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THE FAMILY OF PDOL GROWTH-SETS IS PROPERLY INCLUDED IN THE FAMILY OF DOL GROWTH-SETS

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1. Introduction

L-systems have been introduced for biological purposes (see [3]). However, these have been studied intensively during the last few years from the formal language point of view (see [1] and [6]). A particularly interesting aspect within L-systems is the theory of growth functions. These have been studied for instance in [7] and [8].

The purpose of this note is to show that the family of PDOL growth-sets is properly included in the family of DOL growth-sets. As a corollary of this result we also solve a problem introduced in [5], namely that $\mathcal{L}_{\mathrm{CPDOL}} \subsetneqq \mathcal{L}_{\mathrm{CDOL}}$.

2. Notations

We assume that the reader is familiar with the standard formal language notations. For the definitions of DOL-systems, -languages, and -sequences we refer to [1]. We say that a DOL-system G is λ-free, or a PDOL-system, iff there are no λ-productions in G.

If G is a DOL-system, then $L(G)$ (resp. $E(G)$) means the language (resp. the sequence) generated by G. The *growth-set* generated by G is

$$|L(G)| = \{|P| \quad |P \in L(G)\},$$

where $|P|$ means the length of the word P. Let

$$E(G) = \omega_0, \omega_1, \ldots$$

Then the *growth-sequence* generated by G is

$$|E(G)| = |\omega_0|, |\omega_1|, \ldots$$

We say that a homomorphism $h : V_1^* \to V_2^*$ is a *coding* iff it maps each a letter to another letter. So all codings are length preserving.

A language L is called a CDOL-language (resp. a CPDOL-language) iff there exists a DOL-system G (resp. a PDOL-system G) and a coding h such that

$$L = h(L(G)).$$

The family of CDOL-languages (resp. CPDOL-languages) is denoted by $\mathcal{L}_{\mathrm{CDOL}}$ (resp. $\mathcal{L}_{\mathrm{CPDOL}}$).

3. Lemmas

We need the following three lemmas.

Lemma 1. *Let*

(1) $$|E(G)| = |\omega_0|\,,\ |\omega_1|\,,\ \ldots$$

be a DOL growth-sequence. Then (1) *is ultimately periodic modulo* 2.

Proof. It is well known (see [7]), that, for $n \geq n_0$, (1) satisfies a recursion formula with integer coefficients, say

$$|\omega_n| = \sum_{i=1}^{k} \alpha_i |\omega_{n-i}|\,, \quad \text{for} \quad n \geq n_0\,.$$

Let g be the canonical homomorphism of Z onto Z_2. By applying g to the above equation we obtain in the finite set Z_2 the recursion formula

$$g(|\omega_n|) = \sum_{i=1}^{k} g(\alpha_i) g(|\omega_{n-i}|)\,, \quad \text{for} \quad n \geq n_0\,.$$

Thus, the sequence determined by this recursion formula must be ultimately periodic in Z_2. So we have proved Lemma 1.

Let H be the following DOL-system. The axiom is a, the alphabet is

$$V = \{a\,, b\,, c\,, a_1\,, b_1\,, c_1\,, a_2\,, b_2\,, c_2\}\,,$$

and the productions are as follows:

$$\begin{aligned} &a \to a_1a_2\,, \quad a_1 \to abc^2\,, \quad a_2 \to \lambda\,,\\ &b \to b_1b_2\,, \quad b_1 \to bc^2\,, \quad b_2 \to \lambda\,,\\ &c \to c_1c_2\,, \quad c_1 \to c\,, \quad c_2 \to \lambda\,. \end{aligned}$$

Let

$$L_1 = \{P \in V^* | a \Rightarrow^{2n} P\,, \quad \text{for some} \quad n \geq 0\}$$

and

$$L_2 = \{P \in V^* | a \Rightarrow^{2n+1} P\,, \quad \text{for some} \quad n \geq 0\}\,.$$

