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ON k-PERIODIC QUASIREGULAR MAPPINGS IN R^n

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1. Introduction

Let $f: R^n \to R^n$ be quasiregular i.e. f is continuous, ACL^n, and $|f'(x)|^n \leq K J(x, f)$ for a.e. $x \in R^n$. The mapping f is called periodic with a period $\omega \neq 0$ if $f(x + \omega) = f(x)$ for all $x \in R^n$. If f is non-constant, then it is well-known that f is discrete and open. Hence in this case the module Ω of all periods of f is spanned by k, $1 \leq k < n$, linearly independent periods $\omega_1, \ldots, \omega_k$. The mapping f is then called k-periodic and $\omega_1, \ldots, \omega_k$ are called primitive periods. Let W denote the $(n-k)$-dimensional linear space orthogonal to the linear space spanned by $\omega_1, \ldots, \omega_k$ and let

$$Q_f = \left\{ x = \sum_{i=1}^{k} \alpha_i \omega_i : 0 \leq \alpha_i < 1 \right\}.$$

Then $F_f = Q_f \times W$ is a period strip for f. For $A \subset R^n$ and $y \in R^n$ we let $N(y, f, A)$ denote the number of points, possibly infinite, in the set $A \cap f^{-1}(y)$ and we set $N(f, A) = \sup_{y \in R^n} N(y, f, A)$.

Quasiregular mappings share many common properties with plane analytic functions, see [1 – 3], however, the following theorem only applies to periodic quasiregular mappings in higher dimensions.

1.1. Theorem. *If $f: R^n \to R^n$ is quasiregular and k-periodic, $1 \leq k \leq n-2$, then $N(f, F_f) = \infty$.*

In the above theorem the assumption "$f: R^n \to R^n$ is quasiregular" can be replaced by "$f: \bar{R}^n \to R^n$ is quasimeromorphic". This turns out to be a slight technical difficulty, see 5.3, but all the other assumptions are strictly necessary, for examples and more details on periodic mappings see [4].

The proof of Theorem 1.1 is by contradiction. However, some of the details, especially Chapter 2, are of independent interest. We shall mainly use the terminology of [1], [4], and [6].

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2. Modulus inequalities for periodic mappings

We follow [8] to define the modulus of a family Γ of paths γ in R^n as

$$M(\Gamma) = \inf_{\varrho \in F(\Gamma)} \int_{R^n} \varrho^n \, dm_n$$

where $F(\Gamma)$ is the class of all non-negative Borel-functions $\varrho : R^n \to \dot{R}$ such that

$$\int_\gamma \varrho \, ds \geq 1$$

for all $\gamma \in \Gamma$.

2.1. Suppose that $f: R^n \to R^n$ is quasiregular and k-periodic, $1 \leq k < n$, with the primitive periods $\omega_1, \ldots, \omega_k$. Let Γ be a path family in R^n. We denote by Γ_f a path family such that $f\Gamma_f = \Gamma$. We define the modulus M' of Γ_f as

$$M'(\Gamma_f) = \inf_{\varrho' \in F'(\Gamma_f)} \int_{F_f} \varrho'^n \, dm_n$$

where $F'(\Gamma_f)$ is the family of all non-negative Borel-functions $\varrho' : R^n \to \dot{R}$ such that ϱ' is invariant under Ω i.e. $\varrho'(x + \omega) = \varrho'(x)$ for all $x \in R^n$ and all $\omega \in \Omega$, and

$$\int_\gamma \varrho' \, ds \geq 1$$

for all $\gamma \in \Gamma_f$. Observe that for given Γ there does not, in general, exist Γ_f and in the case Γ_f exists, it is not uniquely determined; moreover, in many cases $M'(\Gamma_f) \neq M(\Gamma_f)$.

For $A \subset R^n$ we denote by ΩA the orbit of A under Ω, i.e.

$$\Omega A = \{ y = x + \omega : x \in A, \ \omega \in \Omega \}.$$

2.2. Theorem. *Let f, Γ, and Γ_f be as in 2.1. If $N(f, F_f) = N < \infty$, then*

$$\frac{1}{K_O(f)\, N} M'(\Gamma_f) \leq M(\Gamma) \leq K_I(f)\, M'(\Gamma_f). \tag{2.3}$$

Proof. At first we shall prove the left hand side of (2.3). Let $\varrho \in F(\Gamma)$. Set

$$\varrho'(x) = \varrho(f(x))\, L(x, f),$$

see [1, p. 16]. Let Γ_f' be the family of all locally rectifiable paths $\gamma \in \Gamma_f$ such that f is locally absolutely continuous on γ. Then slight modifications in the proof of [8, 28.2] show that $M'(\Gamma_f) = M'(\Gamma_f')$. Now for $\gamma \in \Gamma_f'$ and $\alpha = f \circ \gamma$

$$\int_\alpha \varrho' \, ds \geq \int_\gamma \varrho \, ds \geq 1$$

and since ϱ' is invariant under Ω, $\varrho' \in F(\Gamma_f')$. This implies by a transformation formula for Lebesgue integrals

$$\begin{aligned} M'(\Gamma_f) &= M'(\Gamma_f') \leq \int_{F_f} \varrho'^n \, dm_n = \int_{F_f} \varrho(f(x))^n \, L(x, f)^n \, dm_n(x) \\ &\leq K_O(f) \int_{F_f} \varrho(f(x))^n \, J(x, f) \, dm_n(x) \\ &= K_O(f) \int_{R^n} \varrho(y)^n \, N(y, f, F_f) \, dm_n(y) \\ &\leq K_O(f) \, N \int_{R^n} \varrho^n \, dm_n . \end{aligned}$$

Since $\varrho \in F(\Gamma)$ was arbitrary, the first inequality follows.

