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ON THE LOCAL BEHAVIOR OF QUASIREGULAR MAPPINGS

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1. Introduction

Consider a non-constant quasiregular mapping $f: G \to R^n$ from a domain $G \subset R^n$ into the n-dimensional euclidean space R^n, $n \geq 2$. Let B_f be the branch set of f. For any $A \subset R^n$ and $y \in R^n$ let $N(y, f, A)$ be the cardinality of $A \cap f^{-1}(y)$. We call $N(f, A) = \sup \{ N(y, f, A) \mid y \in R^n \}$ the multiplicity of f in A. If $i(x, f)$ denotes the local topological index of f at $x \in G$, then

$$i(x, f) = \min \{ N(f, U) \mid U \quad \text{is a neighborhood of } x \}.$$

Suppose $x \in B_f$ and Δ is an open cone with vertex at x and angle $\alpha \in (0, \pi]$. In [3, 4.4] it is proved that if $n \geq 3$ and $\Delta \cap B_f = \emptyset$, then $i(x, f) \leq C$, where C is a constant depending only on n, α and the maximal dilatation $K(f)$ of f. In this paper we will show that if $n \geq 2$ and f, x and Δ are as above, then $i(x, f) \leq C'$, where C' is a constant depending only on n, α, $N(f, \Delta)$ and $K(f)$. For $n \geq 3$ we will use this result to derive a new proof for the above cone theorem [3, 4.4].

Our notation and terminology is the same as in [2] and [3].

2. Terminology and preliminary results

2.1. We let the notation $f: G \to R^n$ include the assumption that G is a domain in R^n and f is continuous and non-constant. If $f: G \to R^n$ is quasiregular, we write $K_I(f)$, $K_O(f)$ and $K(f)$ for the inner, outer and maximal dilatations, respectively.

Suppose that $f: G \to R^n$ is open and discrete. If $x \in G$ and $r > 0$, we let $U(x, f, r)$ denote the x-component of $f^{-1}B^n(f(x), r)$, and for $0 < r < d(f(x), \partial fG)$ we write, $l^*(x, f, r) = \inf \{ |x-y| \mid y \in \partial U(x, f, r) \}$

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and $L^*(x, f, r) = \sup\{ |x-y| \mid y \in \partial U(x, f, r)\}$. By [2, 2.9] there exists $r_x > 0$ such that $U(x, f, r)$ is a normal neighborhood of x for $0 < r \leq r_x$. If $U(x, f, r_0)$ is a normal neighborhood, it can be shown as in [2, 4.8] that the mapping $r \mapsto l^*(x, f, r)$ is continuous for $r \in (0, r_0]$.

If E, F and D are subsets of R^n, we let $\Delta(E, F; D)$ denote the path family joining E and F in D. We say that path family Γ_1 is minorized by path family Γ_2, abbreviated $\Gamma_1 > \Gamma_2$, if every path in Γ_1 has a subpath which is in Γ_2.

If $x \in R^n$, $e \in S^{n-1}$ and $\alpha \in (0, \pi]$, we let Cone $(x, e; \alpha)$ denote the open infinite cone with vertex at x, axis $\{ x + te \mid t > 0\}$ and angle α, i.e.

$$\text{Cone}\,(x, e; \alpha) = \{ y \in R^n \mid e \cdot (y-x) > |y-x| \cos \alpha \},$$

where $e \cdot (y-x)$ means the scalar product of these vectors. Further we write

$$b_n(\alpha) = m_{n-1}(S^{n-1} \cap \text{Cone}\,(0, e; \alpha)) . \tag{2.2}$$

In [3, 5.2] it was proved that for a quasiregular mapping $f\colon G \to R^n$, the inverse linear dilatation $H^*(x, f)$, $x \in G$, is bounded in G by a constant depending only on n and K. In fact, the proof of [3, 5.2] allows a slightly stronger statement. The next lemma is a combined version of this stronger form of [3, 5.2] and a part of [2, 2.9].

2.3. Lemma. *Let $f\colon G \to R^n$ be open and discrete, and $x \in G$. Then there exists $r_x > 0$ such that for every $r \in (0, r_x]$,*

(1) *$U(x, f, r)$ is a normal neighborhood of x,*
(2) *$\partial U(x, f, r) = U(x, f, r_x) \cap f^{-1}S^{n-1}(f(x), r)$ for $0 < r < r_x$,*
(3) *if, in addition, f is K-quasiregular, then $L^*(x, f, r) \leq C^* \, l^*(x, f, r)$, where*

$$C^* = C^*(n, K) = \exp[c_n K^{2/(n-1)}], \tag{2.4}$$

here c_n being a constant depending only on n.

