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LINEAR PICARD SETS FOR ENTIRE FUNCTIONS

SAKARI TOPPILA

1. Introduction and results

1. In the terminology of Lehto ([5]), a plane set E is a Picard set for entire functions if every transcendental entire function takes all finite values with at most one exception infinitely often in the complement of E. It is proved in [8] that a countable set $E = \{a_n\}$ whose points converge to infinity is a Picard set for entire functions if there exists $\varepsilon > 0$ such that

$$\left\{ z : 0 < |z - a_n| < \frac{\varepsilon\, |a_n|}{\log |a_n|} \right\} \cap E = \phi$$

for all large n. This result is sharp in the sense that, corresponding to each real-valued function $h(r)$ satisfying the condition $h(r) \to \infty$ as $r \to \infty$, there exists $E = \{a_n\}$ with $\lim a_n = \infty$ such that E is not a Picard set for entire functions and

$$\left\{ z : 0 < |z - a_n| < \frac{|a_n|}{h(|a_n|) \log |a_n|} \right\} \cap E = \phi$$

for all large n. We shall prove the corresponding result for linear sets.

Theorem 1. *If the positive numbers* $e < a_1 < a_2 < \ldots$ *satisfy the condition*

$$a_{n+1} \geq a_n + \frac{\varepsilon\, a_n}{(\log a_n)^2}$$

for some $\varepsilon > 0$ *and for all large* n, *then* $E = \{a_n\}$ *is a Picard set for entire functions.*

This theorem is sharp. It is proved in [9] that if $h(r) \to \infty$ as $r \to \infty$, there exist real numbers $0 \leq a_1 < a_2 < \ldots$ such that $E = \{a_n\}$ is not a Picard set for entire functions and

$$a_{n+1} \geq a_n + \frac{a_n}{h(a_n)\,(\log a_n)^2}$$

for all sufficiently large n.

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We denote by $U(z, r)$ the open disc of centre z and radius r. We shall prove the following theorem from which Theorem 1 immediately follows.

Theorem 2. *Let* $0 < \alpha \leq 2$. *Let the positive numbers* $e < a_1 \leq a_2 \leq \ldots$ *satisfy the condition*

$$\text{(a)} \qquad a_{n+1} \geq a_n + \frac{\varepsilon\, a_n}{(\log a_n)^\alpha}$$

for some $\varepsilon > 0$ *and for all large* n. *If the radii of the discs* $C_n = U(a_n, r_n)$ *satisfy the condition*

$$\text{(b)} \qquad \log(1/r_n) \geq K (\log a_n)^{2+\alpha},$$

where $K = 20(1 + 1/\varepsilon)$, *then the union* $\cup C_n$ *is a Picard set for entire functions.*

We denote by $[a, b]$ the closed segment $a \leq x \leq b$, and (a, b) is the segment $a < x < b$. Theorem 2 is sharp in the following sense.

Theorem 3. *Let* $0 < \alpha \leq 2$. *There exists a countable set* $E = \{a_n\}$, $0 \leq a_1 < a_2 < \ldots$, $\lim a_n = \infty$, *satisfying* (a) *for* $\varepsilon = 1/7$ *such that the linear set*

$$\bigcup_{n=1}^{\infty} (a_n - r_n, a_n + r_n),$$

where

$$\log(1/r_n) = \frac{1}{8} (\log a_n)^{2+\alpha},$$

is not a Picard set for entire functions.

The corresponding sharp result for discs C_n whose middle points need not lie on a ray is proved in [10]. If we have $0 < \alpha \leq 1$, $\varepsilon > 0$, and K_0 a sufficiently large constant, then the conditions

$$\text{(A)} \qquad \left\{ z : 0 < |z - a_n| < \frac{\varepsilon\, |a_n|}{(\log |a_n|)^\alpha} \right\} \cap E = \phi$$

and

$$\text{(B)} \qquad \log(1/r_n) \geq K_0 (\log a_n)^{2+2\alpha}$$

guarantee that $\bigcup_{n=1}^{\infty} U(a_n, r_n)$ is a Picard set for entire functions. If K_0 in (B) is taken too small, then $\cup U(a_n, r_n)$ need not be a Picard set.

