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ON THE METHOD OF SUCCESSIVE APPROXIMATIONS FOR VOLTERRA INTEGRAL EQUATIONS

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1. Introduction

Walter has in [4] proved an existence and uniqueness theorem for Volterra integral equations of the form

$$x(t) = y(t) + \int_0^t k(t, s, x(s))\,ds \tag{1}$$

in a Banach space, by the method of successive approximations (see also [3]).

In this paper we shall prove an analogous theorem with more general hypotheses, and derive estimates for the solution of (1), with minimal solutions of the scalar comparison equation

$$u(t) = v(t) + \int_0^t \omega(t, s, u(s))\,ds \tag{2}$$

as estimate functions.

The results so obtained are then applied to the initial value problem

$$x'(t) = y'(t) + f(t, x(t)), \qquad x(0) = y(0), \tag{3}$$

in a Banach space, with the scalar comparison problem

$$u'(t) = v'(t) + g(t, u(t)), \qquad u(0) = v(0). \tag{4}$$

2. Notations

Let X be a Banach space with norm $\|\cdot\|$. Given $T > 0$, denote $J = [0, T]$, and let $C(J)$ and $C_+(J)$ be the spaces of all continuous mappings from J into X and into the nonnegative reals $\mathbf{R}_+$, respectively, with the topology of uniform convergence. For $y \in C(J)$ define $|y| \in C_+(J)$ by

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$$|y|(t) = \|y(t)\|, \quad t \in J,$$

and for $u, v \in C_+(J)$

$$u \leqq v \text{ if and only if } u(t) \leqq v(t) \text{ for each } t \in J.$$

Any constant mapping of $C(J)$ or of $C_+(J)$ is denoted by its value.

Denote by $\mathcal{K}_0^+$ the class of all functions ω from the set $\{(t, s, r) \in J \times J \times \mathbf{R}_+ \mid s \leqq t\}$ into $\mathbf{R}_+$ for which $\omega(t, s, r)$ is measurable in $s \in [0, t]$ for each $(t, r) \in J \times \mathbf{R}_+$, continuous in $(t, r) \in J \times \mathbf{R}_+$ for almost every $s \in [0, t]$, and for each $M > 0$ there is an integrable function $h: J \to \mathbf{R}_+$ such that

$$\omega(t, s, r) \leqq h(s) \tag{2.1}$$

for $0 \leqq s \leqq t \leqq T$ and $0 \leqq r \leqq M$. Correspondingly, denote by $\mathcal{K}_0$ the class of all mappings k from $\{(t, s, z) \in J \times J \times X \mid s \leqq t\}$ into X for which $k(t, s, z)$ is strongly measurable in $s \in [0, t]$ for each $(t, z) \in J \times X$, continuous in $(t, z) \in J \times X$ for almost every $s \in [0, t]$, and for each $M > 0$ there is an integrable $h: J \to \mathbf{R}_+$ such that

$$\|k(t, s, z)\| \leqq h(s) \tag{2.2}$$

whenever $0 \leqq s \leqq t \leqq T$ and $\|z\| \leqq M$.

Applying the Dominated Convergence Theorem for Lebesgue integrals one can show that the integral

$$\Omega u(t) = \int_0^t \omega(t, s, u(s))\,ds \tag{2.3}$$

exists in the Lebesgue sense for $\omega \in \mathcal{K}_0^+$, $u \in C_+(J)$ and $t \in J$, and that (2.3) defines a mapping $\Omega: C_+(J) \to C_+(J)$. Respective properties of Bochner integrals ensure that for $k \in \mathcal{K}_0$,

$$Kx(t) = \int_0^t k(t, s, x(s))\,ds, \quad t \in J, \quad x \in C(J), \tag{2.4}$$

defines a mapping $K: C(J) \to C(J)$.

Via the definitions (2.3) and (2.4) the integral equations (1) and (2) may be written in the forms

$$x = y + Kx, \tag{1'}$$

$$u = v + \Omega u, \tag{2'}$$

respectively.

