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PIECEWISE QUASICONFORMAL MAPS ARE QUASICONFORMAL

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1. *Introduction.* We use the notation and terminology of [7], except that we always assume that a quasiconformal map is sense-preserving. We also extend the concept of a quasiconformal map for arbitrary sets in the n-space R^n. Suppose that $A \subset R^n$ and that $f : A \to R^n$ is a map. If A is open, we say that f is quasiconformal if $f|D$ is quasiconformal for every component D of A. Furthermore, the outer and inner dilatations of f are defined by the well-known formulae

$$K_O(f) = \operatorname*{ess\,sup}_{x \in A} \frac{|f'(x)|^n}{J(x,f)}, \quad K_I(f) = \operatorname*{ess\,sup}_{x \in A} \frac{J(x,f)}{l(f'(x))^n},$$

or equivalently,

$$K_O(f) = \sup_D K_O(f|D), \quad K_I(f) = \sup_D K_I(f|D),$$

where the suprema are taken over all components D of A. If A is not open, we say that f is quasiconformal if it has a quasiconformal extension $g : G \to R^n$ to some open neighborhood G of A, and we set

$$K_O(f) = \inf_g K_O(g), \quad K_I(f) = \inf_g K_I(g)$$

over all such extensions g.

The purpose of this paper is to prove the following result:

2. Theorem. *Suppose that $f : G \to G'$ is a sense-preserving homeomorphism, where G and G' are domains in R^n. Suppose also that $G = \bigcup\{E_k \mid k \in N\}$ such that $K_O(f|E_k) \leq K$ for all k. Then f is quasiconformal, and $K_O(f) \leq K$. Similarly, if $K_I(f|E_k) \leq K$ for all k, then $K_I(f) \leq K$.*

3. *Remarks.* A weaker result has been proved by Rickman [3, Theorem 1]. These results can be applied to extension problems. For example,

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let $f : G \to G$ be a quasiconformal map which extends to a homeomorphism $f^* : \overline{G} \to \overline{G}$ such that $f^*(x) = x$ for all boundary points x of G. Then we can extend f to a quasiconformal map $g : R^n \to R^n$ by setting $g(x) = x$ for $x \notin G$. Furthermore, g and f have the same dilatations. The standard removability argument [7, 35.1, p. 118] applies only if ∂G is of σ-finite $(n-1)$-measure. For another application, see [5, p. 8].

The proof of Theorem 2 is based on a modified version of the analytic definition of quasiconformality.

4. *Definitions.* Suppose that G is an open set in R^n and that $f : G \to R^n$ is a map. We say that f is NL if f satisfies the condition (N) on almost every line L, parallel to the coordinate axes. In other words, if $E \subset L \cap G$ and if E is of linear measure zero, then also fE is of linear measure zero. The *artificial derivative* of f at a point x in G is the linear map $f'_a(x) : R^n \to R^n$ defined as follows: If the partial derivative $\partial_i f_j(x)$ exists, then $e_j \cdot f'_a(x)e_i = \partial_i f_j(x)$. Otherwise $e_j \cdot f'_a(x)e_i = 0$. If f is differentiable at x, then $f'_a(x)$ is equal to the ordinary derivative $f'(x)$ of f. The *upper volume derivative* of f at x is defined by

$$\mu'_f(x) = \limsup_{r \to 0} \frac{m(fB^n(x,r))}{m(B^n(x,r))} .$$

5. Theorem. *Let $f : G \to G'$ be a sense-preserving homeomorphism such that*

(1) $$f \text{ is NL},$$

(2) $$|f'_a(x)|^n \leq K\mu'_f(x) \quad a.e.$$

Then f is quasiconformal, and $K_O(f) \leq K$.

