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ON THE SECOND COEFFICIENT REGION FOR BOUNDED UNIVALENT FUNCTIONS

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1. Introduction

Consider the class $S(b)$ of bounded univalent functions f defined in the unit disc $U = \{ z \in C \mid |z| < 1 \}$ and normalized as follows:

$$\begin{cases} f(z) = b\, z + b_2 z^2 + \dots , \\ a_n = \dfrac{b_n}{b} \quad (n = 1, 2, \dots ; \; b_1 = b) , \\ |f(z)| < 1 , \\ b \text{ is constant} \in (0\, , 1] . \end{cases}$$

For these functions the Power inequality, or P_N-inequality, holds in its quadratic form

$$\sum_{-N}^{\infty} k\, |y_k|^2 + 2\, \mathrm{Re}\, (\bar{x}_0 y_0) \leq \sum_{-N}^{N} k\, |x_k|^2 , \tag{1}$$

derived in [2] under the unessential restriction $x_0 \in R$. The numbers x_ν are supposed to be free complex parameters, determining the combinations

$$y_k = \sum_{\nu=-N}^{k} x_\nu\, c_{\nu k} \qquad (k \geqq -N) .$$

Here the *Power-coefficients* $c_{\nu k}$ are determined by the defining conditions

$$\begin{cases} f(z)^\nu = \displaystyle\sum_{k=\nu}^{\infty} c_{\nu k}\, z^k \qquad (\nu = \pm 1, \pm 2, \dots) , \\ \log \dfrac{f(z)}{z} = \displaystyle\sum_{k=0}^{\infty} c_{0k}\, z^k , \end{cases}$$

and the numbers y_ν are obtained by the aid of the generating function

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$$g(w) = x_0 \log w + \sum_{-N}^{N}{}' x_\nu w^\nu \qquad (\nu \neq 0 \text{ in } \textstyle\sum'),$$

as coefficients of the development

$$g(f(z)) = x_0 \log z + \sum_{k=-N}^{\infty} y_k z^k .$$

In [2] there is the bilinear form of the P_N-inequality

$$(2) \qquad - \operatorname{Re} \{\sum_1^N k\,(y_{-k}\, y_k + x_{-k}\, x_k)\} \leqq \sum_1^N k\,(|y_{-k}|^2 + |x_k|^2)$$

true on the condition that $\operatorname{Re}(\bar{x}_0 y_0) = 0$. We used the bilinear P_3-inequality in [3] for the maximizing of $|a_4|$ in $S(b)$ by choosing the coefficients x_ν in a special optimal way. The same optimization worked also when the first coefficient region (a_3, a_2) was studied by the aid of the quadratic P_1-inequality [1]. Similarly, the optimized bilinear P_3-inequality was able to give some information of the region (a_4, a_3, a_2) on the condition that all the coefficients a_2, a_3, a_4 were supposed to be real.

In the present paper we will make use of the optimization principle in the bilinear P_3 inequality when applied to the function $F \in S(b^{1/2})$, where $F(z) = f(z^2)^{1/2}$, $f \in S(b)$. Our aim is to prove that the result concerning (a_3, a_2) can be generalized: The given pair (a_3, a_2) determines a disc as a range of a_4.

2. The optimized bilinear P_3-inequality for $f(z^2)^{1/2}$

Form first the P_3-inequality for the function $F(z) = f(z^2)^{1/2}$. According to [3], introduce the parameters u_ν,

$$u_\nu = -\nu\, y_{-\nu} \qquad (\nu = 1, 2, 3)$$

and apply the symmetric choice

$$x_{-\nu} = -\bar{x}_\nu \qquad (\nu = 1, 2, 3)$$

together with

$$x_0 = x_2 = 0,$$

which agrees with the odd character of F. Because this implies

$$u_2 = 0; \qquad y_0 = y_2 = 0,$$

we obtain for the P_3-inequality the quadratic truncated form

$$|y_1|^2 + 3\,|y_3|^2 \leqq |u_1|^2 + \tfrac{1}{3}\,|u_3|^2$$

with

$$x_3 = \tfrac{1}{3} b^{3/2} \bar{u}_3 , \qquad x_1 = b^{1/2} \overline{(u_1 + \tfrac{1}{2} a_2 u_3)} .$$

A comparison to (1) shows that equality here requires $y_4 = y_5 = \dots = 0$, i.e. for the extremal F,

$$\begin{aligned} -\tfrac{1}{3} b^{3/2} u_3 F^{-3} + \tfrac{1}{3} b^{3/2} \bar{u}_3 F^3 - b^{1/2}(u_1 + \tfrac{1}{2} a_2 u_3) F^{-1} - b^{1/2}\overline{(u_1 + \tfrac{1}{2} a_2 u_3)} F \\ = -\tfrac{1}{3} u_3 z^{-3} + y_3 z^3 - u_1 z^{-1} + y_1 z \end{aligned} \tag{3}$$

holds necessarily. Here

$$\begin{cases} y_1 = a_1 u_1 + b \bar{u}_1 + d_1 u_3 + e_1 \bar{u}_3 , \\ y_3 = d_1 u_1 + \bar{e}_1 \bar{u}_1 + d_3 u_3 + e_3 \bar{u}_3 , \end{cases} \tag{4}$$

where

$$\begin{cases} a_1 = \tfrac{1}{2} a_2 \\ d_1 = \tfrac{1}{2} a_3 - \tfrac{3}{8} a_2^2 , \\ e_1 = \tfrac{1}{2} b \bar{a}_2 , \\ e_3 = \tfrac{1}{4} b |a_2|^2 + \tfrac{1}{3} b^3 , \\ d_3 = \tfrac{1}{2} a_4 - a_2 a_3 + \tfrac{13}{24} a_2^3 . \end{cases} \tag{5}$$

Because the bilinear form (2) is obtained from the quadratic form (1) by the aid of Schwarz's inequality, we see (cf. [2], (57)) that equality is preserved if, together with the symmetric choice $x_{-\nu} = -\bar{x}_\nu$, we have

$$\begin{cases} y_1 = -\bar{y}_{-1} , \\ y_3 = -\bar{y}_{-3} . \end{cases} \tag{6}$$

The experience gained in [1] in connection with the quadratic P_1-inequality justifies our expecting (6) to hold in the optimized case for P_3, too. Thus, we will base our studies on the bilinear P_3-inequality

$$\operatorname{Re}(u_1 y_1 + u_3 y_3) \leqq |u_1|^2 + \tfrac{1}{3} |u_3|^2 , \tag{7}$$

i.e.

$$\begin{aligned} G = \tfrac{1}{2} a_1 u_1^2 + \tfrac{1}{2} \bar{a}_1 \bar{u}_1^2 + (b - 1) |u_1|^2 + d_1 u_1 u_3 + \bar{d}_1 \bar{u}_1 u_3 \\ = e_1 u_1 \bar{u}_3 + \bar{e}_1 \bar{u}_1 u_3 + \operatorname{Re}(d_3 u_3^2) + (e_3 - \tfrac{1}{3}) |u_3|^2 \leqq 0 . \end{aligned} \tag{8}$$