2. Lehto and Virtanen [7] gave the following definition for normal meromorphic functions: If f is meromorphic in a simply connected domain G, then f is normal if and only if the family $\{ f(S(z)) \}$, where $\zeta = S(z)$

denotes an arbitrary one-to-one conformal mapping onto itself, is normal in the sense of Montel. In multiply connected domains f is said to be normal if it is normal on the universal covering surface. We shall consider the following problem: If E is a closed set, under what conditions does there exist a normal meromorphic function in the complement of E with at least one essential singularity in E ? Lehto and Virtanen [7] proved that if f is normal in G, then f is normal in every subdomain of G, and that a meromorphic function can not be normal in any neighbourhood of its isolated essential singularity. This implies that if E is a finite set and f is normal in $-E$, then f is a rational function. Only finite sets have this property. We shall prove

Theorem 4. *If E is an infinite closed set, there exists a non-rational normal meromorphic function in $-E$.*

We denote

$$\varrho(f(z)) = \frac{|f'(z)|}{1+|f(z)|^2} .$$

The proof of Theorem 4 is based on the following theorem of Lehto and Virtanen [7].

Theorem A. *A non-constant f, meromorphic in a domain G of hyperbolic type, is normal in G if and only if there exists a finite constant C so that for all z in G,*

$$\text{(C)} \qquad \varrho(f(z))\,|dz| \leq C\, d\sigma_G(z) ,$$

where $d\sigma_G(z)$ denotes the element of length in the hyperbolic metric of G.

3. We denote $S(\alpha, \beta) = \{ z : \alpha \leq \arg z \leq \beta \}$ and $L_\alpha = L(\alpha) = \{ z : \arg z = \alpha \}$. Let f be an entire transcendental function. We say that L_α is a line of Julia of f if, for every $\varepsilon > 0$, f takes every finite value except perhaps one infinitely often in $S(\alpha - \varepsilon, \alpha + \varepsilon)$. The set of all Julia lines of f is denoted by J_f. Then $J_f \neq \phi$ and the set $\{ e^{i\alpha} : L_\alpha \in J_f \}$ is closed. On the other hand, if $J = \{ L_\alpha : \alpha \in E \} \neq \phi$, $\{ e^{i\alpha} : \alpha \in E \}$ is closed, and $0 \leq \varrho \leq 1/2$ or $\varrho = \infty$, there exists an entire function f of order ϱ such that $J_f = J$. This is proved by Polya in the case $\varrho = \infty$ and by Anderson and Clunie [1] in the case $\varrho = 0$. In fact, the function f constructed by Anderson and Clunie can be chosen such that f is slowly growing, i.e.

$$\log M(r, f) = O((\log r)^2)$$

as $r \to \infty$. In the case $0 < \varrho \leq 1/2$, we take $L_\alpha \in J$ and construct an entire function g of order ϱ such that, for every $\varepsilon > 0$, $|g(z)|$ tends to infinity uniformly outside $S(\alpha - \varepsilon, \alpha + \varepsilon)$ as $|z| \to \infty$. Let h be a slowly growing entire function such that $J_h = J$. Then $f(z) = g(z)h(z)$ is of order ϱ and $J_f = J$.

Let now f be an entire function of order ϱ, $1/2 < \varrho < \infty$. Cartwright [3] has proved that there exists $S(\alpha, \beta)$ where $\beta = \alpha + \pi/\varrho$, such that if $S(\gamma, \gamma + \pi/\varrho) \cap S(\alpha, \beta) \neq \phi$, then $S(\gamma, \gamma + \pi/\varrho)$ contains at least one Julia line of f. Therefore, if $1/2 < \varrho < 1$, there exist L_α, $L_\beta \in J_f$ such that $2\pi - \pi/\varrho \leq \beta - \alpha \leq \pi$, and if $1 \leq \varrho < \infty$, there exist L_α, L_β, $L_\gamma \in J_f$ such that $\alpha < \beta \leq \gamma$, $\beta - \alpha \leq \pi/\varrho$, $\gamma - \beta \leq \pi/\varrho$ and $\gamma - \alpha \geq \pi/\varrho$. Conversely, if J satisfies these conditions, there exists an entire function of order ϱ such that $J_f \subset J$. We shall prove

Theorem 5. *Let* $\{ e^{i\alpha} : \alpha \in E \}$ *be closed and* $J = \{ L_\alpha : \alpha \in E \} \neq \phi$. *If*

(i) $1/2 < \varrho < 1$ *and there exist* L_α, $L_\beta \in J$ *such that* $2\pi - \pi/\varrho \leq \beta - \alpha \leq \pi$ *or*

(ii) $1 \leq \varrho < \infty$ *and there exist* L_α, L_β, $L_\gamma \in J$ *such that* $\alpha < \beta \leq \gamma$, $\beta - \alpha \leq \pi/\varrho$, $\gamma - \beta \leq \pi/\varrho$ *and* $\gamma - \alpha \geq \pi/\varrho$,

then there exists an entire fuction f *of order* ϱ *such that* $J_f = J$.