To prove the right hand side of (2.3) let $\varrho' \in F'(\Gamma_f)$. Set $E = \{ x \in R^n : \exists f'(x) \text{ and } J(x, f) > 0 \}$ and $E_0 = R^n \setminus E$. Then, by [1, 2.26 and 8.2], $m_n(E_0) = 0$. Define $\sigma : R^n \to \dot{R}$ as

$$\begin{aligned} \sigma(x) &= \varrho'(x)/l(f'(x)) , & x \in E , \\ &= +\infty , & x \in E_0 , \end{aligned}$$

and $\varrho : R^n \to \dot{R}$ as

$$\varrho(y) = \sup \{ \sigma(x) : x \in f^{-1}(y) \} .$$

Then ϱ is a non-negative Borel-function, see [9, p. 6]. Next we shall show that $\varrho \in F(\Gamma_1)$ where $\Gamma_1 = \Gamma \setminus \Gamma_0$ and Γ_0 is the family of all paths β in R^n such that either β is locally non-rectifiable or there is a path α in R^n such that $f \circ \alpha \subset \beta$ and f is not absolutely precontinuous on α in the terminology of [9, Definition 2.4]. Then, by [9, Lemma 2.6], $M(\Gamma_1) = M(\Gamma)$. Fix $\beta \in \Gamma_1$ and assume that β is a closed path. We parametrize β by means of its arc length. Let $\alpha \in \Gamma_f$ be such that $f \circ \alpha = \beta$. Now α is absolutely continuous and for almost every t either $\alpha(t) \in E_0$ or

$$1 = |\beta'(t)| = |f'(\alpha(t))\,\alpha'(t)| \geq l(f'(\alpha(t)))\,|\alpha'(t)|\,.$$

Thus the inequality $\sigma(\alpha(t)) \geq \varrho'(\alpha(t))\,|\alpha'(t)|$ holds almost everywhere. Consequently

$$1 \leq \int_\alpha \varrho'\,ds = \int \varrho'(\alpha(t))\,|\alpha'(t)|\,dt \leq \int \sigma(\alpha(t))\,dt \leq \int_\beta \varrho\,ds\,.$$

If the path β is open or half open, we obtain the result by using a simple limit process, see [8, § 3].

To complete the proof we estimate the integral

$$\int_{R^n} \varrho^n\,dm_n\,.$$

At first it is easy to show using the method of [1, 7.15], the periodicity of f, and the finiteness of $N(f, F_f)$ that there exists a countable net of open disjoint cubes $Q_1, Q_2, \ldots$ such that (1) $R^n \setminus fB_f = \cup \operatorname{cl} Q_i$, (2) the components of $f^{-1}Q_i$ which meet F_f form a finite collection $D_1^i, \ldots, D_k^i$, and (3) f defines quasiconformal mappings $f_j^i \colon D_j^i \to Q_i$. Since, by [1, 2.27], $m_n)fB_f) = 0$, (1) implies

$$\int_{R^n} \varrho^n\,dm_n = \sum_{i=1}^{\infty} \int_{Q_i} \varrho^n\,dm_n\,. \tag{2.4}$$

Fix i and let $E_j^i = D_j^i \cap F_f$. Denote by $j \sim j'$ the equivalence relation $D_j^i \subset \Omega(D_{j'}^i)$. Let the equivalence classes be $a_1, \ldots, a_p$. Clearly $p \leq N$. We define $A_l = \cup E_j^i$, $j \in a_l$, $l = 1, \ldots, p$. Now each A_l is open in F_f and mapped injectively onto Q_i by f. For every $y \in Q_i$ we let $\{x_l\} = f^{-1}(y) \cap A_l$, $l = 1, \ldots, p$. For almost every $y \in Q_i$

$$\begin{aligned}
\varrho(y)^n &= \max_{l=1,\ldots,p} \varrho'(x_l)^n / l(f'(x_l))^n \leq \sum_{l=1}^{p} \varrho'(x_l)^n / l(f'(x_l))^n \\
&= \sum_{l=1}^{p} \varrho'((f|A_l)^{-1}(y))^n / l(f'((f|A_l)^{-1}(y)))^n \\
&\leq K_I(f) \sum_{l=1}^{p} \varrho'((f|A_l)^{-1}(y))^n\, J(y, (f|A_l)^{-1})\,.
\end{aligned} \tag{2.5}$$