Now, keep first u_3 constant and optimize with respect to

$$u_1 = x + i y$$

by determining u_1 from $\partial G / \partial x = \partial G / \partial y = 0$. We will make use of the formulae

$$\begin{cases} \dfrac{\partial |u_1|^2}{\partial x} = u_1 + \bar{u}_1 , & \dfrac{\partial |u_1|^2}{\partial y} = i\,(\bar{u}_1 - u_1) , \\ \dfrac{\partial u_1}{\partial x} = \dfrac{\partial \bar{u}_1}{\partial x} = 1 ; & \dfrac{\partial u_1}{\partial y} = - \dfrac{\partial \bar{u}_1}{\partial y} = i , \end{cases}$$

to give

$$\begin{cases} \dfrac{\partial G}{\partial x} = a_1 u_1 + \bar{a}_1 \bar{u}_1 + (b - 1)(u_1 + \bar{u}_1) + d_1 u_3 + \bar{d}_1 \bar{u}_3 \\ \qquad + e_1 \bar{u}_3 + \bar{e}_1 u_3 = 0 , \\ \dfrac{1}{i} \dfrac{\partial G}{\partial i} = a_1 u_1 - \bar{a}_1 \bar{u}_1 + (b - 1)(\bar{u}_1 - u_1) + d_1 u_3 - \bar{d}_1 \bar{u}_3 \\ \qquad + e_1 \bar{u}_3 - \bar{e}_1 u_3 = 0 . \end{cases}$$

This is equivalent to

$$a_1 u_1 - (1 - b)\bar{u}_1 + d_1 u_3 + e_1 \bar{u}_3 = 0 . \tag{9}$$

In the equality case

$$0 = \tfrac{1}{2}\left(x \frac{\partial G}{\partial x} + y \frac{\partial G}{\partial y}\right) = \tfrac{1}{2} a_1 u_1^2 + \tfrac{1}{2} \bar{a}_1 \bar{u}_1^2 + (b - 1)\,|u_1|^2$$
$$+ \tfrac{1}{2}\,(d_1 u_3 u_1 + \bar{d}_1 \bar{u}_3 \bar{u}_1 + e_1 \bar{u}_3 u_1 + \bar{e}_1 u_3 \bar{u}_1) ,$$

i.e.

$$\begin{aligned} G &= \tfrac{1}{2}\,(d_1 u_1 u_3 + \bar{d}_1 \bar{u}_1 \bar{u}_3 + e_1 u_1 \bar{u}_3 + \bar{e}_1 \bar{u}_1 u_3) \\ &\quad + \operatorname{Re}\,(d_3 u_3^2) + (e_3 - \tfrac{1}{3})\,|u_3|^2 \\ &= \operatorname{Re}\,\{u_1\,(d_1 u_3 + e_1 \bar{u}_3) + d_3 u_3^2 + (e_3 - \tfrac{1}{3})\,|u_3|^2\}_0 \leqq 0 . \end{aligned} \tag{10}$$

From (9) we derive by conjugation a linear system of equations determining u_1 in u_3:

$$\begin{cases} u_1 = \lambda u_3 + \mu \bar{u}_3 ; \\ \lambda = \dfrac{\bar{a}_1 d_1 + (1 - b)\,\bar{e}_1}{(1 - b)^2 - |a_1|^2} = \dfrac{\bar{a}_2\,(a_3 - \frac{3}{4} a_2^2) + 2\,b\,(1 - b)\,a_2}{[2\,(1 - b)]^2 - |a_2|^2} , \\ \mu = \dfrac{\bar{a}_1 e_1 + (1 - b)\,\bar{d}_1}{(1 - b)^2 - |a_1|^2} = \dfrac{b\,\bar{a}_2^2 + 2\,(1 - b)\overline{(a_3 - \frac{3}{4} a_2^2)}}{[2\,(1 - b)]^2 - |a_2|^2} . \end{cases} \tag{11}$$

The inequality (8) preserves its form if both sides are divided by $|u_3|^2 \neq 0$. This means that we may assume without restriction

$$|u_3| = 1 .$$

On this condition we express $\{\ \}_0$ in u_3 by using (11):

$$(12)\qquad \begin{cases} \{\ \}_0 = h\,u_3^2 + k\,\overline{u}_3^2 + l\,; \\ h = d_3 + d_1\,\lambda\,, \\ k = e_1\,\mu\,, \\ l = d_1\,\mu + e_1\,\lambda + e_3 - \tfrac{1}{3}\,. \end{cases}$$

Thus we have

$$\begin{aligned} 2\,G &= 2\operatorname{Re}\{\ \}_0 = (h + \overline{k})\,u_3^2 + (\overline{h} + k)\,\overline{u}_3^2 + 2\operatorname{Re} l \\ &= ((h + \overline{k})^{1/2}u_3 - (\overline{h} + k)^{1/2}\overline{u}_3)^2 + 2\operatorname{Re} l + 2\,|h + \overline{k}| \\ &\leqq 2\operatorname{Re} l + 2\,|h + \overline{k}|\,. \end{aligned}$$

Hence, the optimized bilinear P_3-inequality reads

$$(13)\qquad |h + \overline{k}| + \operatorname{Re} l \leqq 0$$

and the optimizing u_3-choice is

$$(14)\qquad u_3^2 = \left(\frac{\overline{h} + k}{h + \overline{k}}\right)^{1/2} = \frac{\overline{h} + k}{|\overline{h} + k|} = \frac{|h + \overline{k}|}{h + \overline{k}}\,.$$

Determine now y_1 and y_3 in the optimized case. From (4) we obtain, according to (9):

$$y_1 = a_1\,u_1 + b\,\overline{u}_1 + d_1\,u_3 + e_1\,\overline{u}_3 = \overline{u}_1\,;$$

$$y_1 = -\overline{y}_{-1}\,.$$

Similarly, by the aid of (11),

$$y_3 = d_1\,u_1 + \overline{e}_1\,\overline{u}_1 + d_3\,u_3 + e_3\,\overline{u}_3 = \overline{u}_3\,[(h + \overline{k})\,u_3^2 + d_1\,\mu + \overline{e}_1\,\overline{\lambda} + e_3]\,.$$

Here

$$d_1\,\mu + \overline{e}_1\,\overline{\lambda} = \frac{2\operatorname{Re}(a_1\,\overline{d}_1\,\overline{e}_1) + (1 - b)(|e_1|^2 + |d_1|^2)}{(1 - b)^2 - |a_1|^2} \in R\,.$$

Because in the extremum case (13)

$$(h + \overline{k})\,u_3^2 = |h + \overline{k}| = -\operatorname{Re} l\,,$$

we have

$$\begin{aligned} y_3 &= \overline{u}_3\,(-\operatorname{Re} l + d_1\,\mu + \overline{e}_1\,\overline{\lambda} + e_3) \\ &= \overline{u}_3\,[-\tfrac{1}{2}\,(l + \overline{l}) + \tfrac{1}{2}\,(d_1\,\mu + \overline{e}_1\,\overline{\lambda} + \overline{d}_1\,\overline{\mu} + e_1\,\lambda) + e_3] \\ &= \overline{u}_3\operatorname{Re}(-l + d_1\,\mu + e_1\,\lambda + e_3) = \tfrac{1}{3}\,\overline{u}_3\,; \end{aligned}$$

$$y_3 = -\overline{y}_3\,.$$

The validity of the conditions (6) in the extremum case is thus verified and (13) will hence be sharp simultaneously with the quadratic P_3-inequality.

The conditions (6) written in the form

$$(15) \qquad y_1 = \bar{u}_1 , \qquad y_3 = \tfrac{1}{3} \bar{u}_3$$

can now be directly combined with the expressions (4) to give

$$(16) \qquad \begin{cases} a_1 u_1 + (b - 1) \bar{u}_1 + d_1 u_3 + e_1 \bar{u}_3 = 0 , \\ d_1 u_1 + \bar{e}_1 \bar{u}_1 + d_3 u_3 + (e_3 - \tfrac{1}{3}) \bar{u}_3 = 0 . \end{cases}$$

Observe that the first condition is the same as (9), thus giving (11). Therefore, the second condition (16) assumes the form

$$(d_1 \lambda + \bar{e}_1 \bar{\mu} + d_3) u_3 + (d_1 \mu + \bar{e}_1 \bar{\lambda} + e_3 - \tfrac{1}{3}) \bar{u}_3 = 0 ;$$
$$d_3 + d_1 \lambda + \bar{e}_1 \bar{\mu} = (\tfrac{1}{3} - e_3 - d_1 \mu - \bar{e}_1 \bar{\lambda}) u_3^{-2} .$$

Here

$$\tfrac{1}{3} - e_3 - d_1 \mu - \bar{e}_1 \bar{\lambda} = \tfrac{1}{3} - e_3 - \tfrac{1}{2} (d_1 \mu + \bar{e}_1 \bar{\lambda} + \bar{d}_1 \bar{\mu} + e_1 \lambda)$$
$$= \operatorname{Re} (\tfrac{1}{3} - e_3 - d_1 \mu - e_1 \lambda) = - \operatorname{Re} l \geqq 0 .$$

Thus, we may interprete the above d_3-condition as a circumference of the disc in which d_3 is restricted to be. – The results are collected as follows.