3. Existence and uniqueness theorems

Making minor modifications to the proof of Theorem I.7.XII in [4] we obtain

Theorem 3.1. *Assume that* $k \in \mathcal{K}_0$, *and for* $0 \leqq s \leqq t \leqq T$, $z, \bar{z} \in X$,

$$\|k(t, s, z) - k(t, s, \bar{z})\| \leqq \omega(t, s, \|z - \bar{z}\|), \tag{3.1}$$

where $\omega \in \mathcal{K}_0^+$ *such that*

(i) $\omega(t, s, r)$ *is nondecreasing in* $r \in \mathbf{R}_+$ *for* $0 \leqq s \leqq t \leqq T$;
(ii) *the mapping* Ω, *defined by* (2.3), *has* $u = 0$ *as the only fixed point;*
(iii) *for each* $C > 0$ *there is* $w \in C_+(J)$ *satisfying*

$$w \geqq C + \Omega w. \tag{3.2}$$

Then for $y, x_1 \in C(J)$ *the successive approximations*

$$x_{n+1} = y + Kx_n, \qquad n \in \mathbf{N} = 1, 2, \ldots, \tag{3.3}$$

with K *given by* (2.4), *converge on* J *uniformly to a unique solution of* (1).

Proof. From (i) it follows that Ω is nondecreasing, and from (3.1) that

$$|Kx - K\bar{x}| \leqq \Omega(|x - \bar{x}|), \qquad x, \bar{x} \in C(J), \tag{3.4}$$

whence

$$|x_{n+1} - x_1| \leqq |y - x_1| + |Kx_1| + \Omega(|x_n - x_1|), \qquad n \in \mathbf{N}.$$

These properties, together with (iii), imply by induction that

$$|x_n - x_1| \leqq w, \qquad n \in \mathbf{N}, \tag{3.5}$$

for any $w \in C_+(J)$, satisfying (3.2) with

$$C \geqq |y - x_1| + |Kx_1|.$$

From (3.3), (3.4) and (3.5) it follows by induction that

$$|x_{n+m} - x_m| \leqq u_m, \qquad n, m \in \mathbf{N}, \tag{3.6}$$

where the functions u_m are defined by

$$u_1 = w, \; u_{m+1} = \Omega u_m, \qquad m \in \mathbf{N}.$$

The sequence (u_m) is nonincreasing and nonnegative, whence by (ii) it can be shown to converge to 0-function (see the proof of Theorem I.7.XII in [4]), uniformly on J. From (3.6) it then follows that (x_n) converges in $C(J)$. The conditions given for $\mathcal{K}_0$, together with the Dominated Convergence Theorem for Bochner integrals, ensure that the limit mapping of (x_n) is a solution of (1).

The uniqueness can be proved as in Theorem I.7.XII of [4].

Denote by $\mathcal{K}$ the class of those $k \in \mathcal{K}_0$ for which (2.2) holds for some integrable $h\colon J \to \mathbf{R}_+$ and for all $(t, s, z) \in J \times J \times X$, $s \leqq t$.

Corollary 3.1. *Let* $k \in \mathcal{K}$ *satisfy* (3.1) *with* $\omega \in \mathcal{K}_0^+$ *such that the hypotheses* (i) *and* (ii) *of Theorem* 3.1 *hold. Then* (1) *has for each* $y \in C(J)$ *a unique solution on* J.

Proof. Define

$$\bar{\omega}(t, s, r) = \min\{\omega(t, s, r), 2h(s)\}, \qquad 0 \leqq s \leqq t \leqq T, \qquad r \geqq 0,$$

where h is the majorant of k in (2.2). Then $\bar{\omega} \in \mathcal{K}_0^+$, (3.1) holds with ω replaced by $\bar{\omega}$, and $\bar{\omega}$ satisfies the hypotheses (i) and (ii) of Theorem 3.1. Also (iii) holds, because

$$w(t) = C + \int_0^t 2h(s)\,ds$$

satisfies (3.2). Thus the assertion follows from Theorem 3.1.

Remark 3.1. Corollary 3.1 simplifies the Existence and Convergence Theorem I.7.XII of [4] in the sense that the hypothesis

(γ) for each $C > 0$ there is $\varrho \in C_+(J)$ satisfying

$$\varrho \geqq C \text{ and } \varrho \geqq \Omega\varrho,$$

of the Theorem is not specified in the Corollary. In Theorem 3.1 we use the stronger condition (iii) in place of (γ), also to show the boundedness of (x_n) (see (3.5)), whence we may assume that $k \in \mathcal{K}_0$, instead of $k \in \mathcal{K}$. Furthermore, (iii) (but not (γ)) is sufficient for the considerations of the next section.