Here we have used the quasiconformality of the mappings $(f|A_l)^{-1}$ restricted to the components of $f(\operatorname{int} A_l)$ and the fact that ∂F_f is of m_n-measure zero and f preserves sets of measure zero, see [1, 8.4]. Integrating (2.5) over Q_i and using a transformation formula for Lebesgue integrals we obtain

$$\begin{aligned}\int_{Q_i} \varrho^n \, dm_n &\leq K_I(f) \sum_{l=1}^{p} \int_{Q_i} \varrho'((f|A_l)^{-1}(y))^n J(y\,,(f|A_l)^{-1})\, dm_n(y) \\ &= K_I(f) \sum_{l=1}^{p} \sum_{j\in a_l} \int_{fE_j^i} \varrho'(f_j^{i-1}(y))^n J(y\,,f_j^{i-1})\, dm_n(y) \\ &= K_I(f) \sum_{l=1}^{p} \sum_{j\in a_l} \int_{E_j^i} \varrho'^n \, dm_n = K_I(f) \sum_{l=1}^{p} \int_{A_l} \varrho'^n \, dm_n \\ &= K_I(f) \int_{f^{-1}(Q_i)\cap F_f} \varrho'^n \, dm_n\,.\end{aligned}$$

Summing over i yields by (2.4)

$$M(\Gamma) = M(\Gamma_1) \leq \int_{R^n} \varrho^n \, dm_n \leq K_I(f) \int_{F_f} \varrho'^n \, dm_n\,.$$

Since $\varrho' \in F'(\Gamma_f)$ was arbitrary, the result follows.

2.6. *Remarks.* (a) It is possible to prove better modulus estimates than (2.3) in the spirit of [9]. However, we shall only need the right hand side of (2.3).

(b) The right hand side of (2.3) is true without the assumption $N(f\,, F_f) < \infty$. Almost the same proof applies to this case.

(c) The essential idea in the inequalities (2.3) is that instead of f we consider the mapping $g: M \to R^n$ from the orbit space $M = R^n/\Omega$ induced by f. The inequalities (2.3) now become ordinary modulus inequalities for the quasiregular mapping g defined on the manifold M.

2.7. We shall need a modulus estimate for a special path family. Assume that f is as in 2.1 and, moreover, that the primitive periods of f are the coordinate unit vectors $e_1, \ldots, e_k$, $1 \leq k \leq n-1$. Let W be the linear space spanned by $e_{k+1}, \ldots, e_n$ and V its orthogonal space. For $R_0 > 0$ we denote

$$C(R_0) = \{ x \in R^n : d(x\,, V) < R_0 \}\,.$$

Here d means the usual euclidean distance.

2.8. Lemma. *Let $R_0 \geq 1$ and let Γ_f be a path family described in* 2.1 *with two additional properties:*

(a) *Each $\alpha \in \Gamma_f$ is contained in $R^n \setminus C(R_0)$.*

(b) *If $\sup_t d(\alpha(t)\,, V) = r'$, then the length of α is $\geq r'/2$.*

Then $M'(\Gamma_f) \leq C/R_0$ where C depends only on n and k.

Proof. For $x \in R^n$ we use the representation $x = (z, z') \in V \times W$. Let

$$\varrho'(x) = \varrho'(z, z') = 2\,|z'|^{-(1+n-k)/n} \text{ if } |z'| \geq R_0 \text{ and}$$
$$\varrho'(x) = 0 \text{ otherwise.}$$

Let $\alpha \in \Gamma_f$ and suppose that $\sup d(\alpha(t), V) = r'$. Then $r' \geq R_0 \geq 1$. Now by (b)

$$\int_\alpha \varrho'\, ds \geq \int_\alpha 2\, r'^{-(1+n-k)/n}\, ds \geq r'^{-(1+n-k)/n}\, r' = r'^{(k-1)/n} \geq 1 .$$

Clearly ϱ' is invariant under Ω, hence $\varrho' \in F'(\Gamma_f)$. This implies by Fubini's theorem

$$\begin{aligned} M'(\Gamma_f) &\leq \int_{F_f} \varrho'^n\, dm_n = \int_{V \cap \operatorname{cl} F_f} \int_W \varrho'^n\, dm_{n-k}\, dm_k \\ &= 2^n \int_{W \cap \complement C(R_0)} |z'|^{-(1+n-k)}\, dm_{n-k}(z') = 2^n\, \omega'_{n-k-1} \int_{R_0}^\infty r^{-2}\, dr \\ &= 2^n\, \omega'_{n-k-1}/R_0 , \end{aligned}$$

where ω'_p denotes the p-dimensional measure of S^p.

3. Behavior of f at ∞

Suppose that $f\colon R^n \to R^n$ is quasiregular and k-periodic, $1 \leq k \leq n-2$, with $N = N(f, F_f) < \infty$. Assume that the primitive periods of f are $e_1, \ldots, e_k$.