R e s u l t. For $S(b)$-functions with given coefficients a_2 and a_3, the number d_3 lies in the disc

$$(17) \qquad \begin{cases} |d_3 - d_3^0| \leq R ; \\ d_3^0 = - d_1 \lambda - \bar{e}_1 \bar{\mu} , \\ R = \tfrac{1}{3} - e_3 - d_1 \mu - \bar{e}_1 \bar{\lambda} \geqq 0 . \end{cases}$$

The boundary points of this disc are parametrized in $u_3 = e^{i\omega}$; thus

$$(18) \qquad d_3 = d_3^0 + R u_3^{-2} .$$

Equality in (17) is reached in the cases where the necessary extremum condition for $F = F(z) = f(z^2)^{1/2} \in S(b^{1/2})$,

$$(19) \qquad \begin{aligned} &\tfrac{1}{3} b^{3/2} (\bar{u}_3 F^3 - u_3 F^{-3}) + b^{1/2} (\bar{s} F - s F^{-1}) \\ &= \tfrac{1}{3} (\bar{u}_3 z^3 - u_3 z^{-3}) + \bar{u}_1 z - u_1 z^{-1} ; \end{aligned}$$

$$(20) \qquad \begin{cases} u_1 = \lambda u_3 + \mu \bar{u}_3 , \\ s = u_1 + \tfrac{1}{2} a_2 u_3 , \end{cases}$$

defines a $S(b)$-function f. The abbreviations used are

$$(21)\quad \begin{cases} d_3 = \frac{1}{2} a_4 - a_2 a_3 + \frac{13}{24} a_2^3 , \\ d_1 = \frac{1}{2} a_3 - \frac{3}{8} a_2^2 , \\ e_3 = \frac{1}{4} b \, |a_2|^2 + \frac{1}{3} b^3 , \\ e_1 = \frac{1}{2} b \, \bar{a}_2 ; \\ \lambda = \dfrac{\bar{a}_2(a_3 - \frac{3}{4} a_2^2) + 2 b (1 - b) a_2}{[2(1-b)]^2 - |a_2|^2} , \\ \mu = \dfrac{b \bar{a}_2^2 + 2(1-b)\overline{(a_3 - \frac{3}{4} a_2^2)}}{[2(1-b)]^2 - |a_2|^2} . \end{cases}$$

Remark. According to the above result (15) we may, in general, reduce the optimization of parameters in the Power inequality P_N in solving the linear system of equations

$$y_\nu = \frac{\bar{u}_\nu}{\nu} ,$$

the number of which depends on N. The linear structure of this system clearly suggests that the disc result found for $N = 1$ and 3 is independent of the index N.

3. Rotation of the extremum function

Consider the rotation of f to $\tilde{f}$:

$$\tilde{f}(z) = \tau^{-1} f(\tau z) , \qquad |\tau| = 1 .$$

Suppose that f is determined by a defined equation

$$\Phi(f(z), z) = 0 .$$

Because this implies $\Phi(f(\tau z), \tau z) = 0$, we see that for $\tilde{f}$

$$\Phi(\tau \tilde{f}(z), \tau z) = 0 .$$

Apply the above observation to the condition (19), written in the form

$$(22)\quad \begin{aligned} &\tfrac{1}{3} b^{3/2} \bar{u}_3 f(z)^{3/2} + b^{1/2} \bar{s} f(z)^{1/2} - (\tfrac{1}{3} b^{3/2} u_3 f(z)^{-3/2} + b^{1/2} s f(z)^{-1/2}) \\ &= \tfrac{1}{3} \bar{u}_3 z^{3/2} + \bar{u}_1 z^{1/2} - (\tfrac{1}{3} u_3 z^{-3/2} + u_1 z^{-1/2}) . \end{aligned}$$

Suppose that this determines

$$f(z) = b z + b_2 z^2 + \ldots$$

which is connected with the rotated function

$$\tilde{f}(z) = b\,z + \tilde{b}_2 z^2 + \dots ;$$
$$\tilde{b}_\nu = \tau^{\nu-1} b_\nu .$$

Consider now the numbers determined by (18), (20), and (21) for the rotated function $\tilde{f}$:

$$\begin{aligned} \tilde{d}_3 &= \tau^3 d_3 , & \tilde{d}_1 &= \tau^2 d_1 , & \tilde{e}_3 &= e_3, \\ \tilde{e}_1 &= \tau^{-1} e_1 , & \tilde{\lambda} &= \tau \lambda, & \tilde{\mu} &= \tau^{-2} \mu , \\ \tilde{u}_3 &= \tau^{-3/2} u_3 , & \tilde{u}_1 &= \tau^{-1/2} u_1 , & \tilde{s} &= \tilde{\tau}^{-1/2} s . \end{aligned}$$

For the function $\tilde{f}$ we thus obtain from (22)

$$\begin{aligned} \tfrac{1}{3} b^{3/2} \bar{\tilde{u}}_3 \tilde{f}(z)^{3/2} + b^{1/2} \bar{\tilde{s}} \tilde{f}(z)^{1/2} - (\tfrac{1}{3} b^{3/2} \tilde{u}_3 f(z)^{-3/2} + b^{1/2} \tilde{s} \tilde{f}(z)^{-1/2}) \\ = \tfrac{1}{3} \bar{\tilde{u}}_3 z^{3/2} + \bar{\tilde{u}}_1 z^{1/2} - (\tfrac{1}{3} \tilde{u}_3 z^{-3/2} + \tilde{u}_1 z^{-1/2}) . \end{aligned} \tag{23}$$

This means that the extremum condition is formally invariant for the rotation applied to the extremum function.

Especially, we may choose $\tilde{u}_3 = 1$, i.e.

$$u_3 = \tau^{3/2} ; \qquad \tau = u_3^{2/3} . \tag{24}$$

This implies for the corresponding $\tilde{F}(z) = \tilde{f}(z^2)^{1/2}$

$$\tfrac{1}{3} b^{3/2} (\tilde{F}^3 - \tilde{F}^{-3}) + b^{1/2}(\bar{\tilde{s}} \tilde{F} - \tilde{s} \tilde{F}^{-1}) = \tfrac{1}{3} (z^3 - z^{-3}) + \bar{\tilde{u}}_1 z - \tilde{u}_1 z^{-1} . \tag{25}$$

4. Real coefficients a_2, a_3, a_4

In the special case where a_2, a_3, a_4 are real, the maximal a_4 and the corresponding extremum cases are studied in [1]. It is useful here to compare the real and complex cases to each other. Consider therefore the real case in detail. From (18) we deduce that if a_2, a_3, $a_4 \in R$, then $u_3^2 = \pm 1$; i.e. we may take $u_3 = 1$ or $u_3 = i$.

