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EXTREMAL QUASICONFORMAL MAPPINGS OF A CONE

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In the paper [2] we proved the following extremal property for cylinder: Let G be a domain in R^{n-1} with $m_{n-1}(G) < \infty$ and f a quasiconformal mapping of the cylinder $Z = G \times R^1 \subset R^n$ onto itself which satisfies the boundary condition

$$f(x_1, \ldots, x_n) = (x_1, \ldots, x_{n-1}, Kx_n), \tag{1}$$

where $K \geqq 1$ is a constant. Then $K_O(f) \geqq K^{n-1}$ and $K_I(f) \geqq K$, where $K_O(f)$ and $K_I(f)$ are the outer and inner dilatations of the mapping f. In the extremal case $K_O(f) = K^{n-1}$ the lines parallel to x_n-axis go to similar lines, and the image of the section

$$G(t) = \{x + te_n \mid x \in G\}$$

of Z is for every $t \in R^1$ the section $G(Kt)$. However, the mapping need not then be affine. On the other hand, if $K_I(f) = K$, then f is the affine mapping (1).

Now we consider a similar problem for cones. Let G be a domain in S^{n-1}, $G \neq S^{n-1}$, and let C be the cone $\{x \in R^n \mid x/|x| \in G\}$. We consider a homeomorphism $f : \overline{C} \to \overline{C}$ whose restriction to C is quasiconformal and which satisfies on the boundary ∂C the condition

$$f(x) = |x|^{K-1} x, \tag{2}$$

where $K \geqq 1$ is a constant.

By Rickman [1], Theorem 1, or Väisälä [4], Theorem 2, f can be extended to a quasiconformal mapping $\hat{f} : R^n \to R^n$ so that

$$\hat{f}(x) = \begin{cases} f(x), & \text{when } x \in C \\ |x|^{K-1}x, & \text{when } x \in R^n \setminus C. \end{cases}$$

The following distortion theorem is valid for f.

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Theorem 1. *Suppose that* $f : C \to C$ *is a quasiconformal mapping which satisfies the boundary condition* (2). *Then there exist positive constants* λ *and* $M > 1$ *such that*

(3) $$|x|^K / \lambda \leqq |\hat{f}(x)| \leqq \lambda |x|^K$$

holds for $|x| \geqq M$ *or* $|x| \leqq 1/M$. *Here* λ *depends only on* K *and* $K(f)$.

Proof. Denote $\lambda = 2H(0, \hat{f})$, where $H(0, f)$ is the linear dilatation of $\hat{f}$ in the origin. The assertion for small values of $|x|$ follows then immediately.

For big values the result follows from the preceding one by inversion $\varphi(x) = x / |x|^2$. Now $\varphi^{-1} \circ f \circ \varphi$ is a quasiconformal mapping of R^n with the same dilatations and values on ∂C as f itself.

In the following theorem we give the natural lower bounds for the dilatations.

Theorem 2. *If* f *satisfies the boundary condition* (2), *then*

$$K_O(f) \geqq K^{n-1}, \quad K_I(f) \geqq K.$$

Proof. Choose $0 < r_1 < 1/M < M < r_2$ such that $\lambda r_1^K < r_2^K / \lambda$. Let Γ be the curve family which joins the sets $G(r_1) = r_1 G = \{x \mid x / r_1 \in G\}$ and $G(r_2) = r_2 G$ in C. Then by Sections 7.7 and 6.4 of [3]

$$M(\Gamma) = m_{n-1}(G) / (\log (r_2 / r_1))^{n-1}$$

and

$$M(f\Gamma) \leqq m_{n-1}(G) / (\log (\lambda^{-1} r_2^K / \lambda r_1^K))^{n-1}.$$

Letting $r_2 \to \infty$ we obtain by $M(\Gamma) \leqq K_O(f) M(f\Gamma)$

$$K_O(f) \geqq K^{n-1}.$$

Since $K_O(f) \leqq K_I(f)^{n-1}$, we have

$$K_I(f) \geqq K.$$

Theorem 3. *If* $K_O(f) = K^{n-1}$, *then* f *maps each ray* $r(y) = \{ty \mid t > 0\}$, $y \in G$, *onto a similar ray, and the image of*

$$G(t) = tG$$

is for every $t > 0$ *the set* $G(t^K)$. *Further, the volume derivative* $\sigma(x, h_t)$ *of the homeomorphism* $h_t = f \mid G(t)$ *is equal to* $t^{(n-1)(K-1)}$ *for almost every* $x \in G(t)$.