Proof. We first show that f is ACL. Fix i and j in $\{1, \ldots, n\}$. Let P be the set of all x in G such that $\partial_i f_j(x)$ exists, and let L be a line parallel to the x_i-axis such that (i) f_j satisfies the condition (N) on L, (ii) μ'_f is locally integrable on $L \cap G$. Since μ'_f is locally integrable in G [7, 24.2.3, p. 84], almost every line has these properties. It suffices to show that f_j is locally absolutely continuous on $L \cap G$. Let I be a closed line segment on $L \cap G$. Then

$$\int_{P \cap I} |\partial_i f_j|^n \, dm_1 \leq \int_I |f'_a(x)|^n \, dm_1(x) \leq K \int_I \mu'_f \, dm_1 < \infty .$$

Thus $|\partial_i f_j|^n$, and hence also $\partial_i f_j$, is integrable over $P \cap I$. By Bary's theorem [4, p. 285], f_j is absolutely continuous on I. Thus f is ACL.

Since $|f'(x)|^n \leq K\mu'_f(x)$ a.e., f is ACL^n. As an ACL^n-homeomorphism, f is differentiable a.e. [6, Lemma 3]. Thus $\mu'_f(x) = J(x, f)$ a.e. Hence (2) implies $K_O(f) \leq K$.

6. *Proof of Theorem* 2. Assume first that $K_O(f|E_k) \leq K$ for all k. Let $\varepsilon > 0$. For every $k \in N$ choose an extension g_k of $f|E_k$ to a neighborhood D_k of E_k so that $K_O(g_k) \leq K + \varepsilon$. Replacing E_k by $\bar{E}_k \cap D_k$ we may assume that each E_k is a Borel set. We shall show that the conditions of Theorem 5 are satisfied.

If every g_k satisfies the condition (N) on a line L, then f also satisfies the same condition, because for every $E \subset L, fE = \bigcup\{g_k(E \cap E_k) \mid k \in N\}$. Thus f is NL.

We let B_k denote the set of all x in E_k such that (i) x is a point of density of E_k, (ii) x is a point of linear density of E_k in the direction of every coordinate axis, (iii) $g_k(x)$ is a point of density of $g_kE_k = fE_k$, (iv) g_k is differentiable at x. Then $m(E_k \setminus B_k) = 0$. For (i), (ii), and (iii) this follows from standard density theorems and from the fact that g_k^{-1} satisfies the condition (N). For (iv) this follows from the quasiconformality of g_k. Setting $A = \bigcup\{E_k \setminus B_k | k \in N\}$ we have $m(A) = 0$. We shall show that $|f_a'(x)|^n \leq K_1\mu_f'(x)$ for every x in $G \setminus A$ and for $K_1 = n^n(K + \varepsilon)$.

Let $x \in G \setminus A$. Then $x \in E_k \setminus A \subset B_k$ for some k. If $\partial_i f_j(x)$ exists, then (ii) and (iv) imply $\partial_i f_j(x) = \partial_i (g_k)_j(x)$. Thus $|\partial_i f_j(x)| \leq |g_k'(x)|$. Hence $|f_a'(x)| \leq n|g_k'(x)|$, which yields $|f_a'(x)|^n \leq K_1 J(x, g_k)$. Consequently, it suffices to show that $J(x, g_k) \leq \mu_f'(x)$. Using the standard notation (see e.g. [7, p. 78]) we set $y = f(x) = g_k(x)$, $L = L(x, g_k, r)$, $l = l(x, g_k, r)$, where r is so small that $\bar{B}^n(x, r) \subset D_k$. Then

$$J(x, g_k) = \mu_{g_k}'(x) \leq \mu_f'(x) + \limsup_{r \to 0} \frac{m(g_k(B^n(x,r) \setminus E_k))}{m(B^n(x,r))}.$$

Here

$$m(g_k(B^n(x,r) \setminus E_k)) \leq \frac{m(B^n(y,L) \setminus g_kE_k)}{m(B^n(y,L))} \left(\frac{L}{l}\right)^n m(g_kB^n(x,r)) .$$

By (iii), the first factor on the right tends to zero as $r \to 0$. The second factor remains bounded by quasiconformality. Since the third factor is asymptotically equal to $J(x, g_k) m(B^n(x, r))$, we obtain $J(x, g_k) \leq \mu_f'(x)$. By Theorem 5, f is quasiconformal with $K_O(f) \leq K_1$.