The case $u_3 = 1$ was met with in [1]. The extremum condition (19) assumes the form

$$\tfrac{1}{3} b^{3/2} (F^3 - F^{-3}) + b^{1/2} s (F - F^{-1}) = \tfrac{1}{3} (z^3 - z^{-3}) + u_1 (z - z^{-1}) . \tag{26}$$

Differentiation gives

$$b^{3/2} z \frac{F'}{F} = \frac{z^3 + z^{-3} + u_1 (z + z^{-1})}{F^3 + F^{-3} + \frac{s}{b} (F + F^{-1})} ;$$

$$b^{3/2} z \frac{F'}{F} = \frac{F^3}{z^3} \frac{(z - i)(z - z_1)(z - z_2)(z + i)(z + z_1)(z + z_2)}{(F - i)(F - F_1)(F - F_2)(F + i)(F + F_1)(F + F_2)} ; \tag{27}$$

$$z_1\,,\,z_2 \;=\; \pm\tfrac{1}{2}\,(3 - u_1)^{1/2} + \tfrac{1}{2}\,(-u_1 - 1)^{1/2}\,,$$

$$F_1\,,\,F_2 \;=\; \pm\tfrac{1}{2}\left(3 - \frac{s}{b}\right)^{1/2} + \tfrac{1}{2}\left(-\frac{s}{b} - 1\right)^{1/2}.$$

Denote $z_\nu = e^{i\varphi}$, $F_\nu = e^{i\psi}$ $(\nu = 1$ or $2)$. Because $z_\nu^2 + z_\nu^{-2} - 1 + u_1 = 0$, $F_\nu^2 + F_\nu^{-2} - 1 + s/b = 0$, we have

$$u_1 = 1 - 2\cos 2\varphi_\nu\,, \qquad \frac{s}{b} = 1 - 2\cos 2\psi_\nu\,.$$

This implies

(28) $$-1 \leqq u_1 \leqq 3\,,$$

(29) $$-1 \leqq \frac{s}{b} \leqq 3\,.$$

(28) is necessary and sufficient for four unit roots z_ν. Similarly, (29) is connected with four unit roots F_ν.

If (29) holds, then $-1 \leqq s/b$ implies

$$-b - \tfrac{1}{2}a_2 \leqq u_1\,.$$

Because $-(1-b) \leqq -a_2/2$, it follows from the preceding condition that

$$-1 = -b - (1-b) \leqq u_1\,,$$

i.e. the left side of (29) implies the left side of (28). – The factorized condition (27) suggests the type 1° in Figure 1 for the extremum domain.

If $s/b > 3$, the type 2° of Figure 1 emerges from the factorized condition.

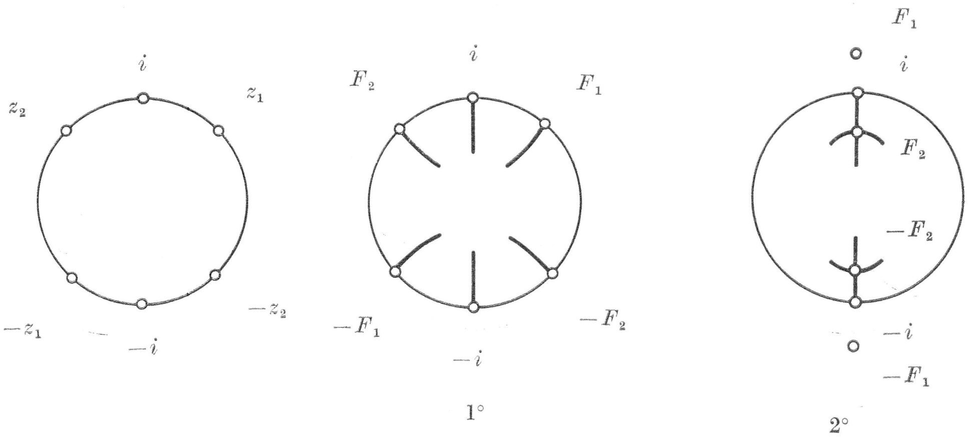


Figure 1.

The type of the possible extremum domain can be further studied by the aid of boundary correspondence defined by (26). Take $z = e^{i\varphi}$ and denote

$$F(e^{i\varphi}) = r(\varphi)\, e^{i\psi(\varphi)}$$

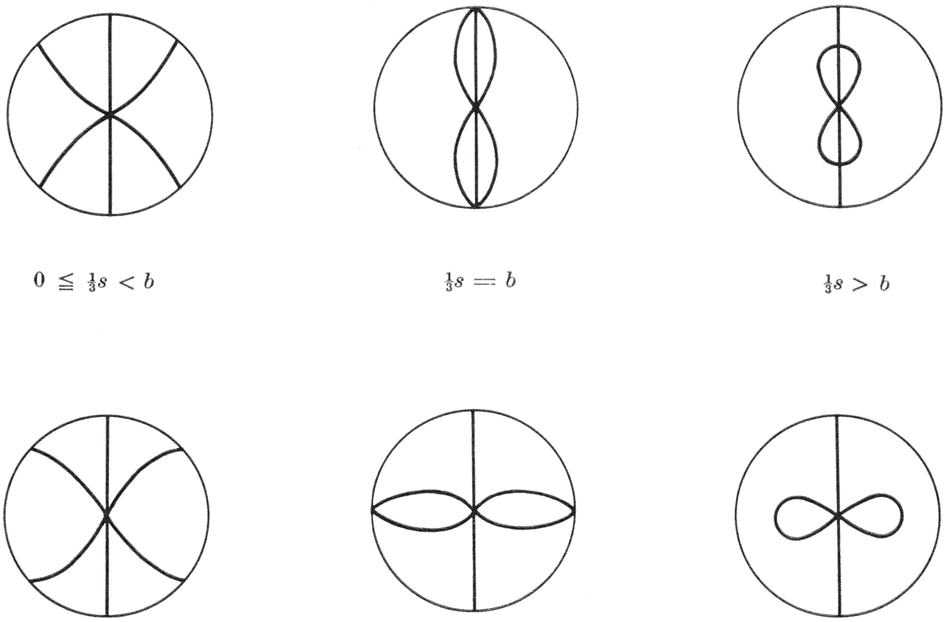
in (26), which gives

$$(30) \quad \begin{cases} (r - r^{-1})[\frac{1}{3}\, b\, (r^2 + 1 + r^{-2}) \cos 3\psi + s \cos \psi] = 0\,, \\ A = \frac{1}{2}\,[\frac{1}{3}\, b^{3/2}\,(r^3 + r^{-3}) \sin 3\psi + b^{1/2}\, s\,(r + r^{-1}) \sin \psi] \\ \quad = \frac{1}{3} \sin 3\varphi + u_1 \sin \varphi = B\,. \end{cases}$$

For $F(e^{i\varphi})$ the first condition (30) implies either that $r(\psi) \equiv 1$ or that $r(\psi)$ satisfies the condition

$$(31) \quad \begin{cases} H = \frac{1}{3}\, b\,(r^2 + 1 + r^{-2}) = -\dfrac{1}{1 - 4 \sin^2 \psi} = K\,, \quad \text{or} \\ \cos \psi \equiv 0\,. \end{cases}$$

The boundary curve candidates are given schematically in Figure 2.



$0 \leqq \frac{1}{3}s < b$ $\qquad$ $\frac{1}{3}s = b$ $\qquad$ $\frac{1}{3}s > b$

$0 \leqq -s < b$ $\qquad$ $-s = b$ $\qquad$ $-s > b$

Figure 2.

It appears that the first condition (30) is connected with the type 1° if $-s/b \leqq 1$ and $s/b \leqq 3$, i.e. if (29) holds. The type 2° occurs for $s/b > 3$.

Up to now we have found the following conditions necessary for the existence of the boundary mapping

$$-1 \leqq \frac{s}{b} \quad \text{and} \quad u_1 \leqq 3 . \tag{32}$$

If a_2, $a_3 \in R$ and $u_1 \leqq 3$ we have

$$u_1 = \lambda + \mu = \frac{a_3 - \frac{3}{4} a_2^2 + b\, a_2}{2\,(1 - b) - a_2} \leqq 3 , \tag{33}$$

which gives us the domain bounded by the parabola $u_1 = 3$.