3.1. Lemma. *Under the above assumptions* $\lim f(x) = \infty$ *as* $x \to \infty$ *in* F_f.

Proof. Choose $y \in fF_f$ with $\operatorname{card}(f^{-1}(y) \cap F_f) = N$ and let $f^{-1}(y) \cap F_f = \{x_1, \ldots, x_N\}$. For each i let U_i denote a normal neighborhood of x_i, see [1, 2.4]. We may assume that $\Omega U_i \cap U_j = \emptyset$ for $i \neq j$. Set $U = \cup U_i$, $V = \cap fU_i$, and $Q_1 = \{x \in R^n : |x_i| < 1, i = 1, \ldots, n\}$. We may assume that $U \subset Q_1$ for if this is not the case, we may take a bigger cube instead of Q_1. Now $f(\complement \Omega Q_1) \subset \complement V$ for if there exists $x \in \complement \Omega Q_1$ such that $f(x) \in V$, then $\operatorname{card}(f^{-1}f(x) \cap F_f) > N$ which is impossible.

Let $Q = 2\,Q_1 \setminus \Omega \operatorname{cl} Q_1$ where we have used the notation

$$pA = \{x \in R^n : x/p \in A\},$$

$p \in R \setminus \{0\}$, for a subset A of R^n. Observe that Q is connected; the proof breaks down here for $k = n-1$. For each integer $m > 0$ we define $g_m : Q \to R^n$ as $g_m(x) = m\,x$ and set $f_m = f \circ g_m$. Then $f_m : Q \to \complement V$, hence $\{f_m\}$ is a normal family [2, Theorem 3.17]. Passing to a subsequence if necessary, we may assume that $\{f_m\}$ converges uniformly on every compact subset of Q. By [5, p. 664] the limit mapping f_0 is quasimeromorphic. We shall first show that f_0 is constant.

Choose a point $x_0 \in Q$ and a number $r_0 \in (0, d(x_0, \partial Q))$. By the periodicity of f for every $m > 2/r_0$ there exists $x_m \in F = \operatorname{cl}(B^n(x_0, r_0)) \setminus B^n(x_0, r_0/2)$ such that $f_m(x_m) = f_m(x_0)$. Since F is compact in Q, there exists a subsequence of $\{x_m\}$ converging to $y_0 \in F$. On the other hand $f_0(y_0) = f_0(x_0)$. Since this is true for all $r_0 \in (0, d(x_0, \partial Q))$, x_0 is not an isolated point of $f_0^{-1}f_0(x_0)$. Thus f_0 cannot be discrete and so it is constant $= a$.

Now we shall show that $\lim f(x) = a$ as $x \to \infty$ in the period strip F_f. For a moment we may assume that $a = \infty$. Let $M > 0$ and denote $E = \Omega((3/2)Q_1)$, $E_m = g_m\, \partial E$, and $F_m = F_f \cap (g_{m+1}E \setminus g_m E)$. Choose m_0 such that $f_m\, \partial E \subset \complement B^n(M)$ for $m > m_0$. Now for $m > m_0$, $\partial f F_m \subset f E_m \cup f E_{m+1} \subset \complement B^n(M)$. Since $f F_m \subset \complement V$ and f is open, $f F_m \subset \complement B^n(M)$. This shows the existence of the limit.

Finally, if $a \neq \infty$, then f is bounded. This violates Liouville's theorem in n dimensions, see [1, p. 29]. The lemma follows.

3.2. *Remark.* The above proof is similar to the proof of Theorem 8.2 in [4].

3.3. Lemma. $fR^n = R^n$.

Proof. This immediately follows from Liouville's theorem [1, p. 29] and Lemma 3.1.

3.4. Let $P : R^n \to R^n$ denote the projection to the linear space W orthogonal to the periods of f.

3.5. Lemma. *Suppose that* $C \subset R^n$ *is such that* PC *is bounded. Then there exists* $r_0 > 0$ *with the properties*

(a) $f^{-1}B^n(r)$ *and* $\complement f^{-1}B^n(r)$ *are connected,*

(b) $C \subset f^{-1}B^n(r)$,

for all $r \geq r_0$.

Proof. By Lemma 3.3 there exists $x \in f^{-1}(0)$. By the periodicity of f and Lemma 3.1 each component of $f^{-1}B^n(r)$ is mapped onto $B^n(r)$ and hence the finiteness of $f^{-1}(0) \cap F_f$ implies that for large r, say $r \geq r'$, the x-component $U(r)$ of $f^{-1}B^n(r)$ is the whole of $f^{-1}B^n(r)$. Take $r_0 \geq r'$ so that $B^n(r_0) \supset fC$. Then $C \subset U(r)$ for $r \geq r_0$. If now $\complement U(r)$ is not connected, let F denote the component of $\complement U(r)$ such that $F \cap F_f$ is not bounded. Observe that since $1 \leq k \leq n-2$ there exists only one such component. The proof breaks down here for $k = n-1$. Let E be a component of $\complement U(r) \setminus F$. Since f is open and does not take the value ∞, the periodicity of f implies $f \operatorname{int} E \subset B^n(r)$, hence $\operatorname{int} E = \emptyset$. Setting $U = \complement F$ we thus have $fU \subset \operatorname{cl} B^n(r)$. Now $fU \subset B^n(r)$ since f is open. But this implies $U \subset U(r)$ which shows $E = \emptyset$. The lemma follows.