We denote $\overset{+}{\log} d = \max\{0, \log d\}$ for $d \geq 0$. We need Schottky's theorem in our considerations. It is proved by Ahlfors in the following form.

Schottky's theorem. *If* g *is regular and* $g(z) \neq 0, 1$ *in* $|z| < 1$, *then*

$$\overset{+}{\log} |g(z)| \leq \frac{1 + |z|}{1 - |z|} (7 + \overset{+}{\log} |g(0)|) .$$

2. Proof of Theorem 2

4. Contrary to our assertion, let us suppose that there exist an entire transcendental function f and $r_0 > 0$ such that

$$f^{-1}(\{0, 1\}) \subset U(0, r_0) \cup \bigcup_{n=1}^{\infty} C_n .$$

It does not mean any restriction to assume that $f(C_n) \cap \{0, 1\} \neq \phi$ for every n. Similarly, we may assume that $f(0) \neq 0$, because if $f(0) = 0$ then we can consider the function $1 - f$. We denote by K_1, K_2, ... constants depending only on the numbers ε and α in Theorem 2, and M_1, M_2, ... are constants depending on f.

Let $r > 2r_0$. Applying Schottky's theorem to the function $h(z) = f(z^2)$ we see that $\overset{+}{\log} |h(i r^{\frac{1}{2}})| \leq M_1 r^{\frac{1}{2}}$ and therefore $\overset{+}{\log} |f(-r)| \leq M_1 r^{\frac{1}{2}}$. Applying Schottky's theorem repeatedly, we see that $\overset{+}{\log} |f(z)| \leq M_2 r^{\frac{1}{2}}$ on

$$\{-r+iy: |y| \leq r\} \cup \{x+iy: |y|=r, \ -r \leq x \leq 2r\}.$$

It follows from (a) and (b) that we can choose s, $r \leq s \leq 2r$, such that, for any n, $U(s, \delta) \cap C_n = \phi$ where

$$\delta = \frac{\varepsilon r}{4 (\log r)^a}.$$

Applying Schottky's theorem in $U(s+ir, r-\delta/4)$ and in $U(s-ir, r-\delta/4)$, we see that

$$\overset{+}{\log} |f(s+iy)| \leq M_3 (\log r)^a r^{\frac{1}{2}};$$

if $\delta/2 \leq |y| \leq r$, and the same theorem applied in $U(s, \delta)$ gives

$$\overset{+}{\log} |f(s+iy)| \leq M_4 (\log r)^a r^{\frac{1}{2}};$$

for $-r \leq y \leq r$. Now it follows from the maximum principle that $\log M(r) \leq M_4 (\log r)^a r^{\frac{1}{2}}$; , where $M(r) = M(r, f) = \max\{|f(z)| : |z| = r\}$. The order of f is at most $1/2$ and we may write

$$f(z) = b \prod_{\nu=1}^{\infty} (1 - z/t_\nu)$$

where $0 < |t_1| \leq |t_2| \leq \ldots$ and $b \neq 0$.

5. Let us suppose that there exists n_0 such that $\{0, 1\} \subset f(C_n)$ for every $n \geq n_0$. Let $r > 10\, a_{n_0}$. If either $a_n \leq 3r/4 - 1$ or $a_n \geq r+1$ for any n, then $n(r) = n(3r/4)$, where $n(t) = n(t, 0)$ is the number of zeros of f in $|z| \leq t$, zeros of order p being counted p times.

