth are different for $\alpha \in [(N-2)^{-1}, 3(N-4)^{-1})$, hence $\sigma = \infty$. Moreover, $h_0 \notin B$, $u_0 \notin B$ for $\alpha \geq 3(N-4)^{-1}$, with

$$h_0(r) \sim (1-r)^{-(N-2)\alpha-1}, \quad \alpha > 3(N-4)^{-1},$$

$$u_0(r) \sim \begin{cases} (1-r)^{-(N-4)\alpha-3}, & \alpha > 3(N-4)^{-1}, \\ \log(1-r), & \alpha = 3(N-4)^{-1}, \end{cases}$$

whereas

$$q_0(r) \sim (1-r)^{2\alpha+2}, \quad \alpha \geq 3(N-4)^{-1}.$$

The rates of growth are all different for $\alpha \geq 3(N-4)^{-1}$, hence $\sigma = \infty$.

The proof of Theorem 3 is herewith complete.

7. We proceed to the proof of the fundamental property of σ referred to in No. 1. Let R be an arbitrary Riemannian manifold, R_0 its regular subregion, and x a point of $S_0 = R - \bar{R}_0$.

Proposition. *The finiteness of the biharmonic measure* $\sigma(x)$ *on* $\bar{S}_0$ *is independent of* R_0 *and of* x.

Proof. For any region G, let $g_G(x, y)$ be the harmonic Green's function on G, with pole y. Denote the harmonic measure on S_0 by ω. The biharmonic measure on S_0 and the biharmonic Green's function on R, if they exist, are

$$\sigma_{S_0}(x) = \int_{S_0} g_{S_0}(x, y)\omega(y)dy,$$

$$\gamma(p, q) = \int_R g_R(p, y)g_R(y, q)dy.$$

We are to prove:

I. *If* $\sigma_{S_0}(x) < \infty$ *for some* $S_0 , x \in S_0$, *then* $\gamma(p , q) < \infty$ *for any* $p , q \in R$.

II. *If* $\gamma(p , q) < \infty$ *for some* $p , q \in R$, *then* $\sigma_{S_0}(x) < \infty$ *for any* $S_0 , x \in S_0$.

Proof of I. Given $\sigma_{S_0}(x) < \infty$ for some $S_0 , x \in S_0$, choose any $p , q \in R$ and regular subregions R_1 , Ω of R with

$$\bar{R}_0 \cup x \cup p \cup q \subset R_1 \subset \bar{R}_1 \subset \Omega .$$

Set $\alpha_0 = \partial R_0$, $\alpha_1 = \partial R_1$, $\beta_\Omega = \partial \Omega$, $S_1 = R - \bar{R}_1$, and take

$$\omega_\Omega \in C(\bar{\Omega} \cap \bar{S}_0) \cap H(\Omega \cap S_0) , \quad \omega_\Omega | \alpha_0 = 1 , \quad \omega_\Omega | \beta_\Omega = 0 .$$

We shall use the following constants:

$$m_{1\Omega} = \min_{y \in \alpha_1} g_{S_0 \cap \Omega}(y , x) , \quad M_{1\Omega} = \max_{y \in \alpha_1} g_{S_0 \cap \Omega}(y , x) ,$$

$$m_{2\Omega} = \min_{\alpha_1} \omega_\Omega , \quad M_{2\Omega} = \max_{\alpha_1} \omega_\Omega ,$$

$$m_{3\Omega} = \min_{y \in \alpha_1} g_\Omega(y , p) , \quad M_{3\Omega} = \max_{y \in \alpha_1} g_\Omega(y , p) ,$$

$$m_{4\Omega} = \min_{y \in \alpha_1} g_\Omega(y , q) , \quad M_{4\Omega} = \max_{y \in \alpha_1} g_\Omega(y , q) ,$$

$$m_i = \lim_{\Omega \to R} m_{i\Omega} , \quad M_i = \lim_{\Omega \to R} M_{i\Omega} , \quad i = 1 , 2 , 3 , 4 ,$$