3.6. *Remark.* If $k = n-1$, then the above lemma holds except the result "$\complement f^{-1}B^n(r)$ is connected", see [4]. This result turns out to be essential in the constructions of the next chapter.

4. Construction lemma

Here we do the main construction for contradiction. The construction of the path family Γ in Lemma 4.2 and its modulus estimate are based on Rickman's method in [6], and we shall frequently refer to his work and omit some details.

4.1. Let f be as in 2.7. Recall that $C(r)$ denotes the cylinder $\{ x \in R^n : d(x, V) < r \}$ and V is the linear subspace spanned by the periods $e_1, \ldots, e_k$ of f. Fix $r_0 > 0$ so that Lemma 3.5 holds with $C = C(1)$. Let $r > r_0$. Then by Lemmas 3.5 and 3.1 $f^{-1}S^{n-1}(r)$ meets the line $L = \{ t e_n : t > 0 \}$ at some point x. Set $y = f(x) \in S^{n-1}(r)$. For $\varphi \in (0, \pi]$ we let $C(y, \varphi)$ be the open spherical cap of angle φ centered at y on the sphere $S^{n-1}(r)$.

4.2. Lemma. *There exists a family Γ of paths $\gamma : [0, t_\gamma] \to S^{n-1}(r)$ such that*

(a) $\gamma(0) = y$.

(b) $M_n^S(\Gamma) \geq d/rN^{n+1}$ *where* $d = d(n) > 0$ *and* M_n^S *means the n-modulus on a sphere* [8, § 10].

(c) γ *has a lift* $\alpha : [0, t_\gamma] \to \complement C(1)$ *starting at* x *and such that the length of* α *is* $\geq |x|$.

For the proof it is convenient to consider the orbit space $M = R^n / \Omega$ which is a connected and oriented n-manifold obtained from $\operatorname{cl} F_f$ by identifying its opposite faces. Let $\pi : R^n \to M$ be the canonical projection. Then π is a covering mapping. Now f induces a discrete and open mapping $g : M \to R^n$ such that $f = g \circ \pi$ and $\operatorname{card}(g^{-1}(y)) \leq N$ for all $y \in R^n$. Let $M^* = M \cup \{\infty\}$ denote the one point compactification of M and $i : M \to M^*$ the natural inclusion mapping. By Lemma 3.1 g has a continuous extension $g^* : M^* \to \overline{R}^n$ with $g^*(\infty) = \infty$.

We remark that all the local topological results in [1, pp. 8–11] concerning discrete and open mappings $f : G \to R^n$, $G \subset R^n$ a domain, remain valid if instead of f we consider $g : M \to R^n$. Especially we shall use the notation $U(z, g, r)$ for the z-component of $g^{-1}B^n(g(z), r)$ of a point z in M. Similarly we shall use the concept "normal neighborhood" in M and the notation $N(y, g, A) = \operatorname{card}(g^{-1}(y) \cap A)$, $A \subset M$.

We shall use the following notation for paths: Let M' be M or R^n. If $\alpha : [a, b] \to M'$ and $\beta : [a', b'] \to M'$ are paths with $\alpha(b) = \beta(a')$ we denote by $\alpha * \beta$ the path obtained by going at first along α and then along β. By α' we mean the inverse path of α.

The following path lifting lemma can easily be obtained from [3, 3.11] by replacing G by M.

4.3 Lemma. *Suppose that $\beta : [a, b] \to R^n$ is a path and $z \in g^{-1}(\beta(a))$. Then two cases are possible: Either there is a path $\alpha : [a, b] \to M$ such that $g \circ \alpha = \beta$ and $\alpha(a) = z$ or there is $t \in (a, b]$ and a path $\alpha : [a, t) \to M$ such that $g \circ \alpha = \beta|[a,t)$, $\alpha(a) = z$, $\lim_{\tau \to t} i \circ \alpha(\tau) = \infty$, and if $\gamma : [a, t') \to M$ is any partial lift of β, i.e. $g \circ \gamma = \beta|[a, t')$, starting at z, then $t' \leq t$.*

In both cases α is called a maximal lift of β starting at z.

In the following lemmas we study inverse images of caps and their components. Fix $\varphi \in (0, \pi]$. Let E be a component of $g^{-1}C(y, \varphi)$, let $C = \operatorname{cl} g^{-1}C(y, \varphi)$, and let C' be a component of C. By $C^*(u, \varphi)$ we denote the u-component of $g^{-1}C(y, \varphi)$ if $u \in g^{-1}(y)$.