The type 1° is connected with values for which $-1 \leqq s/b \leqq 3$, i.e.

$$-b \leqq \frac{a_3 - \frac{5}{4} a_2^2 + a_2}{2\,(1 - b) - a_2} \leqq 3\,b . \tag{34}$$

This defines the domain bounded by two parabolas $s = -b$ and $s = 3\,b$. – All the limiting parabolas mentioned meet at the point

$$a_2 = 2(1 - b) , \qquad a_3 = 3 - 8\,b + 5\,b^2$$

connected with the radial slit mapping.

The type 2° occurs if $s > 3\,b$, i.e. in the domain bounded by the parabolas $s = 3\,b$ and $u_1 = 3$. These domains are illustrated in Figure 7 of [1], where the notation u means the same as u_1 here.

There remains the second condition (30) which gives for $r(\psi) \equiv 1$

$$A = \tfrac{1}{3}\, b^{3/2} \sin 3\psi + b^{1/2}\, s \sin \psi = \tfrac{1}{3} \sin 3\varphi + u_1 \sin \varphi = B . \tag{35}$$

For the boundary correspondence to be of the type 1° or 2° it is sufficient that each $\psi \in [0\,,\,2\pi]$ has a uniquely determined pre-image $\varphi \in [0\,,\,2\pi]$. The correspondence becomes clear from Figures 3 and 4.

Consider first the alternative 1°, connected with Figure 3. There are two conditions to be satisfied for the existence of a pre-image of each ψ.

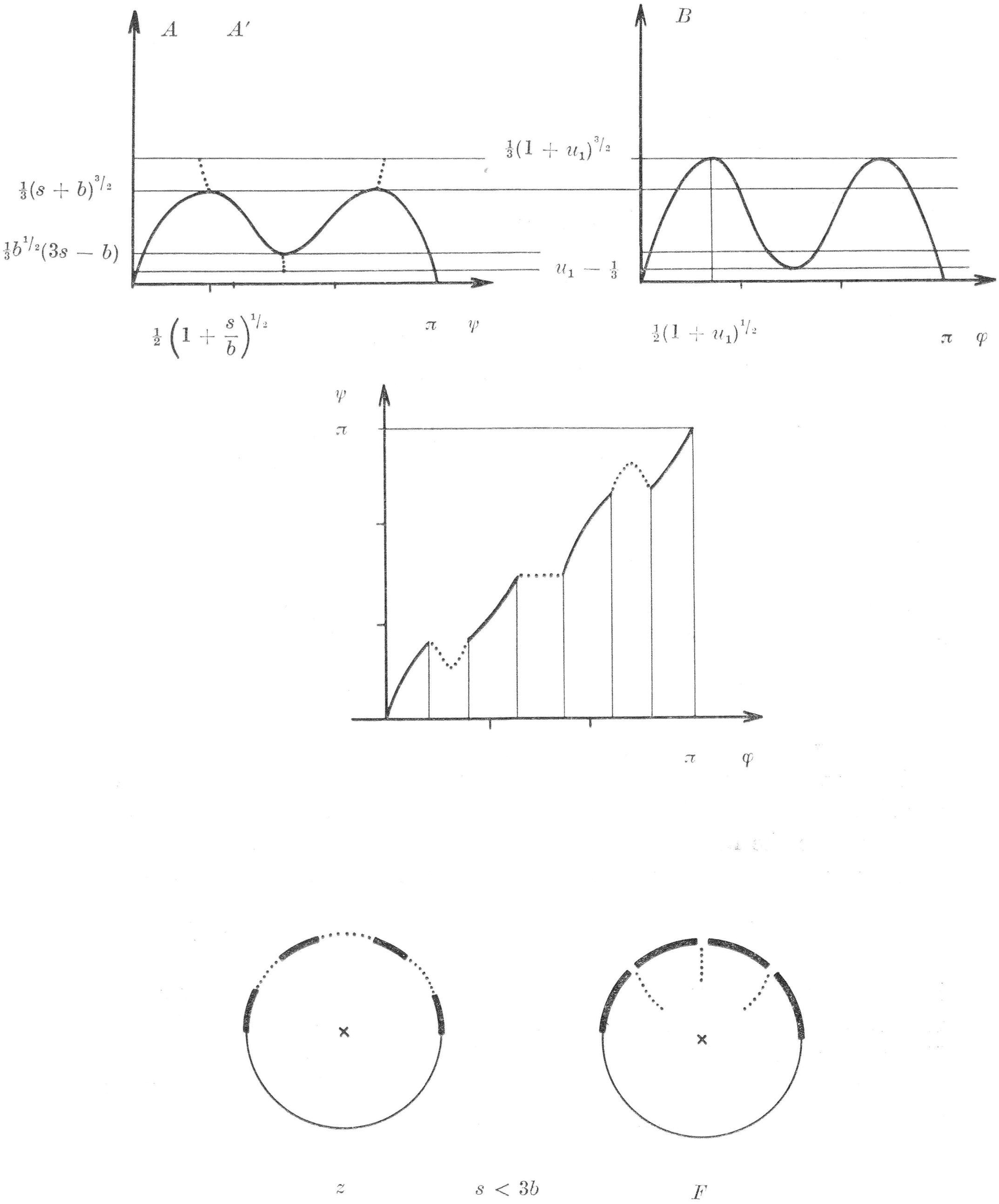
1)

$$\tfrac{1}{3}\,(u_1 + 1)^{3/2} \geqq \tfrac{1}{3}\,(s + b)^{3/2} \tag{36}$$

$\Leftrightarrow$

$$1 - b \leqq s - u_1 = \tfrac{1}{2}\, a_2 .$$

This condition is always true



A
A'
B
$\frac{1}{3}(s + b)^{3/2}$
$\frac{1}{3}b^{1/2}(3s - b)$
$\frac{1}{3}(1 + u_1)^{3/2}$
$u_1 - \frac{1}{3}$
$\frac{1}{2}\left(1 + \frac{s}{b}\right)^{1/2}$
π
ψ
$\frac{1}{2}(1 + u_1)^{1/2}$
π
φ
ψ
π
π
φ
z
$s < 3b$
F

Figure 3.