Let $3r/4 - 1 < a_n < r + 1$. We choose $\xi \in C_n$ such that $f(\xi) = 1$. We denote $q_n = n(a_n + 1) - n(a_n - 1)$; here $q_n \geq 1$ because C_n contains at least one zero of f. It is seen that $|1 - \xi/t_\nu| < 1$ for $|t_\nu| \geq 5r/8$, and therefore

$$0 = \log |f(\xi)| \leq n(5r/8) \log r + q_n \log r_n.$$

Now it follows from (b) that

$$q_n \leq \frac{n(5r/8) \log r}{\log (1/r_n)} \leq \frac{2\, n(5r/8)}{K (\log r)^{1+a}}. \tag{1}$$

Let b_r be the number of the points a_n satisfying the condition $3r/4 - 1 < a_n < r + 1$. It follows from (a) that $b_r \leq \varepsilon^{-1} (\log r)^a$, and we see from (1) that

$$n(r) - n(3r/4) \leq \frac{2\, n(3r/4)}{\varepsilon K \log r}.$$

Therefore $n(r) \leq n(3r/4)\,(1 + 1/(10 \log r))$ for all large values of r, and we get

$$n(r) \leq n((3/4)^4 r)\left(1 + \frac{1}{5 \log r}\right)^4 \leq n(r/e)\left(1 + \frac{1}{\log r}\right).$$

Now it is seen that

$$n(e^m r) \leq n(r) \prod_{t=1}^{m}\left(1 + \frac{1}{t + \log r}\right) \leq n(r)\,(m + 1)\,.$$

This implies that $n(R) = O(\log R)$ as $R \to \infty$ and so $q_n = 0$ for all large n. We are led to a contradiction and conclude that there exists an infinite subsequence C_{n_k} of the discs C_n such that $\{0\,,\,1\} \not\subset f(C_{n_k})$.

6. If r is large, then $|1 + r/t_m| > 1$ for every m and $|1 + r/t_m| \geq 7/4$ for $m = 1, 2, \ldots, n(r)$. Therefore we obtain

$$\log |f(-r)| \geq \log |b \prod_{m=1}^{n(r)} (1 + r/t_m)| \geq n(r) \log (3/2)\,,$$

and see that

(2) $$n(r) \leq K_1 \log M(r)$$

for all large values of r.

We choose n such that $\{0\,,\,1\} \not\subset f(C_n)$, and denote

$$\delta_n = \frac{\varepsilon\, a_n}{2\,(\log a_n)^a}\,.$$

If $|f(z)| \geq 3$ for every $z \in \Gamma_n = \{\, z : |z - a_n| = \delta_n \,\}$, then it follows from Rouche's theorem that $\{0\,,\,1\} \subset f(C_n)$ because $f(C_n) \cap \{0\,,\,1)\} \neq \phi$. Therefore, there exists $\zeta \in \Gamma_n$ such that $|f(\zeta)| \leq 3$, and we deduce from Schottky's theorem that $\overset{+}{\log} |f(z)| \leq K_2$ on Γ_n. Applying Schottky's theorem in $U(a_n + i\,a_n\,,\, a_n - \delta_n/2)$, we get $\overset{+}{\log} |f(a_n + i\,a_n)| \leq K_3\,(\log a_n)^a$, and because $|f(z)| \leq f(-2\,|z|)$ if $|z|$ is large, it follows from Schottky's theorem that

(3) $$\log M(4\,a_n) \leq K_4\,(\log a_n)^a$$

if n is large enough, say $n \geq n_1$, and $\{0\,,\,1\} \not\subset f(C_n)$.

If possible, we choose $n > n_1$ such that $\{0\,,\,1\} \not\subset f(C_n)$ and $\{0\,,\,1\} \subset f(C_{n+1})$, and $z_1\,,\; z_2 \in C_{n+1}$ such that $f(z_1) = 0$ and $f(z_2) = 1$. We have $|z_1 - z_2| < 2\,r_{n+1}$ and $|1 - z_2/t_m| < 1$ for $m > n(a_n + 1)$. Therefore we obtain

$$\begin{aligned} 0 = \log |f(z_2)| &\leq \log r_{n+1} + \log |b \prod_{m=1}^{n(a_n+1)} (1 - z_2/t_m)\,| \\ &< \log r_{n+1} + n(a_n + 1) \log a_{n+1}\,, \end{aligned}$$

and it follows from (b) that

(4) $$n(a_n + 1) > K (\log a_{n+1})^{1+a} .$$

On the other hand, it follows from (2) and (3) that $n(a_n + 1) \leq K_1 K_4 (\log a_n)^a$. This is in contradiction with (4), and we see that there exists $n_2 > n_1$ such that $\{0, 1\} \not\subset f(C_n)$ if $n \geq n_2$.

Let now $a_n - 1 \leq r < a_{n+1} - 1$. Then $n(r) \leq n(a_n + 1)$ and we see from (2) and (3) that

(5) $$n(r) < K_5 (\log r)^a$$

for all large values of r.