$$k_1 = \frac{M_3 M_4}{m_1 m_2} , \quad k_2 = \frac{M_1 M_2}{m_3 m_4} .$$

We obtain

$$g_\Omega(y , p) \le \frac{M_{3\Omega}}{m_{1\Omega}} g_{S_0 \cap \Omega}(y , x) \quad \text{on} \quad \alpha_1 \cup \beta_\Omega , \quad \text{hence on} \quad \bar{\Omega} \cap \bar{S}_1 ,$$

$$g_R(y , p) \le \frac{M_3}{m_1} g_{S_0}(y , x) \quad \text{on} \quad \bar{S}_1 ,$$

$$g_\Omega(y , q) \le \frac{M_{4\Omega}}{m_{2\Omega}} \omega_\Omega(y) \quad \text{on} \quad \bar{\Omega} \cap \bar{S}_1 ,$$

$$g_R(y , q) \le \frac{M_4}{m_2} \omega(y) \quad \text{on} \quad \bar{S}_1 .$$

Therefore,

$$\int_{S_1} g_R(p , y) g_R(y , q) dy = \int_{S_1} g_R(y , p) g_R(y , q) dy$$

$$\leq k_1 \int_{S_1} g_{S_0}(y, x)\omega(y)dy$$

$$= k_1 \int_{S_1} g_{S_0}(x, y)\omega(y)dy$$

$$< k_1 \int_{S_0} g_{S_0}(x, y)\omega(y)dy < \infty ,$$

and a fortiori

$$\gamma(p, q) = \int_R g_R(p, y)g_R(y, q)dy$$

$$= C_1 + \int_{S_1} g_R(p, y)g_R(y, q)dy$$

$$= C_1 + k_1\sigma_{S_0}(x) < \infty .$$

Proof of II. Suppose $\gamma(p, q) < \infty$ for some $p, q \in R$. Take any regular region R_0 and an $x \in S_0 = R - \bar{R}_0$. For R_1, Ω chosen as before,

$$g_{S_0 \cap \Omega}(y, x) \leq \frac{M_{1\Omega}}{m_{3\Omega}} g_\Omega(y, p) \quad \text{on} \quad \alpha_1 \cup \beta_\Omega , \quad \text{hence on} \quad \bar{S}_1 \cap \bar{\Omega} ,$$

$$g_{S_0}(y, x) \leq \frac{M_1}{m_3} g_R(y, p) \quad \text{on} \quad \bar{S}_1 ,$$

$$\omega_\Omega(y) \leq \frac{M_{2\Omega}}{m_{4\Omega}} g_\Omega(y, q) \quad \text{on} \quad \alpha_1 \cup \beta_\Omega , \quad \text{hence on} \quad \bar{S}_1 \cap \bar{\Omega} ,$$

$$\omega(y) \leq \frac{M_2}{m_4} g_R(y, q) \quad \text{on} \quad \bar{S}_1 .$$

Therefore,

$$\sigma_{S_0}(x) = \int_{S_0} g_{S_0}(x, y)\omega(y)dy$$

$$= C_2 + \int_{S_1} g_{S_0}(x, y)\omega(y)dy$$

$$\leq C_2 + k_2 \int_{S_1} g_R(y, p)g_R(y, q)dy$$

$$= C_2 + k_2 \left(C_3 + \int_R g_R(p\,, y) g_R(y\,, q)\; dy \right)$$

$$= C_2 + k_2(C_3 + \gamma(p\,, q)) < \infty\,.$$

8. In view of the Proposition, we may introduce the class of Riemannian manifolds R with ideal boundaries of infinite biharmonic measure

$$O_{\Sigma} = \{R \, \sigma = \infty\}\,.$$

The class of Riemannian manifolds which do not carry biharmonic Green's functions γ is denoted by O_Γ (cf. [22]). Properties I and II of σ and γ provide us with our main result:

Theorem 4. $O_{\Sigma} = O_\Gamma$.

As a consequence, e.g. the values of α in Theorem 3 characterize the Poincaré balls in O_Γ. Moreover, known properties of O_Γ carry over to O_{Σ}. E.g., parabolicity implies $\sigma = \infty$.

The author is indebted to Professor Cecilia Wang for a careful checking of the manuscript.

A bibliography of recent work in the field is attached.

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Printed November 1974