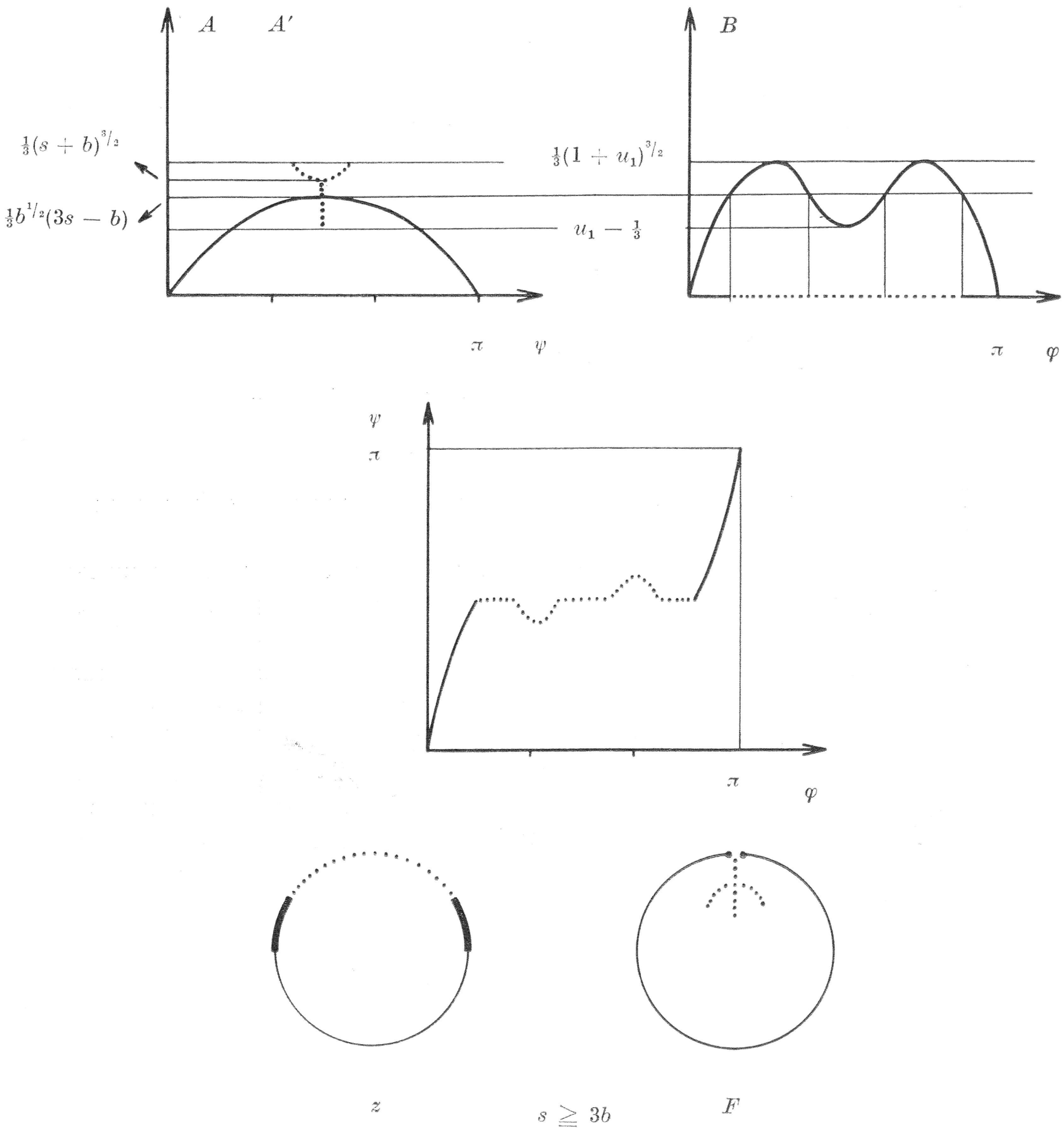


Figure 4.

[2])

$$(37) \qquad u_1 - \tfrac{1}{3} \leqq \tfrac{1}{3}\, b^{1/2}\,(3\,s - b)$$

$\Leftrightarrow$

$$u_1 \leqq \tfrac{1}{3}\,(1 + b^{1/2} + b) + \tfrac{1}{2}\,\frac{b^{1/2}}{1 - b^{1/2}}\,a_2\,.$$

Because $|a_2| \leqq 2(1 - b)$, this implies $u_1 \leqq 3$. According to (33), the condition can be rewritten in the form

$$(38) \quad a_3 \leqq \tfrac{1}{4} \frac{3 - 5\,b^{1/2}}{1 - b^{1/2}} a_2^2 - \tfrac{1}{3}(1 - b^{1/2})^2 a_2 + \tfrac{2}{3}(1 + b^{1/2})(1 - b^{3/2}) .$$

The points of the slits where $\cos\psi \neq 0$ are controlled by the last equation (30) and (31), which give

$$(39) \quad \begin{cases} \varkappa = r + r^{-1} = (1 - \nu)^{1/2} ; \\ \nu = \dfrac{3s}{b} \dfrac{1}{1 - 4\sin^2\psi} , \\ A' = \tfrac{1}{2}\left[\tfrac{1}{3} b^{3/2} (\varkappa^3 - 3\varkappa) \sin 3\psi + b^{1/2} s \varkappa \sin\psi\right] \\ \quad = \tfrac{1}{3}\sin 3\varphi + u_1 \sin\varphi = B . \end{cases}$$

The dotted arcs in Figure 3 indicate points connected with the slits, determined by (39).

As a result of the above considerations we have found a triangle bounded by the parabolas $s = -b$, $s = 3b$ and (38), in the coefficient body (a_3, a_2) where the extremum condition (26) defines functions $F \in S(b^{1/2})$ of the type 1°. The corresponding functions $f \in S(b)$ are thus 3-slit functions. On the boundary curves of the triangle there exist certain degenerated limit cases of those 3-slit functions. The type of these is readable in Figure 3.

The alternative 2° is connected with Figure 4. In this case $A(\psi)$ has only one maximum in the interval $[0, \pi]$ which is bigger than minimum of $B(\varphi)$, provided that (37) continues to hold. This order is needed to guarantee the 3-fork structure of the slit. Again, the dotted arcs belong to the slits and are governed by $A'(\psi)$ of (39). Especially the branch point is connected with

$$\begin{cases} r = \dfrac{(s + b)^{1/2} - (s - 3b)^{1/2}}{2\,b^{1/2}} , \\ \psi = \dfrac{\pi}{2} , \end{cases}$$

obtained from F_2 of (27). Thus $\varkappa = (s/b + 1)^{1/2}$ and (39) give

$$A'\left(\frac{\pi}{2}\right) = \tfrac{1}{3}(b + s)^{3/2} .$$

This is $\leqq \tfrac{1}{3}(u_1 + 1)^{3/2}$, which condition, according to (36), is always true.

Thus, in the case 2° the parabola $s = 3\,b$ together with the parabola (38) determines a domain in the coefficient body (a_3, a_2), where the functions $f \in S(b)$ given by (26) are 3-fork-slit functions. Actually, this type continues to hold slightly behind the parabola (38). The bounadry curve on which the extremum function degenerates into a 2-fork-slit function is given by

$$(38)' \qquad -\tfrac{1}{3} + u_1 \leqq \tfrac{1}{3}(b+s)^{3/2}.$$

This defines a third degree arc in a_2, a_3 with the end points at the intersection of the curves

$$\begin{cases} s = 3b, \\ -\tfrac{1}{3} + u_1 = \tfrac{1}{3} b^{1/2}(3s - b). \end{cases}$$

Thus, these end points are

$$\begin{cases} a_2 = -\tfrac{2}{3}(1 - 9b + 8b^{3/2}), \\ a_3 = \tfrac{5}{9}(1 - 9b + 8b^{3/2})^2 + (3b+1)\tfrac{2}{3}(1 - 9b + 8b^{3/2}) + 6(b - b^2); \end{cases}$$

$$\begin{cases} a_2 = 2(1-b), \\ a_3 = 3 - 8b + 5b^2. \end{cases}$$

In the case $b = 1/2$ the curve (38)′ differs very little from that defined by (38). Hence, the Figure 7 of [1] continues to give a schematic presentation of the extremum domain.*

In the extremum domain, defined above by (32), (38) and (38)′ the inequality (17) is sharp and we obtain from (18) the maximum for d_3:

$$\begin{aligned} \max d_3 &= d_3^0 + R = -d_1\lambda - e_1\mu + 1/3 - e_3 - d_1\mu - e_2\lambda \\ &= -(d_1+e_1)(\lambda+\mu) + 1/3 - e_3 \\ &= \tfrac{1}{3}(1-b^3) - \tfrac{1}{4} b a_2^2 - \tfrac{1}{2}\frac{(a_3 - \tfrac{3}{4}a_2^2 + b a_2)^2}{2(1-b) - a_2} \qquad (40) \\ &= M(a_2, a_3). \end{aligned}$$

This is the result derived already in [1]. From the above inequality $\min d_3$ also follows. Instead of $f(z)$ consider

$$-f(-z) = bz - b_2 z^2 + b_3 z^3 - b_4 z^4 + \dots .$$

This means the following transformation:

$$f(z),\ a_2,\ a_3,\ a_4,\ d_3 \Rightarrow -f(-z),\ -a_2,\ a_3,\ -a_4,\ -d_3.$$

Thus,

$$-d_3(a_2, a_3) \leqq M(-a_2, a_3),$$

i.e.