7. We denote $z_n = a_n + (a_n + i\, a_n)/4$ and $s_n = a_n - \delta_n$. It follows from Schottky's theorem that $\log |f(z_n)| \geq 2 K_6 \log M(a_n)$ where $K_6 > 0$. Because $s_n \in \Gamma_n$, we have $\overset{+}{\log} |f(s_n)| \leq K_2$, and so

(6) $$\log \left| \frac{f(z_n)}{f(s_n)} \right| \geq K_6 \log M(a_n) .$$

Further, $|(t_m - z_n)/(t_m - s_n)| < 1$ for $m > n(3\, a_n)$, and therefore

$$\log \left| \frac{f(z_n)}{f(s_n)} \right| \leq \log \prod_{m=1}^{n(3a_n)} \left| \frac{t_m - z_n}{t_m - s_n} \right| \leq n(3a_n) \log (6\, a_n/\delta_n) .$$

Because $\delta_n = \dfrac{\varepsilon\, a_n}{2 (\log a_n)^a}$, we obtain

$$\log \left| \frac{f(z_n)}{f(s_n)} \right| \leq K_7\, n(3\, a_n) \log \log a_n$$

and we see from (6) that for all sufficiently large n,

(7) $$n(3\, a_n) \geq \frac{K_8 \log M(a_n)}{\log \log a_n}$$

where $K_8 > 0$.

It follows from (5) and (7) that $n(r) = O((\log r)^2)$ as $r \to \infty$ and $n(r) \neq O((\log r)^d)$ if $d < 1$. Therefore we can choose d, $1 \leq d \leq 2$, such that $n(r) = O((\log r)^d)$ and $n(r) \neq O((\log r)^{d-\frac{1}{2}})$. Then $\log M(r) \neq O((\log r)^{d+\frac{1}{2}})$ as $r \to \infty$.

Let us suppose that $a_{n+1} < a_n^7$ for all large n. Let $a_{n-1} < r \leq a_n$. Then it is seen from (7) that

$$\log M(r) \leq \log M(a_n) \leq 2 K_8^{-1}\, n(r^8) \log \log r ,$$

and hence $\log M(r) = O((\log r)^{d+\frac{1}{4}})$. We are led to a contradiction and conclude that there exist arbitrarily large values of n such that $a_{n+1} \geq a_n^7$.

8. We choose n such that $a_{n+1} \geq a_n^7$ and set $g(z) = f(a_n + z^2)$. Because $a_n + \delta_n \in \Gamma_n$, we see that $\overset{+}{\log} |g(\delta_n^{\frac{1}{2}})| \leq K_2$. Applying Schottky's theorem in

$$U(a_n^{\frac{1}{2}}/4 \,,\, a_n^{\frac{1}{2}}/4 - \delta_n^{\frac{1}{2}}/2) \,,$$

we get

$$\overset{+}{\log} |g(a_n^{\frac{1}{2}}/4)| \leq K_9 (\log a_n)^{a/2} \,,$$

and therefore $\overset{+}{\log} |f(a_n + a_n/16)| \leq K_9 \log a_n$. Because f omits the values 0 and 1 in $33a_n/32 < |z| < 35a_n/32$, we now see from Schottky's theorem that

$$\log M(a_n + a_n/16) \leq K_{10} \log a_n \,.$$

This implies that

$$\liminf_{r\to\infty} \frac{\log M(r)}{\log r} < \infty$$

and we are led to a contradiction. Theorem 2 is proved.

3. Proof of Theorem 3

9. Let $0 < a \leq 2$. We set $t = 1/(1 + a)$, $a_n = e^{n^t}$ and $f(z) = \prod_{m=1}^{\infty} (1 - z/a_m)$. It is easily seen that

$$a_{n+1} > a_n + \frac{a_n}{7 (\log a_n)^a}$$

for all large n. Let $n > 100$ and $z \in U(a_n \,,\, d_n)$ where

$$d_n = \frac{a_n}{14 (\log a_n)^a} \,.$$

We choose positive integers k and p such that $a_{k-1} < a_n/4 \leq a_k$ and $a_{p-1} < 3\, a_n \leq a_p$. Set

$$f(z) = H(z)\, Q(z)\, (1 - z/a_n)\, S(z) \,,$$

where $H(z) = \prod_{m=1}^{k-1} (1 - z/a_m)$, $S(z) = \prod_{m=p}^{\infty} (1 - z/a_m)$ and