Then the corresponding sequences are

$$E_1 = a\,,\ abc^2\,,\ abc^2bc^4\,,\ abc^2bc^4bc^6\,,\ \ldots$$

and

$$E_2 = a_1a_2\,,\, a_1a_2b_1b_2(c_1c_2)^2\,,\;\; a_1a_2b_1b_2(c_1c_2)^2b_1b_2(c_1c_2)^4\,,\,\ldots$$

So the language generated by H is

$$L(H) = L_1 \cup L_2 = \{a, abc^2bc^4 \ldots bc^{2n} | n \geq 1\} \cup \{h(P) | P \in L_1\}\,,$$

where h is the homomorphism of $\{a\,,b\,,c\}$ into V defined by $h(y) = y_1y_2$. Because

$$1 + n + 2 + 4 + \ldots + 2n = (n + 1)^2$$

the growth-set determined by H is

$$|L(H)| = \{n^2\,,\, 2n^2 | n \geq 1\}\,.$$

We now put the elements of $|L(H)|$ in increasing order and let X be this sequence. Denote

(2) $$X = x_1\,,\, x_2\,,\,\ldots$$

Lemma 2. X *is not a DOL growth-sequence.*

Proof. Assume the contrary: that a DOL-system H_1 generates the sequence X. Then, by Lemma 1, (2) is ultimately periodic modulo 2. So there exist natural numbers r and s such that

$$x_r \text{ is an odd square}$$

and for each i and j, $i \geq 0$, $0 \leq j \leq s - 1$,

$$x_{r+j+is} \equiv x_{r+j+(i+1)s} \pmod 2\,.$$

Let $x_r = k^2$. Note that in (2) all odd integers are squares. So if m is the number of odd integers in the period, then for each $i \geq 0$

$$x_{r+is} = (k + i2m)^2\,.$$

But this implies that in each period there must also be a fixed number of integers of the form $2n^2$. This however leads to a contradiction, as we shall now show.

For all $i \geq 0$, consider the natural numbers z_i satisfying the condition

$$(k + i2m)^2 < 2z_i^2 < (k + (i + 1)2m)^2\,,$$

or equivalently,

(3) $$\frac{k}{\sqrt{2}} + i\sqrt{2}\,m < z_i < \frac{k}{\sqrt{2}} + i\sqrt{2}\,m + \sqrt{2}\,m\,.$$

By what we have shown, the number of such z_i's is the same for all $i \geq 0$.

Trivially this number is either $[\sqrt{2}\,m]$ or $[\sqrt{2}\,m]+1$. Let δ_0 and δ_1 be positive real numbers defined by

$$\delta_0 = \sqrt{2}\,m - [\sqrt{2}\,m]$$

and

$$\delta_1 = [\sqrt{2}\,m] + 1 - \sqrt{2}\,m\,.$$

First, assume that for all $i \geq 0$ the number of z_i's is $[\sqrt{2}\,m]$. Choose i_0 such that $i_0\delta_0 > 1$. Then the length of the interval

$$\left(\frac{k}{\sqrt{2}}\,,\ \frac{k}{\sqrt{2}} + i_0\sqrt{2}\,m\right)$$

is

$$i_0\sqrt{2}\,m = i_0[\sqrt{2}\,m] + i_0\delta_0 > i_0[\sqrt{2}\,m] + 1\,.$$

So the number of z_i's in this interval is at least $i_0[\sqrt{2}\,m]+1$. On the other hand, by our assumption, their number is $i_0[\sqrt{2}\,m]$.

Secondly, assume that the number of z_i's is $i_0[\sqrt{2}\,m]+1$, for all $i \geq 0$. Now choose i_1 such that $i_1\delta_1 > 2$. Thus, the length of the interval

$$\left(\frac{k}{\sqrt{2}}\,,\ \frac{k}{\sqrt{2}} + i_1\sqrt{2}\,m\right)$$

is

$$i_1\sqrt{2}\,m = i_1[\sqrt{2}\,m] + i_1 - i_1\delta_1 < i_1[\sqrt{2}\,m] + i_1 - 2\,.$$

Thus, in this interval there are at most $i_1[\sqrt{2}\,m] + i_1 - 1$ numbers z_i. But by our assumption, in this interval there must be $i_1[\sqrt{2}\,m] + i_1$ numbers of this kind.