* In this context another inaccuracy of [1] can be mentioned. Unfortunately the direction of rotation of the extremum domains in Figure 12 of [1] is reversed.

$$(41) \qquad -M(-a_2\,,\,a_3) \;\leq\; d_3(a_2\,,\,a_3)\,.$$

Here the point $(-a_2\,,\,a_3)$ must lie in the domain defined above. Thus the range of $(a_2\,,\,a_3)$ for which (41) is sharp is symmetric with the preceding domain with respect to the a_3 axis.

We may check that our general result (17) implies (41) by substituting $u_3 = i$ in (18) for $a_2\,,\ a_3\,,\ a_4 \in R$:

$$\begin{aligned}
d_3 \geq d_3^0 - R &= -d_1\,\lambda - e_1\,\mu - 1/3 + e_3 + d_1\,\mu + e_1\,\lambda \\
&= (d_1 - e_1)(\mu - \lambda) - 1/3 + e_3 \\
&= -\tfrac{1}{3}\,(1 - b^3) + \tfrac{1}{4}\,b\,a_2^2 + \tfrac{1}{2}\,\frac{(a_3 - \frac{3}{4}\,a_2^2 - b\,a_2)^2}{2\,(1 - b) + a_2} \\
&= -M(-a_2\,,\,a_3)\,.
\end{aligned}$$

5. The special case $a_2 = 0$

Before discussing the general case by the aid of numerical examples, we want to find an easy case, different from the preceding one, where boundary points of the d_3-disc give a sharp inequality. For $a_2 = 0$, $a_3 \neq 0$ (21) gives

$$\begin{cases}
d_3 = \tfrac{1}{2}\,a_4\,, \quad d_1 = \tfrac{1}{2}\,a_3\,, \quad e_3 = \tfrac{1}{3}\,b^3\,, \quad e_1 = 0\,; \\[4pt]
\lambda = 0\,, \quad \mu = \dfrac{\bar{a}_3}{2\,(1 - b)}\,, \quad u_1 = \mu\,\bar{u}_3 = \dfrac{\bar{a}_3}{2\,(1 - b)}\,u_3^{-1}\,.
\end{cases}$$

Now F is determined by (19). Apply to it the rotation $\tau = u_3^{2/3}$, giving $\tilde{u}_3 = 1$. The rotated function $\tilde{f}$ has coefficients $\tilde{a}_\nu$ and defines $\tilde{F} = \tilde{f}(z^2)^{1/2}$, for which there holds, according to (25),

$$\begin{aligned}
&\tfrac{1}{3}\,b^{3/2}\,(\tilde{F}^3 - \tilde{F}^{-3}) + \frac{b^{1/2}}{2\,(1 - b)}\,(\bar{\tilde{a}}_3\,\tilde{F} - \tilde{a}_3\,\tilde{F}^{-1}) \\
&= \tfrac{1}{3}\,(z^3 - z^{-3}) + \frac{1}{2\,(1 - b)}\,(\bar{\tilde{a}}_3\,z - \tilde{a}_3\,z^{-1})\,.
\end{aligned}$$

Now choose τ such that

$$\tilde{a}_3 = \tau^2\,a_3 = |a_3|\,,$$

i.e.

$$(42) \qquad \tau = \left(\frac{|a_3|}{a_3}\right)^{1/2}, \qquad u_3^2 = \tau^3 = \left(\frac{|a_3|}{a_3}\right)^{3/2}.$$

At the boundary point we obtain from (18) for d_3

$$d_3 = \tfrac{1}{2} a_4 = \left[\tfrac{1}{3}(1 - b^3) - \frac{|a_3|^2}{4(1-b)}\right]\left(\frac{a_3}{|a_3|}\right)^{3/2}, \tag{43}$$

and $\tilde{F}$ satisfies the condition

$$\begin{cases} \tfrac{1}{3} b^{3/2} (\tilde{F}^3 - \tilde{F}^{-3}) + b^{1/2} s (\tilde{F} - \tilde{F}^{-1}) = \tfrac{1}{3}(z^3 - z^{-3}) + s(z - z^{-1}), \\ s = \dfrac{|a_3|}{2(1-b)}. \end{cases}$$

Thus, we may use the results true for the equation (26). Both the first condition (32) and (38) must be satisfied. In the present case $-1 \leqq s/b$ is automatically true and (38) reduces into the form

$$|a_3| \leqq \tfrac{2}{3}(1 + b^{1/2})(1 - b^{3/2}). \tag{44}$$

Hence we see that if $a_2 = 0$ and a_3 satisfies (44), the corresponding boundary point of the d_3-disc is sharp in the sense that (43) holds at this boundary point.

6. Real coefficients a_2, a_3

In the general case, where all the coefficients a_2, a_3, a_4 are complex, we may normalize the mapping by rotation into the form (25). For brevity, drop tilda in the notations. Differentiation gives

$$b^{3/2} z \frac{F'}{F} = \frac{z^3 + z^{-3} + \bar{u}_1 z + u_1 z^{-1}}{F^3 + F^{-3} + b^{-1}\bar{s} F + b^{-1} s F^{-1}}.$$

The factorization depends on the roots of the equation having the type

$$z^3 + z^{-3} + \bar{c} z + c z^{-1} = 0.$$

If z is a root here, so are $-z$, $\bar{z}^{-1}$ and $-\bar{z}^{-1}$. This includes two alternatives.

1°. If there are two unit roots (i.e. they have the absolute value 1) not lying on the same diameter of the unit circle $|z| = 1$, there must be four, and hence six, unit roots.

2°. If z is a non-unit root ($|z| \neq 1$), there are four non-unit roots on the diameter where z lies and two unit roots again on one diameter. This implies that the type of domains obtained is the same as before, except that they are not twice axially symmetric any more.

We are not too far from the most general case if we assume a_2 and a_3 to be real and a_4 complex. In Section 4 we actually studied those boundary points of the d_3-disc which were connected with that special case and were given by the values $u_3 = 1$ and $u_3 = i$. Therefore we may restrict ourselves to the values

$$u_3 = e^{i\omega}, \qquad \omega \in \left(0, \frac{\pi}{2}\right).$$

For brevity, normalize by the rotation where

$$\tau = u_3^{2/3} = e^{i\frac{2\omega}{3}}.$$

The connection (25) holds for $\tilde{F}$. The coefficients $\tilde{u}_1$ and $\tilde{s}$ are determined by the following formulae:

$$\begin{cases} u_1 = \lambda e^{i\omega} + \mu e^{-i\omega} = (\lambda + \mu)\cos\omega + i(\lambda - \mu)\sin\omega, \\ s = u_1 + \frac{1}{2} a_2 e^{i\omega} = (\lambda + \mu + \frac{1}{2} a_2)\cos\omega + i(\lambda - \mu + \frac{1}{2} a_2)\sin\omega; \end{cases}$$

$$\begin{cases} \tilde{u}_1 = u_3^{-1/3} u_1 = \xi e^{i\eta}, \\ \tilde{s} = u_3^{-1/3} s = \sigma e^{i\chi}; \end{cases}$$

$$\begin{cases} \lambda + \mu = \dfrac{a_3 - \frac{3}{4} a_2^2 + b a_2}{2(1-b) - a_2}, \\[2ex] \lambda - \mu = \dfrac{a_3 - \frac{3}{4} a_2^2 - b a_2}{2(1-b) + a_2}, \\[2ex] \lambda + \mu + \frac{1}{2} a_2 = \dfrac{a_3 - \frac{5}{4} a_2^2 + a_2}{2(1-b) - a_2}, \\[2ex] \lambda - \mu + \frac{1}{2} a_2 = \dfrac{a_3 - \frac{1}{4} a_2^2 + (1 - 2b) a_2}{2(1-b) + a_2}. \end{cases}$$