$$Q(z) = (1 - z/a_n)^{-1} \prod_{m=k}^{p-1} (1 - z/a_m) \,.$$

We have

$$\log |H(z)| \geq \log \prod_{m=1}^{k-1} \frac{a_n}{2a_m} \geq (k - 1) \log (a_n/2) - \sum_{m=1}^{k-1} m^t$$

and so $\log |H(z)| \geq n^{t+1}/6$. It follows that $\log |Q(z)| > -n$ and $\log |S(z)| > -n$. Therefore,

$$(5) \qquad \log |H(z)\, Q(z)\, S(z)\,| \geq n^{t+1}/7$$

in $U(a_n, d_n)$. Now we see that $|f(z)| \geq 2$ on $|z - a_n| = d_n$, and f has therefore exactly one 1-point in $U(a_n, d_n)$. Let us denote by ζ this 1-point of f. Because f takes on the segment $[a_n - d_n, a_n + d_n]$ every value w satisfying $-2 \leq w \leq 2$, we see that ζ lies on the real axis. It follows from (5) that

$$\log \left| \frac{a_n}{\zeta - a_n} \right| \geq n^{t+1}/7$$

and hence $\log |\zeta - a_n|^{-1} > n^{t+1}/8 = (\log a_n)^{2+a}/8$. Let I_n denote the segment $(a_n - r_n, a_n + r_n)$ where

$$\log (1/r_n) = (\log a_n)^{2+a}/8 .$$

Then f has only a finite number of 1-points outside $\cup I_n$ and we see that $\cup I_n$ is not a Picard set for entire functions. Theorem 3 is proved.

4. Proof of Theorem 5

10. We construct the desired counterexamples with the aid of Mittag-Leffler's function

$$E_a(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(1 + a\, n)}$$

where $0 < a < 2$. E_a is of order $1/a$, E_a is bounded on $S(-a\,\pi/2, 2\,\pi - a\,\pi/2)$ and $a\, E_a(z) - \exp(z^{1/a})$ is bounded on $S(-a\,\pi/2, a\,\pi/2)$.

Set $a = 1/\varrho$. Let us suppose that there exist L_β, $L_\gamma \in J$ such that $\gamma = \beta + a\,\pi$. We may suppose that $\beta = -a\,\pi/2$. We choose a slowly growing entire function h such that $J_h = J$ and set $f(z) = h(z)\,(E_a(z) + M)$, where $M > 0$ is chosen such that $|E_a(z)| < M - 1$ outside $S(-a\,\pi/2, a\,\pi/2)$. Then $J_f = J$ and f is of order ϱ.

Let $1/2 < \varrho < 1$ and $a = 1/\varrho$. If $\gamma \neq \beta + a\,\pi$ for every L_β, $L_\gamma \in J$, then it follows from (i) that there exist L_β, $L_\gamma \in J$ such that $2\,\pi - a\,\pi < \gamma - \beta \leq \pi$. We may assume that $\beta = -\gamma$ where $\pi - a\,\pi/2 < \gamma \leq \pi/2$. Set

$$g(z) = E_a(z) + E_a(-t^a\, z)$$

with

$$t = \frac{\cos(\gamma\,\varrho)}{\cos(\gamma\,\varrho - \varrho\,\pi)} .$$

Let $z = r\, e^{i\psi} \in S(\pi - a\,\pi/2, a\,\pi/2)$. Then

$$\overset{+}{\log} |E_a(z)| = R_1(z) + r^\varrho \cos(\psi\,\varrho)$$

and

$$\overset{+}{\log} |E_\alpha(-t^\alpha z)| = R_2(z) + t\, r^\varrho \cos(\psi\,\varrho - \varrho\,\pi)\,,$$

where $R_1(z)$ and $R_2(z)$ are bounded functions defined on $S(\pi - \alpha\,\pi/2\,,\ \alpha\,\pi/2)$. Now we see easily that for every $\varepsilon > 0$, $|g(z)|$ tends to infinity uniformly in $S(\beta + \varepsilon\,, \gamma - \varepsilon) \cup S(\gamma + \varepsilon\,, 2\,\pi - \gamma - \varepsilon)$ as $|z| \to \infty$. If h is a slowly growing function such that $J_h = J$, then $f(z) = h(z)\,g(z)$ satisfies $J_f = J$.