Before the proof of Theorem 3 we introduce some preliminary lemmas.

Lemma 1. *If* $K_O(f) = K^{n-1}$, *then*

$$0 \leqq \int_C (L_f(x)^n - |\partial_x |f(x)||^n) |f(x)|^{-n} \leqq 2 \log \lambda (1+n) K^{n-1} m_{n-1}(G)$$

where $\partial_x |f(x)|$ *is the directional derivative of* $|f|$ *at* x *in the direction of* x.

Proof. Let $j \in N$, $j > M$ and $j^K > \lambda$; then Theorem 1 is valid for $|x| = j$, and $j^K/\lambda > \lambda/j^K$. On almost every ray $r(y) = \{ty \mid t > 0\}$, $y \in G$, f and thus also $|f|$ are locally absolutely continuous. Moreover, f is differentiable at almost all points of $r(y)$. Theorem 1 implies

$$\begin{aligned} \log (j^K/\lambda) - \log (\lambda/j^K) &\leqq \log |f(jy)| - \log |f(y/j)| \\ &= \int_{1/j}^{j} (\partial_y |f(ty)| / |f(ty)|)\, dt . \end{aligned}$$

By Hölder's inequality we obtain

$$\begin{aligned} (2K \log j - 2 \log \lambda)^n &\leqq (2 \log j)^{n-1} \int_{1/j}^{j} |\partial_y |f(ty)||^n |f(ty)|^{-n} t^{n-1} dt \\ &\leqq (2 \log j)^{n-1} \int_{1/j}^{j} L_{|f|}(ty)^n |f(ty)|^{-n} t^{n-1} dt \\ &\leqq (2 \log j)^{n-1} \int_{1/j}^{j} L_f(ty)^n |f(ty)|^{-n} t^{n-1} dt ; \end{aligned}$$

the inequality $L_{|f|}(ty) \leqq L_f(ty)$ follows from the triangle inequality. Since $L_f(x)^n \leqq K_O(f) J(x,f)$ at almost every point of

$$C_j = \{x \in C \mid 1/j \leqq |x| \leqq j\},$$

if follows by integrating over G

$$\begin{aligned} &(2K \log j - 2 \log \lambda)^n m_{n-1}(G) \\ &\leqq (2 \log j)^{n-1} \int_{C_j} |\partial_x |f(x)||^n |f(x)|^{-n} dm(x) \\ &\leqq (2 \log j)^{n-1} \int_{C_j} L_f(x)^n |f(x)|^{-n} dm(x) \\ &\leqq (2 \log j)^{n-1} K^{n-1} \int_{C_j} J(x,f) |f(x)|^{-n} dm(x) \\ &= (2 \log j)^{n-1} K^{n-1} \int_{fC_j} |z|^{-n} dm(z) \\ &\leqq (2K \log j)^{n-1} \int_G dm_{n-1} \int_{\lambda^{-1} j^{-K}}^{\lambda j^K} t^{-1} dt \\ &= (2K \log j)^{n-1} m_{n-1}(G) (2K \log j + 2 \log \lambda) . \end{aligned}$$

This implies

$$0 \leqq \int_{C_j} (L_f(x)^n - |\partial_x |f(x)||^n) |f(x)|^{-n} dm(x)$$
$$\leqq m_{n-1}(G)(2(n+1)\log \lambda K^{n-1} + \varepsilon_j),$$

where $\lim_{j\to\infty} \varepsilon_j = 0$. Letting $j \to \infty$ yields the lemma.

Lemma 2. *If* $K_0(f) = K^{n-1}$, *then the function*

$$((L_f - \partial_x |f|)/|f|)^n$$

is integrable in C.