An essential point where the method of Walter, used in the proof for the uniform convergence of (x_n), differs from other methods (see for ex. [2]) is that the equicontinuity of (x_n) is not needed.

4. Integral inequalities

The estimates derived in the present section for solutions of (1) are based on

Lemma 4.1. *Let* $\omega \in \mathcal{K}_0^+$ *satisfy the hypotheses* (i) *and* (iii) *of Theorem* 3.1. *Then given* $v \in C_+(J)$ *the equation*

(2′) $$u = v + \Omega u,$$

with Ω *given by* (2.3), *has the minimal solution* u *on* J. *If* (y_n) *is a convergent sequence in* $C(J)$ *such that*

(4.1) $$|y_1| \leqq v \text{ and } |y_{n+1}| \leqq v + \Omega(|y_n|), \qquad n \in \mathbf{N},$$

then the limit mapping $\tilde{y}$ *of* (y_n) *satisfies*

(4.2) $$|\tilde{y}| \leqq u.$$

Proof. From (i) it follows that Ω is nondecreasing, whence the sequence (u_n) defined by

$$u_1 = v, \quad u_{n+1} = v + \Omega u_n, \quad n \in \mathbf{N}, \tag{4.3}$$

is nondecreasing and bounded above by any w satisfying (3.2) with $C \geqq |v|$. Since (u_n) is also equicontinuous (cf. the proof of Theorem I.2.II in [4]), it converges uniformly on J. The continuity of $\omega(t, s, r)$ in r, together with the Dominated Convergence Theorem, implies that the limit u of (u_n) is a solution of (2′). The solution u is minimal, since

$$u_n \leqq \bar{u}, \quad n \in N,$$

for any solution $\bar{u}$ of (2′).

The last conclusion of lemma is a consequence of

$$|y_n| \leqq u_n, \quad n \in \mathbf{N},$$

which follows from (4.1) and (4.3) by induction.

Theorem 4.1. *Let k and ω satisfy the hypotheses of Theorem* 3.1, *and let x be the solution of* (1) *with a given $y \in C(J)$. Assume further that $z \in C(J)$ and $v \in C_+(J)$ satisfy*

$$|z - y - Kz| \leqq v. \tag{4.4}$$

Then

$$|z - x| \leqq u \tag{4.5}$$

where u is the minimal solution of (2′).

Proof. Let (x_n) be the sequence of successive approximations, with $x_1 = y + Kz$ as the first approximation, converging to x. Then it is easy to see that

$$|z - x_1| \leqq v \text{ and } |z - x_{n+1}| \leqq v + \Omega(|z - x_n|), \quad n \in \mathbf{N}.$$

Thus (4.1) holds for $y_n = z - x_n$, which by Lemma 4.1 implies the assertion.

Corollary 4.1. *With the hypotheses of Theorem* 3.1,

$$|x - y| \leqq u, \tag{4.6}$$

where u is the minimal solution of (2′) *with $v = |Ky|$. If $\bar{x}$ is the solution of* (1) *with y replaced by another mapping $\bar{y}$ from $C(J)$, then*

$$|\bar{x} - x| \leqq u, \tag{4.7}$$

where u is the minimal solution of (2′) *with $v = |\bar{y} - y|$.*

Proof. Choose first $z = y$ and then $z = \bar{x}$ in Theorem 4.1.

As another consequence of Theorem 3.1. and Lemma 4.1 we obtain

Theorem 4.2. *Assume that k satisfies the hypotheses of Theorem* 3.1, *and that for* $(t, s, z) \in J \times J \times X$, $s \leqq t$,

$$\|k(t, s, z)\| \leqq \bar{\omega}(t, s, \|z\|),$$

where $\bar{\omega} \in \mathcal{K}_0^+$ *such that the hypotheses* (i) *and* (iii) *of Theorem* 3.1 *hold for* $\omega = \bar{\omega}$. *Then the solution of* (1) *with a given* $y \in C(J)$ *satisfies*

$$|x| \leqq \bar{u}, \tag{4.8}$$

where $\bar{u}$ *is the minimal solution of*

$$\bar{u} = |y| + \bar{\Omega}\bar{u} \tag{4.9}$$

with

$$\bar{\Omega}\bar{u}(t) = \int_0^t \bar{\omega}(t, s, \bar{u}(s))ds, \quad t \in J, \quad \bar{u} \in C_+(J). \tag{4.10}$$

Proof. Let (x_n) be the sequence of the successive approximations given by (3.3) with $x_1 = y$. Then (4.1) holds for $y_n = x_n$, $v = |y|$ and $\Omega = \bar{\Omega}$, so that (4.8) follows from (4.2).