4.4. Lemma. *Let $\beta : [a, b] \to \operatorname{cl} C(y, \varphi)$ be a path such that $\beta(a, b] \subset C(y, \varphi)$, $\beta(b) = y$, and let $u \in g^{-1}(\beta(a)) \cap \operatorname{cl} E$. Then there exists a lift $\alpha : [a, b] \to \operatorname{cl} E$ of β such that (1) $\alpha(a) = u$, (2) $\alpha(a, b] \subset E$, and (3) $\alpha(b) \in g^{-1}(y)$. Moreover, $\alpha[a, b] \subset E$ if and only if $\beta(a) \in C(y, \varphi)$.*

Proof. Assume at first that $g(u) = \beta(a) = z \in \operatorname{cl}(C(y, \varphi)) \setminus C(y, \varphi)$. Let $U = U(u, g, \delta)$ be a normal neighborhood of u. Fix $t \in (a, b)$ with $\beta(a, t] \subset C(y, \varphi) \cap B^n(z, \delta)$. Using [6, Lemma 3.3] which also holds for the mapping g there exists $v \in U \cap E$ such that $g(v) = \beta(t)$. Let $\alpha_1 : [a, t] \to U$ be a lift of $\beta|[a,t]$ terminating at v, see [1, 2.7]. Then $\alpha_1(a) = u$ and $\alpha_1(a, t] \subset E$. Let α_2 be the maximal lift of $\beta_2 = \beta|[t,b]$ starting at v. Then by Lemma 4.3 either (i) $\alpha_2 : [t, b] \to E$ or (ii) $\alpha_2 : [t, t') \to E$ for some $t' \leq b$ and $i \circ \alpha_2(\tau) \to \infty$ as $\tau \to t'$. Now the case (ii) is impossible by Lemma 3.1. Hence the required lift α of β is $\alpha_1 * \alpha_2$.

If $\beta[a, b] \subset C(y, \varphi)$, then we can select α to be the maximal lift of β starting at u. The rest of the proof then follows as above. The last assertion of the lemma is trivial.

4.5. Corollary. *$E = C^*(u, \varphi)$ for some $u \in g^{-1}(y)$.*

4.6. Lemma. *$gE = C(y, \varphi)$.*

Proof. The proof is similar to [6, Lemma 3.5 (d)].

4.7. **Lemma.** $C' = \bigcup_{u \in g^{-1}(y) \cap C'} \operatorname{cl} C^*(u, \varphi) = F$.

Proof. Clearly $F \subset C'$. Let $z \in C'$. Then $g(z) \in \operatorname{cl} C(y, \varphi)$. Select a path $\beta : [a, b] \to \operatorname{cl} C(y, \varphi)$ such that $\beta(a) = g(z)$, $\beta(a, b] \subset C(y, \varphi)$, and $\beta(b) = y$. By Lemma 4.4 there exists a lift $\alpha : [a, b] \to \operatorname{cl} C^*(\alpha(b), \varphi)$ of β where $\alpha(a) = z$ and $\alpha(b) \in g^{-1}(y)$. Now the locus of α belongs to C', hence $\alpha(b) \in C'$, and since $z \in \operatorname{cl} C^*(\alpha(b), \varphi)$, the lemma follows.

4.8. **Corollary.** $C' = \bigcup_{i=1}^{k} \operatorname{cl} C^*(u_i, \varphi)$ *where* $u_i \in g^{-1}(y)$ *and* $k \leq N$.

Proof of Lemma 4.2: We start with the following construction: Let $L^* = \pi\{ t e_n : t \leq 0 \}$ and denote $x' = \pi(x)$. We recall that x is the point selected in 4.1. Consider the set

$$\Phi_0 = \{ \varphi \in (0, \pi] : L^* \cap C^*(x', \varphi) = \emptyset \}$$

and set $\varphi_0 = \sup \Phi_0$. The first step is to prove:

$$(4.9) \quad \begin{array}{ll} \text{a)} & \Phi_0 \neq \emptyset , \\ \text{b)} & \varphi_0 \in \Phi_0 , \\ \text{c)} & \text{the } x'\text{-component of } \operatorname{cl}(g^{-1}C(y, \varphi_0)) \text{ intersects } L^* . \end{array}$$

The proofs of a) and b) are easy and similar to [6, Lemma 3.5]. To prove c) suppose that this is not true. Let C' denote the x'-component of $C = \operatorname{cl} g^{-1}C(y, \varphi_0)$. Since C' is compact in M and since by Corollary 4.8 there exist only a finite number of components of C, we can find a neighborhood V' of C' such that $\partial V' \cap C = \emptyset$, $\operatorname{cl} V'$ is compact in M, and $\operatorname{cl}(V') \cap L^* = \emptyset$. Suppose that $\varphi_0 < \pi$. Then there exists $\varphi > \varphi_0$ such that $C(y, \varphi) \subset gV'$ and $C(y, \varphi) \cap g\partial V' = \emptyset$. But then $C^*(x', \varphi) \subset V'$ and so $\varphi \in \Phi_0$, a contradiction. If $\varphi_0 = \pi$, $g\partial V' \cap S^{n-1}(r) = \emptyset$ and $\partial V' \cap g^{-1}S^{n-1}(r) = \emptyset$. This implies by Lemma 3.5 $g^{-1}B^n(r) \subset V'$ and so $g^{-1}B^n(r) \cap L^* = \emptyset$. But by Lemma 3.5 L^* meets $g^{-1}B^n(r)$.