Proof. Let A and B be those subsets of C where $\partial_x |f(x)| \geqq 0$, resp. $\partial_x |f(x)| < 0$. By $0 \leqq \partial_x |f(x)| \leqq L_f(x)$ the inequality

$$(L_f(x) - \partial_x |f(x)|)^n \leqq L_f(x)^n - (\partial_x |f(x)|)^n$$

holds in A and the integral of $((L_f - \partial_x |f|)/|f|)^n$ over A is finite by Lemma 1.

To obtain the respective result for the set B we prove that

$$\int_B L_f(x)^n |f(x)|^{-n} dm(x)$$

is finite. If $j \geqq M$, then for almost all rays $r_j(y) = \{ty \mid 1/j \leqq t \leqq j\}$, $y \in G$,

$$\log(j^K/\lambda) - \log(\lambda/j^K) \leqq \int_{r_j(y)} (\partial_y |f(x)|/|f(x)|)\, dm_1(x)$$
$$\leqq \int_{r_j(y)\cap A} (\partial_y |f(x)|/|f(x)|)\, dm_1(x) \leqq \int_{r_j(y)\cap A} (L_f(x)/|f(x)|)\, dm_1(x).$$

By Hölder's inequality and integration over G we obtain

$$(2K \log j - 2\log\lambda)^n m_{n-1}(G) \leqq (2\log j)^{n-1} \int_{C_j\cap A} L_f(x)^n |f(x)|^{-n} dm(x).$$

On the other hand,

$$\int_{C_j} L_f(x)^n |f(x)|^{-n} dm(x) \leqq K^{n-1}(2K\log j + 2\log\lambda)\, m_{n-1}(G).$$

From these inequalities it follows that

$$\int_{C_j\cap B} L_f(x)^n |f(x)|^{-n} dm(x) \leqq [2(n+1)K^{n-1}\log\lambda + \varepsilon_j]\, m_{n-1}(G).$$

Letting $j \to \infty$ yields

$$\int_B L_f(x)^n |f(x)|^{-n} dm(x) \leqq 2(n+1)K^{n-1}\log\lambda\, m_{n-1}(G).$$

The inequality $|\partial_x |f(x)|| \leqq L_f(x)$ implies

$$\int_B ((L_f(x) - \partial_x |f(x)|) / |f(x)|)^n \, dm(x) \leqq 2^{n+1}(n+1) K^{n-1} \log \lambda \, m_{n-1}(G).$$

Hence $((L_f - \partial_x |f|)/|f|)^n$ is integrable over C.

We consider again the extension $\hat{f}$ of f and define a sequence $g_j : R^n \to R^n$ of quasiconformal mappings by

$$g_j(x) = \hat{f}(jx) / j^K, \; j = 1, 2, \ldots .$$

For every j is $K_O(g_j \mid C) = K_O(f) = K^{n-1}$ and $g_j \mid \partial C = f \mid \partial C$. By Väisälä [3], 20.5, 21.3, 37.2 and by the above Theorem 2 the sequence g_j has a subsequence g_j, $j \in J \subset N$, which converges uniformly on compact subsets of R^n and whose limit mapping is quasiconformal with $K_O(g \mid C) = K^{n-1}$. The formula (3) is still valid for g with the same bound λ.

L e m m a 3. *The mapping* g *maps each ray* r_y *onto a ray* r_z.

Proof. Suppose $a < b$. We choose a set $E \subset G$, $m_{n-1}(E) = m_{n-1}(G)$, such that g_j is for every $j \in J$ locally absolutely continuous on every ray r_y, $y \in E$. Denote $L_j = L_{g_j}$. By Fatou's Lemma

$$\int_G \liminf_{j \to \infty, j \in J} \left(\int_a^b (L_j(ty) - \partial_y |g_j(ty)|) \, dt \right)^n dm_{n-1}(y)$$

$$\leqq \liminf_{j \to \infty, j \in J} \int_G \left(\int_a^b (L_j(ty) - \partial_y |g_j(ty)|) \, dt \right)^n dm_{n-1}(y).$$