The first conclusion of Corollary 4.1 yields the following local version of Theorem 3.1:

Corollary 4.2. *Let B be an open subset of X, and let* k: $\{(t, s, z) \in J \times J \times B \mid s \leqq t\} \to X$ *satisfy the hypotheses of Theorem* 3.1 *for all* (t, s, z), $(t, s, \bar{z}) \in J \times J \times B$, $s \leqq t$. *Then for each continuous mapping* y: $J \to B$, *the integral equation* (1) *has a unique solution on* $[0, T_1)$ *with*

$$T_1 = \sup\{t \in J \mid u(s) \leqq d(y(s), B^c) \text{ for } 0 \leqq s \leqq t\}, \tag{4.11}$$

where u is the minimal solution of (2′) *with* $v = |Ky|$, *and* $d(y(t), B^c)$ *denotes the distance between* $y(t)$ *and the complement* B^c *of* B *in* X.

This corollary shows an advantage of minimal solutions of (2′) as estimating functions, compared to corresponding maximal ones, obtained by other methods (see for ex. [4], I.4.I).

5. Consequences for differential equations

Denote by $AC_+(J)$ the class of all absolutely continuous functions u: $J \to \mathbf{R}_+$, and by $AC(J)$ the class of those $u \in C(J)$ which are strongly differentiable almost everywhere on J, and for which $|u| \in AC_+(J)$. Let $\mathcal{J}_0$ denote the class of such mappings of $\mathcal{K}_0$ which do not depend on t, and $\mathcal{J}_0^+$ the respective subclass of $\mathcal{K}_0^+$.

From the properties of Bochner integrals (see [1] Chapter 3) it follows that for each $y \in AC(J)$ the formula

(5.1) $$y(t) = y(0) + \int_0^t y'(s)\,ds$$

holds on J, and that the mapping $t \to \int_0^t f(s, x(s))\,ds$, $t \in J$, belongs to $AC(J)$ whenever $f \in \mathfrak{F}_0$ and $x \in C(J)$. Analogous properties hold trivially in the scalar case. These facts imply that the initial value problem

(3) $$x'(t) = y'(t) + f(t, x(t)), \qquad x(0) = y(0),$$

with $y \in AC(J)$ and $f \in \mathfrak{F}_0$, is representable in the form

(3′) $$x(t) = y(t) + \int_0^t f(s, x(s))\,ds,$$

and similarly,

(4) $$u'(t) = v'(t) + g(t, u(t)), \qquad u(0) = v(0),$$

with $v \in AC_+(J)$ and $g \in \mathfrak{F}_0^+$, in the form

(4′) $$u(t) = v(t) + \int_0^t g(s, u(s))\,ds.$$

Thus the results of Sections 3 and 4 are applicable for (3). From Theorem 3.1 we have

Theorem 5.1. *Assume $f \in \mathfrak{F}_0$ and for $(s, z), (s, \bar{z}) \in J \times X$*

$$\|f(s, z) - f(s, \bar{z})\| \leqq g(s, \|z - \bar{z}\|),$$

where $g \in \mathfrak{F}_0^+$ satisfying

(i) *for each $s \in J$, $g(s, r)$ is nondecreasing in $r \in \mathbf{R}_+$;*
(ii) *$u(t) \equiv 0$ is the maximal solution of*

$$u'(t) = g(t, u(t)), \qquad u(0) = 0;$$

(iii) *for each $C > 0$ there is $w \in C_+(J)$ for which*

$$w(t) \geqq C + \int_0^t g(s, w(s))\,ds, \qquad t \in J.$$

Then given $y \in AC(J)$, (3) has a unique solution x (a. e.) on J, and x can be obtained as the uniform limit of successive approximations with any mapping of $C(J)$ as the first approximation.