Let now $1 \leq \varrho < \infty$ and $\alpha = 1/\varrho$. If $\gamma \neq \beta + \alpha\,\pi$ for every L_β, $L_\gamma \in J$, then by (ii) there exist L_β, L_γ, $L_\psi \in J$ such that $\beta < \gamma < \psi$, $\gamma - \beta < \alpha\,\pi$, $\psi - \gamma < \alpha\,\pi$ and $\psi - \beta > \alpha\,\pi$. We may assume that $\beta = -\,\alpha\,\pi/2$. Set

$$g(z) = E_\alpha(z) + E_\alpha(t^\alpha\, z\, e^{-i\varphi})$$

where $\varphi = \psi - \alpha\,\pi/2$ and

$$t = \frac{\cos(\gamma\,\varrho)}{\cos(\psi\,\varrho - \gamma\,\varrho - \pi/2)}\,.$$

Then $g(z)$ is bounded on $S(\psi\,, 2\,\pi + \beta)$ and, for every $\varepsilon > 0$, $|g(z)|$ tends to infinity uniformly on $S(\beta + \varepsilon\,, \gamma - \varepsilon) \cup S(\gamma + \varepsilon\,, \psi - \varepsilon)$ as $|z| \to \infty$. If h is a slowly growing function such that $J_h = J$, then

$$f(z) = h(z)\,(g(z) + M)$$

satisfies $J_f = J$, provided the constant M is chosen sufficiently large. Theorem 5 is proved.

5. Proof of Theorem 4

11. Let E be an infinite closed set. We choose a linear mapping L such that $\{0\,, \infty\} \subset F = L(E)$ and 0 is a limit point of F. Then F contains an infinite countable set $A = \{a_n\}$ such that $|a_1| < 1/4$ and $|a_{n+1}| < |a_n|^5$ for $n \geq 1$. We set

$$f(z) = \prod_{n=1}^{\infty} \left(\frac{z - a_n}{z + a_n}\right). \tag{1}$$

In order to prove Theorem 4, it is sufficient to prove that f is normal in the complement of $B = A \cup \{0\,, \infty\}$. If f is normal in $-B$, then it is normal in $-F$, and because both $\varrho(f(z))\,|dz|$ and $d\sigma_G(z)$ are conformal invariants, it follows from Theorem A that $f(L(\zeta))$ is normal in $-E$.

It follows from Theorem 1 of Lehto [6] that

$$\limsup_{z \to 0} |z|\, \varrho(f(z)) = \frac{1}{2}\,.$$

Therefore there exists $M_1 \geq 1/2$ such that

$$\varrho(f(z)) \leq \frac{M_1}{|z|} \tag{2}$$

for all z satisfying the condition $0 < |z| \leq 10$. Differentiation yields

$$\varrho(f(z)) = \frac{|f(z)|}{1 + |f(z)|^2} \left| \sum_{k=1}^{\infty} \frac{2\, a_k}{z^2 - a_k^2} \right| .$$

Because $|a_1| < 1/4$ and $|a_{n+1}| < |a_n|^5$, we see easily that

$$\varrho(f(z)) \leq \frac{4\, |a_1|}{|z|^2} \tag{3}$$

in $|z| \geq 2\, |a_1|$, and

$$\varrho(f(z) \leq \frac{4}{|a_n|} + \frac{4\, |a_{n+1}|}{|z|^2} \tag{4}$$

in $2\, |a_{n+1}| \leq |z| \leq |a_n|/2$ for any $n \geq 1$.

We denote by D the complement of the points 0, 1 and ∞, and let $\sigma_D(w, w')$ be the hyperbolic metric of D. Constantinescu [4] has proved that

$$\lim_{w \to 0} |w| \left(\log \left|\frac{1}{w}\right|\right) \frac{d\sigma_D(w)}{|dw|} = \frac{1}{2} .$$

Therefore there exists δ, $0 < \delta \leq 1/2$, such that

$$\frac{|dw|}{d\sigma_D(w)} \leq 4\, |w| \log \left|\frac{1}{w}\right| \tag{5}$$

in $0 < |w| \leq \delta$. The transformation $w = 1/z$ defines a conformal mapping of D onto itself. Therefore $d\sigma_D(w) = d\sigma_D(z)$, and if $0 < z \leq \delta$, we get from (5)

$$\frac{|d\,w|}{d\sigma_D(w)} = \frac{|dz|}{|z|^2\, d\sigma_D(z)} \leq \frac{4}{|z|} \log \left|\frac{1}{z}\right| = 4\, |w| \log |w| .$$

This implies that

$$\frac{|dw|}{d\sigma_D(w)} \leq 4\, |w| \log |w| \tag{6}$$

in $1/\delta \leq |w| < \infty$. Similarly, we see by means of a linear transformation that there exists $M_2 > 0$ such that

$$\frac{|dw|}{d\sigma_D(w)} < M_2 \tag{7}$$

in $0 < |w - 1| \leq \delta$. Because the set

$$T = \{ z : |z| \geq \delta , \quad |z - 1| \geq \delta , \quad |z| \leq 1/\delta \}$$

is a compact subset of D, there exists $M_3 \geq M_2$ such that

$$\frac{|dw|}{d\sigma_D(w)} < M_3 \tag{8}$$

in T.