Since $L_j(ty) = j^{1-K} L_f(jty)$ and $\partial_y |g_j(ty)| = j^{1-K} \partial_y |f(jty)|$, the latter integral is at most

$$\int_G \left(\int_a^b (L_j(ty) - \partial_y |g_j(ty)|)^n t^{n-1} \, dt \right) dm_{n-1}(y) \left(\int_a^b t^{-1} dt \right)^{n-1}$$

$$= (\log b/a)^{n-1} \int_{C(a,b)} \left((L_f(jx) - \partial_x |f(jx)|) j^{1-K} \right)^n dm(x),$$

where $C(a, b) = \{x \in C \mid a \leqq |x| \leqq b\}$. Setting $jx = z$ we obtain

$$(\log(b/a))^{n-1} \int_{C(ja, jb)} ((L_f(z) - \partial_y |f(z)|) j^{-K})^n \, dm(z)$$

$$= (\log(b/a))^{n-1} b^{Kn} \int_{C(ja, jb)} ((L_f(z) - \partial_y |f(z)|)(jb)^{-K})^n \, dm(z)$$

$$\leqq \lambda^n b^{Kn} (\log(b/a))^{n-1} \int_{C(ja, jb)} ((L_f(z) - \partial_y |f(z)|) / |f(z)|)^n \, dm(z)$$

for $jb \geqq M$, because then $|f(z)| \leqq \lambda (jb)^K$. By Lemma 2 this converges to zero when $j \to \infty$. Thus a set $E_1 \subset E$ can be chosen so that $m_{n-1}(E_1) = m_{n-1}(E)$ and

$$\liminf_{j\to\infty,\, j\in J} \int_a^b (L_j(ty) - \partial_y\,|\,g_j(ty)\,|\,)\,dt = 0$$

for every $y \in E_1$.

Fix $y \in E_1$. For a subsequence (j_k) of J

$$\lim_{k\to\infty} \int_a^b (L_{j_k}(ty) - \partial_y\,|\,g_{j_k}(ty)\,|\,)\,dt = 0\,. \tag{4}$$

Now
$$\int_a^b \partial_y\,|\,g_{j_k}(ty)\,|\,dt = |\,g_{j_k}(by)\,| - |\,g_{j_k}(ay)\,|$$

converges to $|\,g(by)\,| - |\,g(ay)\,|$. Hence the length of the g_{j_k}-image of the interval $\{ty \mid a \leqq t \leqq b\}$ also converges by (4) to the same limit. Since $g_{j_k} \to g$, an elementary argument implies that the g-image of $\{ty \mid a \leqq t \leqq b\}$ is a radial interval. By the continuity of g this holds for every $y \in G$. Since a and b were arbitrary, the lemma follows.

Let $B \subset G$ be a Borel set. By Lemma 3 the g-image of the cone $C(B) = \{ty \mid y \in B\,,\ t > 0\}$ is again a cone. Let $\sigma_g(y)$ be the volume derivative of the homeomorphism $h_g = P_r \circ (g \mid G) : G \to G$, where P_r denotes the radial projection of C onto G. The number $\sigma_g(y)$ is finite for almost every $y \in G$.

Lemma 4. *The number* $\sigma_g(y) = 1$ *a.e. in* G.

Proof. Let $y \in G$ such that $\sigma_g(y) < \infty$. For $\varepsilon > 0$ choose $r > 0$ so small that $U = B^n(y\,,r) \cap S^{n-1} \subset G$ and

$$m_{n-1}(h_g(U)) < (\sigma_g(y) + \varepsilon)\, m_{n-1}(U)\,.$$

For arbitrary $t_1 \leqq 1/M$, $t_2 \geqq M$, $\lambda t_1 < t_2/\lambda$ we consider the curve family

$$\Gamma = \triangle(t_1 U\,,\ t_2 U\,;\ C(U\,;\,t_1\,,\,t_2))\,,$$

where $C(U\,;\,t_1\,,\,t_2) = \{tz \mid z \in U\,,\ t_1 < t < t_2\}$. Then

$$M(\Gamma) = m_{n-1}(U)/(\log(t_2/t_1))^{n-1}$$

and

$$M(g\Gamma) = (\sigma_g(y) + \varepsilon)\, m_{n-1}(U)\;/\;(\log\,(t_2^K\;/\;\lambda^2 t_1^K))^{n-1}\,.$$