The next step is to prove that the following situation holds: There exists a sequence $A_1, \ldots, A_p$, $1 \leq p \leq N$, of different components of $g^{-1}C(y, \varphi_0)$ such that

$$(4.10) \quad \begin{array}{ll} \text{a)} & A_1 = C^*(x', \varphi_0) . \\ \text{b)} & \operatorname{cl}(A_i) \cap \operatorname{cl}(A_{i+1}) \neq \emptyset , \quad i = 1, \ldots, p-1 . \\ \text{c)} & \operatorname{cl}(A_p) \cap L^* \neq \emptyset . \\ \text{d)} & \operatorname{cl}(A_i) \cap A_j = \emptyset , \quad i \neq j . \end{array}$$

This can be easily done by using Corollary 4.8 and (4.9) c), see also [6, Lemma 3.6]. Let $A_1, \ldots, A_p$ be the sequence in (4.10). Let

$\beta_{v,b} : [0, t_{v,b}] \to \mathrm{cl}\, C(y, \varphi_0)$ be the path described in [6, 3.8] and parametrized by means of arc length. Here $v \in T = T_{e_1}$ [6, 3.8] and $\beta_{v,b}$ has the properties (1) $\beta_{v,b}(0) = b \in \mathrm{cl}\,(C(y, \varphi_0)) \setminus C(y, \varphi_0)$, (2) $\beta_{v,b}(t_{v,b}) = y$, and (3) $\beta_{v,b}(0, t_{v,b}] \subset C(y, \varphi_0)$.

Fix $v \in T$. Let γ_i be the lift given by Lemma 4.4 of $\beta_i = \beta_{v,b_i}$, $b_i = f(a_i)$, in $\mathrm{cl}\, A_i$ starting at $a_i \in \mathrm{cl}\,(A_i) \cap \mathrm{cl}\,(A_{i+1}) \setminus A_i \cup A_{i+1}$, $i = 1, \ldots, p-1$. Let α_i be the corresponding lift of β_i in $\mathrm{cl}\, A_{i+1}$ starting at a_i, $i = 1, \ldots, p-1$. Suppose that the following conditions are satisfied:

a) γ_1 terminates at x'.

(4.11) b) α_{i-1} and γ_i terminate at a common point for $i = 2, \ldots, p-1$.

c) α_{p-1} meets L^*.

Then $\alpha_v^* = \gamma_1' * \alpha_1 * \gamma_2' * \ldots * \gamma_{p-1}' * \alpha_{p-1}$ is a path which connects x' and L^*, and $\gamma_v = g \circ \alpha_v^*$ is a path on $S^{n-1}(r)$. By the properties of $\pi : R^n \to M$ there is now a lift α_v of γ_v with respect to the mapping f and starting at x such that α_v connects x and $\pi^{-1}L^* = \Omega(\{ z \in R^n : z = t\, e_n, \ t \leq 0\})$, hence the length of α_v is $\geq |x|$. By Rickman's reasoning in [6] we have the property b) of Lemma 4.2 for the path family $\Gamma = \{ \gamma_v : v \in T \}$. Thus it is enough to show the existence of the paths in (4.11). However, this is not possible in general, hence we shall give an outline how to continue the construction in (4.9) and (4.10) if some of the conditions in (4.11) fails and still achieve the samelike situation as in (4.11).

Suppose now that (4.11) a) is not true. Then $N(y, g, A_1) \geq 2$ and $\Phi_1 \neq \emptyset$ where

$\Phi_i = \{ \varphi \in (0, \varphi_0] : g^{-1}C(y, \varphi)$ has more than one component in $A_i \}$, $i = 1, \ldots, p-1$.

Set $\varphi_1 = \sup \Phi_1$. As in (4.10) we can now form a finite number of components $A_{11}, \ldots, A_{1q}$ of $g^{-1}C(y, \varphi_1)$ in A_1 so that

$$A_{11} \cup \ldots \cup A_{1q} \cup \{a_{11}, \ldots, a_{1,q-1}\}$$

is connected for some $a_{1j} \in \mathrm{cl}\,(A_{1j}) \cap \mathrm{cl}\,(A_{1,j+1})$. Setting $b_{1j} = g(a_{1j})$ and $\beta_{1j} = \beta_{v,b_{1j}}$ and lifting as above we finally obtain, repeating this process, if necessary, in A_{1j} and so on, a path with the property (4.11) a). Observe that the above repeating process ends after at most N steps.

If (4.11) b) is not true, we do the above construction in each A_i.

If (4.11) c) is not satisfied, then the terminal point of α_{p-1} lies in $A_p \cap g^{-1}(y)$. Two cases are now possible:

Case 1. $A_p \cap L^* = \emptyset$. In this case we choose a point $a_p \in \mathrm{cl}\,(A_p) \cap L^*$

and define the path γ_p similarly as the paths $\gamma_1, \ldots, \gamma_{p-1}$. The construction in A_p can be continued similarly as in the components $A_1, \ldots, A_{p-1}$.