The numbers $\xi, \sigma; \eta, \chi$ are determined as follows:

$$\text{(45)} \qquad \begin{cases} \xi^2 = (\lambda + \mu)^2 \cos^2\omega + (\lambda - \mu)^2 \sin^2\omega, \\ \sigma^2 = (\lambda + \mu + \frac{1}{2} a_2)^2 \cos^2\omega + (\lambda - \mu + \frac{1}{2} a_2)^2 \sin^2\omega. \end{cases}$$

$$\text{(46)} \qquad \eta = \arg u_1 - \omega/3; \quad \begin{cases} \cos\arg u_1 = \dfrac{\lambda + \mu}{\xi}\cos\omega, \\[2ex] \sin\arg u_1 = \dfrac{\lambda - \mu}{\xi}\sin\omega. \end{cases}$$

$$\text{(47)} \qquad \chi = \arg s - \omega/3; \quad \begin{cases} \cos\arg s = \dfrac{\lambda - \mu + \frac{1}{2} a_2}{\sigma}\cos\omega, \\[2ex] \sin\arg s = \dfrac{\lambda - \mu + \frac{1}{2} a_2}{\sigma}\sin\omega. \end{cases}$$

By substituting in (25) $z = e^{i\varphi}$, $\tilde{F} = r\,e^{i\psi}$, and splitting it into real and imaginary parts, we end up with the conditions

$$(48)\quad \begin{cases} \frac{1}{3}\,b^{3/2}(r^3 - r^{-3})\cos 3\psi + b^{1/2}\,\sigma\,(r - r^{-1})\cos(\psi - \chi) = 0\,, \\ \frac{1}{3}\,b^{3/2}(r^3 + r^{-3})\sin 3\psi + b^{1/2}\,\sigma\,(r + r^{-1})\sin(\psi - \chi) \\ = \frac{2}{3}\sin 3\varphi + 2\,\xi\sin(\varphi - \eta)\,. \end{cases}$$

The first condition gives a unit circumference $r = 1$ and, for $r < 1$, curves in the disc $|\tilde{F}| < 1$, determined by the equation

$$(49)\qquad H = \tfrac{1}{3}\,b\,(r^2 + 1 + r^{-2}) = -\sigma\,\frac{\cos(\psi - \chi)}{\cos 3\psi} = K\,.$$

The second condition (48) defines for $r = 1$ the connection between the unit circumferences in the z- and $\tilde{F}$-planes. Especially, for $r = 1$ each $\psi \in [0\,, 2\pi]$ must obtain a uniquely determined pre-image $\varphi \in [0\,, 2\pi]$, according to the connection

$$(50)\qquad \begin{aligned} A &= \tfrac{1}{3}\,b^{3/2}\sin 3\psi + b^{1/2}\,\sigma\sin(\psi - \eta) \\ &= \tfrac{1}{3}\sin 3\varphi + \xi\sin(\varphi - \eta) = B\,. \end{aligned}$$

For $r < 1$, (49) together with the second condition (48) determines, the connection between φ and ψ on the slits. From (49) we deduce analogously to (39) in Section 4, that

$$(51)\qquad \begin{cases} \varkappa = r + r^{-1} = (1 - \nu)^{1/2}\,; \\ \nu = \dfrac{3\,\sigma}{b}\,\dfrac{\cos(\psi - \chi)}{\cos 3\psi}\,. \end{cases}$$

The second condition (48) thus assumes the form

$$(52)\qquad \begin{aligned} A' &= \tfrac{1}{2}\,[\tfrac{1}{3}\,b^{3/2}\,(\varkappa^3 - 3\,\varkappa)\sin 3\psi + b^{1/2}\,\sigma\,\varkappa\sin(\psi - \chi)] \\ &= \tfrac{1}{3}\sin 3\varphi + \xi\sin(\varphi - \eta) = B\,. \end{aligned}$$

For each fixed ω and the given pair $(a_2\,, a_3)$ we can study the above conditions similarly to the previous case $\omega = 0$ (and $\omega = \pi/2$). However, the solutions of the extremum problems involved are, in general, only numerically possible.

In the case

$$b = \tfrac{1}{2}, \qquad a_2 = a_3 = \tfrac{1}{4}, \qquad \omega = \pi/4$$

the above connections are illustrated in Figure 5 by the aid of graphs determined numerically. In Figure 6 there is the resulting extremum domain.

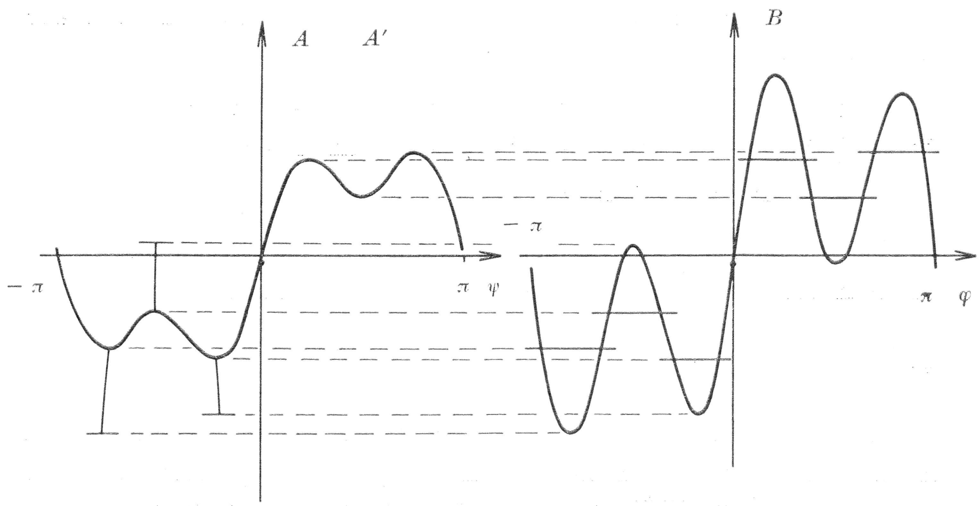


Figure 5.

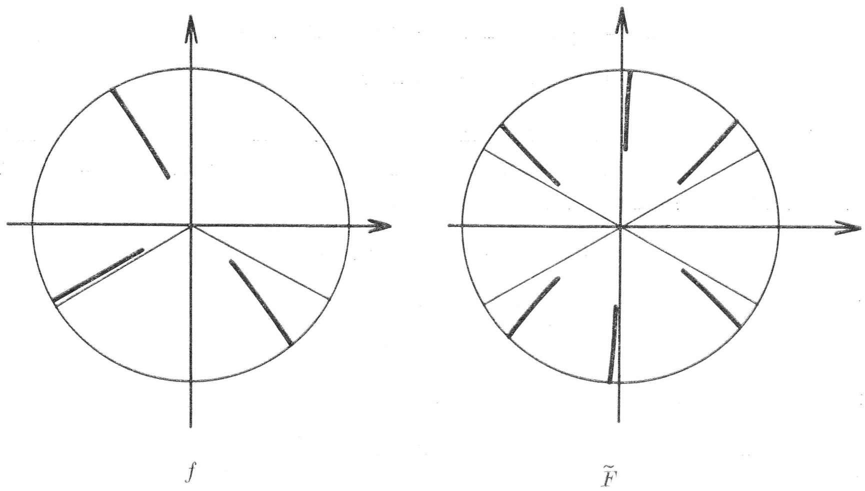


Figure 6.

By determining the range of (a_2, a_3) for various values of ω, one could obtain, as an intersection, the domain round the origin where all the boundary points of the d_3-disc are reached.

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