Letting $t_2 \to \infty$, $M(\Gamma) \leqq K^{n-1}\, M(g\Gamma)$ now implies

$$\sigma_g(y) + \varepsilon \geqq 1\,,$$

and thus, since $\varepsilon > 0$ is arbitrary,

$$\sigma_g(y) \geqq 1\,.$$

On the other hand, by the inequality

$$\int_G \sigma_g(y)\ dm_{n-1}(y) \leqq m_{n-1}(G),$$

$\sigma_g(y) \leqq 1$ a.e. in G. Thus $\sigma_g(y) = 1$ a.e. in G.

Remark. It will be later proved that also f maps the rays r_y onto rays. From the above proof it follows that then also $\sigma_f(y) = 1$ a.e. in G.

Lemma 5. *For almost every* $x \in C$

$$L_g(x) = K\,|\,g(x)\,|\,/\,|\,x\,|\,.$$

Proof. Let $B \subset G$ be a Borel set. Denote $C(B\,;a\,,b) = \{ty \mid y \in B\,, a < t < b\}$. Since h_g satisfies the condition (N) and g maps the rays onto rays, we have

$$\int_{C(B\,;a\,,b)} J(x\,,g)\ dm(x) = m(g(C(B\,;a\,,b)))$$

$$= \int_{h_g B} dm_{n-1}(v) \int_{|g(ay)|}^{|g(by)|} u^{n-1}\,du$$

$$= \int_B \sigma_g(y)\,dm_{n-1}(y) \int_a^b t^{n-1}\,(\,|\,g(ty)\,|\,/\,t\,)^{n-1}\,\partial_x|\,g(ty)\,|\,dt$$

$$= \int_{C(B\,;a\,,b)} (\,|\,g(x)\,|\,/\,|\,x\,|\,)^{n-1}\,\partial_x|\,g(x)\,|\,dm(x)\,.$$

Thus $J(x\,,g) = (\,|\,g(x)\,|/|\,x\,|\,)^{n-1}\,\partial_x|\,g(x)\,|$ and consequently $L_g(x)^n \leqq K^{n-1}J(x\,,g) = K^{n-1}(\,|\,g(x)\,|\,/\,|\,x\,|\,)^{n-1}\,\partial_x\,|\,g(x)\,| \leqq K^{n-1}(\,|\,g(x)\,|\,/\,|\,x\,|\,)^{n-1}\,L_g(x)$ a.e. in C. The lemma follows.

We repeat now the above process with respect to the mapping g. We get a sequence $\varphi_j: R^n \to R^n\,, j \in J$,

$$\varphi_j(x) = g(jx)\,j^{-K}\,.$$

This sequence has a subsequence $\varphi_j\,,\ j \in J_1 \subset J$, which converges uniformly on compact subsets of R^n, and its limit mapping φ is quasiconformal with $K_O(\varphi \mid C) = K^{n-1}$. Further φ has the boundary values (2) on ∂C and Theorem 1 as well as Lemmas 3–5 are valid for it. We will prove now that φ sends the domain tG onto $t^K G$.

Lemma 6. *For every* $x \in C$

$$|\,\varphi(x)\,| = |\,x\,|^K\,.$$

Proof. For a.e. $y \in G$ the mapping g is locally absolutely continuous on the ray $r_y = \{ty \mid y \in G\,, t > 0\}$ and $L_g(ty) \leqq K\,|\,g(ty)\,|\,/\,t$ for almost all $t > 0$, see Lemma 5. Choose such y. Let M be as in Theorem 1. Then for $u > a > M$

$$K(\log u - \log a) - 2 \log \lambda \leq \log |g(uy)| - \log |g(ay)|$$

$$= \int_a^b (\partial_y |g(ty)| / |g(ty)|) \, dt$$

and consequently

$$\lim_{a \to \infty} \int_a^\infty (K/t - \partial_y |g(ty)| / |g(ty)|) \, dt = 0 , \tag{5}$$

because the integrand is by Lemma 5 nonnegative; the nonnegativity of the integrand is not yet known for f and this is the main reason for the above process.