Let x be a point in $G \setminus A$ a twhich f is differentiable. Then $x \in B_k$ for some k, and (iv) together with (i) or (ii) implies $f'(x) = g_k'(x)$. Thus $|f'(x)|^n = |g_k'(x)|^n \leq (K+\varepsilon)J(x,g_k) = (K+\varepsilon)J(x,f)$. Since this holds a.e. in G, $K_O(f) \leq K+\varepsilon$. Since ε is arbitrary, $K_O(f) \leq K$.

Finally assume that $K_I(f|E_k) \leq K$ for all k. Since $K_O \leq K_I^{n-1}$, it follows from the first part of the theorem that f is quasiconformal. Repeating the above argument with K_O replaced by K_I we obtain $J(x,f) \leq (K+\varepsilon)l(f'(x))^n$ a.e. Hence $K_I(f) \leq K$.

7. *Quasiregular maps.* The above result can easily be extended to quasiregular maps. For the definition and the basic properties of these maps we refer to [1]. If A is any set in R^n, we say that a map $f : A \to R^n$ is quasiregular if it has a quasiregular extension to some neighborhood of A. Then a slight modification of the above proof yields:

8. Theorem. *Suppose that $f : G \to R^n$ is a sense-preserving discrete open map of a domain $G \subset R^n$. Suppose also that $G = \bigcup\{E_k \mid k \in N\}$ such that $K_O(f \mid E_k) \leq K$ for all k. Then f is quasiregular and $K_O(f) \leq K$. Similarly, if $K_I(f \mid E_k) \leq K$ for all k, then $K_I(f) \leq K$.*

9. *Open questions.* 1. Is Theorem 8 true for all (continuous) maps, without any condition on discreteness or openness? A positive answer would give as a very special case a theorem of Radó [2]: If $f : G \to R^2$ is continuous and if f is analytic in $G \setminus f^{-1}(0)$, then f is analytic.

2. Suppose that $f : G \to R^n$ is sense-preserving, discrete, and open, and suppose that f is locally K-quasiconformal outside the branch set B_f. Is f K-quasiregular? The answer is known to be affirmative if B_f is of σ-finite $(n-1)$-measure. We remark that Theorem 8 can be sometimes used if we know something about $f \mid B_f$. For example, if $f(x) = x$ for all x in B_f, then f is K-quasiregular.

References

[1] MARTIO, O., S. RICKMAN, and J. VÄISÄLÄ: Definitions for quasiregular mappings. - Ann. Acad. Sci. Fenn. Ser. A I 448, 1969, 1—40.

[2] RADÓ, T.: Über eine nicht fortsetzbare Riemannsche Mannigfaltigkeit. - Math. Z. 20, 1924, 1—6.

[3] RICKMAN, S.: Removability theorems for quasiconformal mappings. - Ann. Acad. Sci. Fenn. Ser. A I 449, 1969, 1—8.

[4] SAKS, S.: Theory of the integral. - Warsaw, 1937.

[5] TAARI, O.: Einige Extremalprobleme für n-dimensionale quasikonforme Abbildungen. - Ann. Acad. Sci. Fenn. Ser. A I 553, 1973, 1—19.

[6] VÄISÄLÄ, J.: Two new characterizations for quasiconformality. - Ann. Acad. Sci. Fenn. Ser. A I 362, 1965, 1—12.

[7] —»— Lectures on n-dimensional quasiconformal mappings. - Lecture Notes in Mathematics 229, Springer-Verlag, Berlin—Heidelberg—New York, 1971.

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