Case 2. $A_p \cap L^* \neq \emptyset$. In this case we consider the non-empty set

$$\Phi_0^z = \{ \varphi \in (0, \varphi_0] : C^*(z, \varphi) \cap L^* = \emptyset \}$$

where z is the terminal point of α_{p-1}. We repeat all replacing x by z and φ_0 by $\varphi_0^z = \sup \Phi_0^z$. This process ends after a finite number of steps either in a case like (4.11) c) or like Case 1.

Since $N(y, g, M) \leq N$, it is clear from the above construction that for each $v \in T$ we end with a path in $S^{n-1}(r)$ of the form

$$\gamma_v = \beta'_{v,z_1} * \beta_{v,z_1} * \ldots * \beta_{v,z_k}$$

where $z_i \in S^{n-1}(r) \setminus \{y\}$ does not depend on v and, moreover, $k \leq N$, and such that γ_v has a lift with respect to the mapping g connecting x' and L^*. For more details we refer to [6]. Lemma 4.2 follows.

5. Proof of Theorem 1.1.

Suppose that the theorem is not true. Performing an auxiliary quasiconformal transformation we may assume that f is as in 4.1. We fix $r_0 > 0$ so large that Lemma 3.5 holds with $C = C(1)$, see 2.7 and 4.1. For each $r > r_0$ let Γ_r be the path family on $S^{n-1}(r)$ of Lemma 4.2. We set

$$\Gamma = \bigcup_{r > r_0} \Gamma_r .$$

Now if $\varrho \in F(\Gamma)$, then $\varrho | S^{n-1}(r) \in F_S(\Gamma_r)$, and by Lemma 4.2

$$\int_{S^{n-1}(r)} \varrho^n \, dS \geq M_n^S(\Gamma_r) \geq \frac{d}{N^{n+1} r} .$$

Integrating from r_0 to $t > r_0$ gives

$$\int_{B^n(t) \setminus B^n(r_0)} \varrho^n \, dm_n \geq \frac{d}{N^{n+1}} \log \frac{t}{r_0} .$$

Letting $t \to \infty$ implies

$$M(\Gamma) = \infty . \tag{5.1}$$

For each $\gamma \in \Gamma$ let α denote the lift of γ described in Lemma 4.2 c). Set $\Gamma_f = \{ \alpha : \gamma \in \Gamma \}$. Now a path $\alpha : [0, t_\gamma] \to R^n$ in Γ_f satisfies the condition b) of Lemma 2.8; to see this let

$$\sup \{ d(\alpha(\tau), V) : \tau \in [0, t_\gamma] \} = r' .$$

If $d(\alpha(0), V) > r'/2$, then $l(\alpha) \geq r'/2$ by Lemma 4.2 c). If

$$d(\alpha(0), V) \leq r'/2 ,$$

we let $P : R^n \to R^n$ denote the projection $P(x) = x - (x_1 e_1 + \ldots + x_k e_k)$ and estimate

$$\begin{aligned} l(\alpha) &\geq |\alpha(0) - \alpha(t)| \geq |P(\alpha(0)) - P(\alpha(t))| \\ &\geq |P(\alpha(t))| - |P(\alpha(0))| \geq r' - r'/2 = r'/2 \end{aligned}$$

where t is such that $d(\alpha(t), V) = \sup d(\alpha(\tau), V) = r'$. The other conditions of Lemma 2.8 are true for $R_0 = 1$, hence

$$M'(\Gamma) \leq C < \infty . \tag{5.2}$$

But (5.1) and (5.2) together with the right hand side of (2.3) yield a contradiction. This proves the theorem.

5.3. *Remarks.* (a) In Theorem 1.1 the condition "$f : R^n \to R^n$ is quasiregular" can be replaced by "$f : R^n \to \overline{R}^n$ is quasimeromorphic", for the terminology see [2]. The proof goes along the same lines. Theorem 2.2 holds as it is. In Lemma 3.1 the limit can be finite, but using an auxiliary Möbius transformation we may still assume that the limit is ∞. Lemma 3.3 now reads $fR^n = \overline{R}^n$, if f really takes the value ∞ at some point. Lemma 3.5 cannot be proved in its original form, however, we can easily avoid the points of $f^{-1}(\infty)$ since there are, by assumption, only finitely many of them in F_f, and a look at the later proofs will show that we only need the component F of $\complement f^{-1}B^n(r)$ for which $F_f \cap F$ is unbounded. The rest of the proof requires only minor changes.

(b) Theorem 1.1 or its quasimeromorphic form have some applications to the theory of space quasiconformal mappings. For instance, the domain

$$G_t = \{ x \in R^n : 0 < x_n < t \} , \qquad t > 0 ,$$

in R^n, $n \geq 3$, cannot be mapped onto B^n by a quasiconformal mapping f. For if this is possible, then f can be extended by reflection to a quasimeromorphic mapping $f^* : R^n \to \overline{R}^n$ such that f^* is 1-periodic, $\operatorname{int} F_{f^*} = G_{2t}$, and $N(f^*, F_{f^*}) = 1$. This phenomenon was observed by J. Väisälä in [7].

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