For arbitrary $z > 0$

$$K \log z - \log (|\varphi_j(zy)| / |\varphi_j(y)|) = \int_1^z (K/t - \partial_y |\varphi_j(ty)| / |\varphi_j(ty)|) dt$$

$$= \int_1^z (K/t - j \, \partial_y |g(jty)| / |g(jty)|) \, dt .$$

Substitution $jt = u$ and (5) yield

$$\int_1^z (K/t - j \, \partial_y |g(jty)| / |g(jty)|) \, dt$$

$$= \int_j^{jz} (K/u - \partial_y |g(uy)| / |g(uy)|) \, du$$

$$= K \log z - \log (|g(jzy)| / |g(jy)|) \to 0 \quad \text{when} \quad j \to \infty , \quad j \in J_1 .$$

Since $\varphi_j \to \varphi$ and φ is continuous, the last two formulas give

$$|\varphi(zy)| = z^K |\varphi(y)|$$

for all $y \in G$, $z > 0$, and consequently

$$\partial_y |\varphi(zy)| = K z^{K-1} |\varphi(y)| .$$

Since Lemma 5 can also be applied to φ,

$$L_\varphi(zy) \leq K |\varphi(zy)| / z = K z^{K-1} |\varphi(y)|$$

and thus

$$L_\varphi(zy) = \partial_y |\varphi(zy)|$$

for a.e. $zy \in C$. At these points $\operatorname{grad} |\varphi(zy)| = \partial_y |\varphi(zy)| \, y$ and hence $\partial_s |\varphi(zy)| = \operatorname{grad} |\varphi(zy)| \cdot s = 0$ in every to y orthogonal direction s.

This implies that $|\varphi(x)| = |x|^K$ in C, for if $|\varphi(x_1)| \neq |\varphi(x_2)|$ at the points $x_1, x_2 \in C$, $|x_1| = |x_2|$, then by a theorem of Fuglede (Väisälä [3], p. 95) the points $x_1' \in C$ and $x_2' \in C$ can be chosen such that a) $|x_1'| = |x_2'|$, b) $|\varphi(x_1')| \neq |\varphi(x_2')|$, c) φ is absolutely continuous on a regular curve c which lies on the set $\{x \mid x/|x_1'| \in G\}$ and which joins the points x_1' and x_2', d) the tangential derivative $\partial_s |\varphi(x)|$ vanishes a.e. in C. The integration leads then to contradiction. Hence $|\varphi(x_1)| = |\varphi(x_2)|$ and the boundary condition (2) gives the result.

Remark. The last part of the above proof implies: If $\partial_s |f(x)| = 0$ a.e. in C, where s is any direction orthogonal to x, then $|f(x)| = |x|^K$ in C. This fact will be used later.

Lemma 7. *There is a sequence* $J_2 \subset N$ *such that for every* $y \in G$

$$\lim_{\substack{j \to \infty \\ j \in J_2}} |g_j(y)| = 1 .$$

Proof. Let $k \in N$. Since $|\varphi_j(y)| \to |\varphi(y)| = 1$, $j \in J_1$, uniformly in S^{n-1}, we can choose $i_k \in J_1$ such that

$$1 - 1/k < |\varphi_{i_k}(y)| < 1 + 1/k$$

for every $y \in G$. Further $g_j \to g$, $j \in J_1$, uniformly in $i_k S^{n-1}$. Thus there is $j_k \in J_1$ so that

$$1 - 1/k < |g_{j_k}(i_k y)| / |g(i_k y)| < 1 + 1/k$$

for every $y \in G$. Denote $J_2 = (i_k j_k)$, $k = 1, 2, \ldots$. Now the proposition follows immediately from

$$|f(i_k j_k y)| (i_k j_k)^{-K} = |g_{j_k}(i_k y)| i_k^{-K} =$$
$$|g_{j_k}(i_k y)| |g(i_k y)|^{-1} |g(i_k y)| i_k^{-K} =$$
$$|g_{j_k}(i_k y)| |g(i_k y)|^{-1} |\varphi_{i_k}(y)| .$$