Because both the cases lead to a contradiction, we have proved Lemma 2.

We also need the following lemma, which is Lemma 5.4. of [4].

Lemma 3. *Let G be a PDOL-system generating an infinite language. Then there exists a PDOL-system G_1 such that*

(i) $|L(G)| = |L(G_1)|$,

(ii) *the sequence* $|E(G_1)|$ *is strictly increasing.*

4. Results

Now we are ready to establish our results.

Theorem 1. *The family of PDOL growth-sets is a proper subset of the family of DOL growth-sets.*

Proof. The inclusion is trivial. It is proper because $|L(H)|$ is a DOL growth-set, but is not, by Lemmas 2 and 3, a PDOL growth-set.

As an immediate corollary of Theorem 1 we can solve a problem proposed in [5].

Theorem 2. *The family* $\mathcal{L}_{\mathrm{CPDOL}}$ *is properly included in the family* $\mathcal{L}_{\mathrm{CDOL}}$.

Examples of languages which lie in the difference $\mathcal{L}_{\mathrm{CDOL}} \setminus \mathcal{L}_{\mathrm{CPDOL}}$ are the languages $L(H)$ and $L = \{a^{n^2}, a^{2n^2} | n \geq 1\}$.

We can generalize Theorem 1 to cover all *growing* DOL-systems, i.e., systems with an increasing growth-sequence. Of course any PDOL-system is a growing DOL-system.

Theorem 3. *The family of growth-sets generated by growing DOL-systems is properly included in the family of DOL growth-sets.*

Proof. It suffices to show that Lemma 3 can be generalized for growing DOL-systems. Assume that G is a growing DOL-system with $E(G) = \omega_0, \omega_1, \ldots$ In the following we use the notations of [4].

Let M be the growth-matrix of G, π the Parikh-vector associated with the axiom of G, and η the column vector with all elements equal to 1. Then the sequence

$$d_n = \pi(M^n - M^{n-1})\eta = \pi(M - I)M^{n-1}\eta\,, \quad n = 1, 2, \ldots \tag{4}$$

tells us how much the length of the word grows during the nth step of the derivation. By (4), the d_n's satisfy a recursion formula with integer coefficients. Thus, by the Theorem proved in [2], zeros occur in (4) ultimately periodically.

For all $i \geq 0$, let $Min(\omega_i)$ denote the set of symbols occurring in ω_i. It is well known that the sequence

$$Min(\omega_0)\,, Min(\omega_1)\,, \ldots \tag{5}$$

is ultimately periodic.

Consider now the sequence consisting of ordered pairs

$$(s(d_0)\,, Min(\omega_0))\,, (s(d_1)\,, Min(\omega_1))\,, \ldots\,, \tag{6}$$

where $s(0) = 0$ and $s(n) = 1$, for $n \geq 1$. Because the component sequences of (6) are ultimately periodic, so is the whole sequence (6).

From this point on the proof is a straightforward modification of the proof of Lemma 5.4. in [4] (it uses only the periodicity of (5)). We omit the details.

Remark 1. The proof of Lemma 3 in [4] is constructive. However, our analogous proof for growing DOL-systems is not constructive, because we need the Theorem of [2].

Remark 2. In [4] M. Nielsen solves the growth-set equivalence problem for PDOL-systems by changing the considered PDOL-sequences effectively to strictly increasing PDOL-sequences (with the same growth-set). Lemma 2 shows that we cannot solve the growth-set equivalence problem for DOL-systems by this method. In fact, it is not known if this problem is decidable at all.

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