We denote by G the complement of $B = A \cup \{0, \infty\}$ and D_n in the complement of the points 0, a_n and ∞. Because $D_n \supset G$, then $d\sigma_{D_n}(z) \leq d\sigma_G(z)$. Since D is mapped conformally onto D_n by $z = a_n w$, we have $d\sigma_{D_n}(z) = d\sigma_D(w)$, and therefore

$$\frac{|dz|}{d\sigma_G(z)} \leq \frac{|dz|}{d\sigma_{D_n}(z)} = \frac{|a_n| \, |dw|}{d\sigma_D(w)} . \tag{9}$$

We denote $Q_1 = \{ z : |a_1|/\delta \leq |z| < \infty \}$, and set for $n \geq 2$

$$Q_n = \{ z : |a_n|/\delta \leq |z| \leq |a_n a_{n-1}|^{\frac{1}{2}} \} .$$

It follows from (9) and (6) that

$$\frac{|dz|}{d\sigma_G(z)} \leq 4 \, |z| \log \left| \frac{z}{a_n} \right| \tag{10}$$

in Q_n. Let $z \in Q_n$. If $n = 1$ we get from (10) and (3)

$$\varrho(f(z)) \frac{|dz|}{d\sigma_G(z)} \leq 16 \left| \frac{a_1}{z} \right| \log \left| \frac{z}{a_1} \right| \leq 8 .$$

Let $n \geq 2$. It follows from (4) that

$$\varrho(f(z)) \leq \frac{4}{|a_{n-1}|} + \frac{4 \, |a_n|}{|z|^2}$$

in Q_n and we obtain from (10)

$$\begin{aligned} \varrho(f(z)) \frac{|dz|}{d\sigma_G(z)} &\leq 16 \left| \frac{a_n}{z} \right| \log \left| \frac{z}{a_n} \right| + 16 \left| \frac{z}{a_{n-1}} \right| \log \left| \frac{z}{a_n} \right| \\ &\leq 8 + 16 \, |a_n/a_{n-1}|^{\frac{1}{2}} \log |a_{n-1}/a_n|^{\frac{1}{2}} \leq 16 . \end{aligned}$$

Let $\delta \, |a_n| \leq |z| \leq |a_n|/\delta$ and $z \in G$. It follows from (9), (7) and (8) that

$$\frac{|dz|}{d\sigma_G(z)} \leq M_3 \, |a_n| ,$$

and we obtain from (2) and (3)

$$\varrho(f(z)) \frac{|dz|}{d\sigma_G(z)} \leq \frac{M_1 \, M_3 \, |a_n|}{|z|} \leq \frac{M_1 \, M_3}{\delta} = M_4 .$$

Let $|a_{n+1} \, a_n|^{\frac{1}{2}} \leq |z| \leq \delta \, |a_n|$. We get from (9) and (5)

$$\frac{|dz|}{d\sigma_G(z)} \leq 4 \, |z| \log \left| \frac{a_n}{z} \right| .$$

Together with (4) this implies that

$$\varrho(f(z))\,\frac{|dz|}{d\sigma_G(z)} \le 16\left|\frac{z}{a_n}\right|\log\left|\frac{a_n}{z}\right| + 16\left|\frac{a_{n+1}}{z}\right|\log\left|\frac{a_n}{z}\right|$$
$$\le 8 + 16\,|a_{n+1}/a_n|^{\frac{1}{2}}\log|a_n/a_{n+1}|^{\frac{1}{2}} \le 16\,.$$

We have proved that

$$\varrho(f(z))\,|dz| \le (M_4 + 16)\,d\sigma_G(z)$$

for any $z \in G$. It follows from Theorem A that f is normal in G. Theorem 4 is proved.

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University of Helsinki
Department of Mathematics
SF-00100 Helsinki 10
Finland

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