The mapping $h = I \circ \hat{f} \circ I : R^n \to R^n$, where I is the inversion $I(x) = x / |x|^2$, satisfies the conditions $K_O(h \mid C) = K_O(f)$ and $h \mid \partial C = f \mid \partial C$. The above results can thus also be applied to h and to respective sequences. If we denote

$$g_{1/j}(y) = j^K f(y/j) ,$$

then we see immediately the validity of the following lemma.

Lemma 8. *There is a subsequence* J_3 *of* J_2 *such that for every* $y \in G$

$$\lim_{\substack{j \to \infty \\ j \in J_3}} |g_j(y)| = 1 , \quad \lim_{\substack{j \to \infty \\ j \in J_3}} |g_{1/j}(y)| = 1 .$$

Now we are ready to prove Theorem 3.

Proof of Theorem 3. We show first that

$$\int_C (L_f(x)^n - |\partial_x |f(x)||^n) |f(x)|^{-n} dm(x) = 0 .$$

Denote for $j \in J_3$

$$a_j(y) = \min_{x \in G} \{ |g_{1/j}(x)| \mid P_r f(x/j) = y\} ,$$

$$b_j(y) = \max_{x \in G} \{ |g_j(x)| \mid P_r f(jx) = y\} ,$$

where P_r is the radial projection of R^n onto S^{n-1}. Since

$$|f(jy)| = |g_j(y)| j^K , \quad |f(y/j)| = |g_{1/j}(y)| / j^K ,$$

we obtain as in the proof of Lemma 1

$$\int_G (2K \log j + \log |g_j(y)| - \log |g_{1/j}(y)|)^n \, dm(y)$$

$$\leq (2 \log j)^{n-1} \int_{C_j} |\partial_x |f(x)||^n |f(x)|^{-n} dm(x)$$

$$\leq (2 \log j)^{n-1} \int_{C_j} L_f(x)^n |f(x)|^{-n} dm(x)$$

$$\leq (2K \log j)^{n-1} \int_G dm_{n-1}(y) \int_{a_j(y)/j^K}^{b_j(y)j^K} t^{-1} dt$$

$$= (2K \log j)^{n-1} \int_G (2K \log j + \log b_j(y) - \log a_j(y)) \, dm_{n-1}(y) .$$

Hence

$$0 \leq \int_{C_j} (L_f(x)^n - |\partial_x |f(x)||^n) |f(x)|^{-n} dm(x)$$

$$\leq K^{n-1} \int_G (\log b_j(y) - \log a_j(y)) \, dm_{n-1}(y)$$

$$- \sum_{k=1}^{n} \binom{n}{k} K^k (2 \log j)^{k-n+1} \int_G (\log |g_j(y)| - \log |g_{1/j}(y)|)^{n-k} dm_{n-1}(y) .$$

Since $g_j \to 1$, $g_{1/j} \to 1$, $j \in J_3$, uniformly on S^{n-1}, we have for every $y \in G$

$$\lim_{\substack{j \to \infty \\ j \in J_3}} a_j(y) = \lim_{\substack{j \to \infty \\ j \in J_3}} b_j(y) = 1 .$$