The results of Section 4 and formula (5.1) yield

Theorem 5.2. *Let f and g satisfy the hypotheses of Theorem 5.1, and let $x(t) = x(t, y'(t), y(0))$ denote the solution of (3), and $u(t) = u(t, v'(t), v(0))$ the minimal solution of (4). If $z \in AC(J)$ and $v \in AC_+(J)$ satisfy*

$$\left\| z(t) - y(t) - \int_0^t f(s, z(s))\,ds \right\| \leqq v(t), \qquad t \in J,$$

then

$$\|z(t) - x(t)\| \leqq u(t), \qquad t \in J.$$

In particular,

$$\|y(t) - x(t)\| \leqq u(t, \|f(t, y(t))\|, 0)$$

and

$$\|y(0) - x(t)\| \leqq u(t, \|y'(t) + f(t, y(0))\|, 0).$$

If $\bar{x}(t) = x(t, \bar{y}'(t), \bar{y}(0))$ *is another solution of* (3), *then*

$$\|\bar{x}(t) - x(t)\| \leqq u(t, \|\bar{y}'(t) - y'(t)\|, \|\bar{y}(0) - y(0)\|)$$

and

$$\|\bar{x}(t) - x(t) - \bar{y}(0) + y(0)\| \leqq u(t, \|\bar{y}'(t) - y'(t)\|, 0).$$

If there is $\bar{g} \in \mathfrak{F}_0^+$ *satisfying the hypotheses* (i) *and* (iii) *imposed on* g *in Theorem* 5.1, *and if*

$$\|f(t, z)\| \leqq \bar{g}(t, \|z\|), \qquad (s, z) \in J \times X,$$

then

$$\|x(t)\| \leqq \bar{u}(t), \qquad t \in J,$$

where $\bar{u}$ *is the minimal solution of*

$$\bar{u}'(t) = \|y'(t)\| + \bar{g}(t, \bar{u}(t)), \qquad \bar{u}(0) = \|y(0)\|.$$

Examples 5.1. The Osgood function

$$g(t, r) = p(t)\psi(r), \qquad (t, r) \in J \times \mathbf{R}_+$$

where $p: J \to \mathbf{R}_+$ is integrable and $\psi: \mathbf{R}_+ \to \mathbf{R}_+$ a continuous and non-decreasing function for which the integrals $\int_0^1 dr/\psi(r)$ and $\int_1^\infty dr/\psi(r)$ diverge, belongs to $\mathfrak{F}_0^+$ and satisfies the conditions (i) and (ii) of Theorem 5.1. Also (iii) holds with any w satisfying

$$\int_c^{w(t)} \frac{dr}{\psi(r)} \geqq \int_0^t p(s)ds, \qquad t \in J.$$

To get an example of (4) with nonunique solutions, define

$$g(t, r) = r \quad \text{for } 0 \leqq r < e^{-t}, \quad t \in J;$$
$$g(t, r) = e^{-t} \quad \text{for } e^{-t} \leqq r < 3 - 2e^{-t}, \quad t \in J;$$
$$g(t, r) = e^{-t} + (r - 3 + 2e^{-t})^{\frac{1}{2}} \quad \text{for } r \geqq 3 - 2e^{-t}, \quad t \in J.$$

this g belongs to $\mathfrak{F}_0^+$ and satisfies the hypotheses (i)—(iii) of Theorem 5.1, and the initial value problem

$$u'(t) = e^{-t} + g(t, u(t)), \qquad u(0) = 1,$$

has

$$u_*(t) = 3 - 2e^{-t} \text{ and } u^*(t) = 3 - 2e^{-t} + \frac{t^2}{4}$$

as the minimal and maximal solutions, respectively.

Remark 5.1. The closed interval $J = [0, T]$ can be replaced in above considerations by $[t_0, T)$, $-\infty < t_0 < T \leqq \infty$, the convergence of the successive approximations being uniform on every compact subset of $[t_0, T)$. In case $T = \infty$, the example above shows that the minimal solutions of (3.2), which are as estimators in (4.5)—(4.8), may be bounded, whereas the corresponding maximal solutions may be unbounded.

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