Moreover $|g_j(y)| \leq \lambda$, $|g_{1/j}(y)| \geq 1/\lambda$, $b_j(y) \leq \lambda$, $a_j(y) \geq 1/\lambda$ for $j \geq$ M, hence by the Lebesgue convergence Theorem

$$\int_C (L_f(x)^n - |\partial_x |f(x)||^n) |f(x)|^{-n} dm(x) = 0 .$$

Thus $L_f(x) = |\partial_x |f(x)||$ and consequently $\partial_s |f(x)| = 0$ a.e. in C for every direction s orthogonal to x. Remark after Lemma 6 implies that

f maps every domain $G(t)$ onto $G(t^K)$. Thus $L_f(x) = \partial_x |f(x)| = K|x|^{K-1}$ for a.e. $x \in C$, and the length of the image of an arbitrary segment $\{ty \mid a \leq t \leq b\}$ is for a.e. $y \in G$ equal to $b^K - a^K$, i.e. the distance between $G(a^K)$ and $G(b^K)$. Now $P_r(f(ay)) = P_r(f(by))$ and by the continuity of f even for all $y \in G$. This means that f carries every ray r_y onto a ray. Finally, by Remark after Lemma 4 $\sigma_f(x) = 1$ a.e. in G. The theorem is proved.

We show now that in the case $K_O(f) = K^{n-1}$ the mapping f need not be defined by (2) in C.

Example. Let $w(z) = e^i z e^{-2i|z|}$ be a quasiconformal mapping of the plane disc $D = \{z \mid |z| < 1/2\}$ onto itself. Then $w(z) = z$ on the boundary of D and $J(z, w) = 1$ for every $z \in D$, see [2, p. 17]. Let

$$G = \{x \in R^3 \mid x_1^2 + x_2^2 + x_3^2 = 1,\ x_1^2 + x_2^2 < 1/4,\ x_3 > 0\}$$

and let P be the orthogonal projection of G onto D. Then the composed mapping $\varphi = P^{-1} \circ w \circ P : G \to G$ also satisfies the conditions $\varphi(x) = x$ for $x \in \partial G$ and the area derivative $\sigma(x, \varphi) = 1$ for $x \in G$, because $|w(z)| = |z|$ for every $z \in D$. Denote the supremum of the maximal stretching $L_\varphi(y)$ by M, where the supremum is taken over all $y \in G$.

We define now the mapping $f : C \to C$ by

$$f(x) = \varphi(x/|x|)\,|x|^K,$$

where $K > M$ is a constant. Then $f(x) = |x|^{K-1}x$ on the boundary of C and the volume derivative at x is

$$K|x|^{K-1}\,(|x|^{K-1})^2\,\sigma(x/|x|, \varphi) = K|x|^{3(K-1)}.$$

Since $K > M$, the maximal stretching of f at x is $K|x|^{K-1}$. Thus

$$K_O(f) = K^2,$$

but the mapping φ is not the identity mapping.

Theorem 4. *If $K_I(f) = K$, then f is composed of the mapping* (2) *and a rotation.*

Proof. Theorem 2 and the inequality $K_O(f) \leq K_I(f)^{n-1}$ imply $K_O(f) = K^{n-1}$ and hence Theorem 3 is valid. Let $x \in C$ be a regular point of f such that $\sigma_f(P_r(x)) = 1$. Let $\lambda_1 = |\partial_x f(x)| = K|x|^{K-1}$, $\lambda_2, \ldots, \lambda_n$ be the semiaxis of the dilatation ellipsoid $E(f'(x))$, $\lambda_2 \geqq \ldots \geqq \lambda_n$. By Theorem 3

$$\lambda_2 \ldots \lambda_n = |x|^{(n-1)(K-1)} \tag{6}$$

and hence $\lambda_n \leq |x|^{K-1}$. The definition of $K_I(f)$ gives

$$\lambda_1 \ldots \lambda_n = K|x|^{K-1}|x|^{(n-1)(K-1)} \leqq K\lambda_n^n,$$

i.e. $\lambda_n \geqq |x|^{K-1}$. Thus $\lambda_n = |x|^{K-1}$. The equality (6) implies then, since $\lambda_2 \geqq \ldots \geqq \lambda_n$,

$$\lambda_2 = \lambda_3 = \ldots = \lambda_n = |x|^{K-1} .$$

Hence $f | G$ is a rotation, and the theorem follows.

Remark. The rotation in Theorem 4 can appear only if ∂C is contained in an $(n\text{-}2)$-dimensional linear subspace of R^n.

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