Using (3) of the above lemma, we can restate [3, 4.3] in a quantitatively better form:

2.5. Lemma. *Suppose that $f\colon G \to R^n$ is K-quasiregular and $x \in G$. Let $r_x > 0$ and $C^* = C^*(n, K)$ be as in Lemma 2.3. Then for every $r \in (0, r_x]$, we have: If $|x-y| < l^*(x, f, r)/C^*$, then*

$$|f(x) - f(y)| \leq r \left(\frac{C^*}{l^*(x, f, r)}\right)^{\mu} |x-y|^{\mu},$$

where $\mu = (i(x, f)/K_I(f))^{1/(n-1)}$.

Proof. By Lemma 2.3, for every $r \in (0, r_x]$, $U(x, f, r)$ is a normal neighborhood of x and $L^*(x, f, r)/l^*(x, f, r) \leq C^*$. Fix $r \in (0, r_x]$ and choose $y \in G$ with $0 < |y-x| < l^*(x, f, r)/C^*$. Then we proceed as Martio in [1, 6.1]. Let $s = |f(x) - f(y)| > 0$. Hence,

$$L^*(x, f, s) \leq C^* l^*(x, f, s) \leq C^* |x-y| < l^*(x, f, r) . \tag{2.6}$$

Thus, $\mathrm{cl}\, U(x, f, s) \subset B^n(x, l^*(x, f, r))$. For the condenser $(U(x, f, r), \mathrm{cl}\, U(x, f, s)) = E$, we have, by [1, 5.13 and 5.15],

$$\begin{aligned} \omega_{n-1}\left[\log \frac{r}{s}\right]^{1-n} &= \mathrm{cap}\, fE \leq \frac{K_I(f)}{i(x, f)} \mathrm{cap}\, E \\ &\leq \frac{K_I(f)\, \omega_{n-1}}{i(x, f)} \left[\log \frac{l^*(x, f, r)}{L^*(x, f, s)}\right]^{1-n} . \end{aligned}$$

It follows that $s \leq r\, l^*(x, f, r)^{-\mu} L^*(x, f, s)^{\mu}$, which implies with (2.6), $|f(x) - f(y)| \leq r\, l^*(x, f, r)^{-\mu} C^{*\mu} |x-y|^{\mu}$. The lemma is proved.

3. On multiplicity and local index

3.1. **Lemma.** *Suppose that $f: G \to R^n$ is K-quasiregular and $x \in G$. Let $r_x > 0$ and $C^* = C^*(n, K)$ be as in Lemma 2.3. If $m > 1$, $0 < t < L^*(x, f, r_x)$ and A is a non-empty Borel set in $U(x, f, r_x)$, then*

$$i(x, f) \leq \omega_{n-1} K_I(f) K_O(f) \frac{N(f, A)}{M(\Gamma) (\log m)^{n-1}} , \tag{3.2}$$

where $\Gamma = \Delta(S^{n-1}(x, t/(mC^{*2})), S^{n-1}(x, t); A)$.

Proof. By [2, 3.2],

$$M(\Gamma) \leq K_O(f) N(f, A) M(f\Gamma) . \tag{3.3}$$

To derive (3.2) from this inequality, we estimate $M(f\Gamma)$. By Lemma 2.3 and the assumptions of the theorem, $t < L^*(x, f, r_x) \leq C^* l^*(x, f, r_x)$. Hence, by the continuity of the mapping $r \mapsto l^*(x, f, r)$, $0 < r \leq r_x$, we can choose $r \in (0, r_x)$ such that $l^*(x, f, r) = t/C^*$. Set $l^* = l^*(x, f, r)$. The relation $t/(m C^{*2}) = l^*/(m C^*) < l^*/C^*$ implies, by Lemma 2.5, $fS^{n-1}(x, t/(m C^{*2})) \subset \mathrm{cl}\, B^n(f(x), r')$, where $r' = r (C^*/l^*)^{\mu} (l^*/(m C^*))^{\mu} = r m^{-\mu}$, and $\mu = (i(x, f)/K_I(f))^{1/(n-1)}$. Let Γ be as in (3.2). Since $L^*(x, f, r) \leq C^* l^*(x, f, r) = t$, $f\Gamma$ is minorized by

$$\Gamma' = \Delta(S^{n-1}(f(x), r'), S^{n-1}(f(x), r); B^n(f(x), r)) .$$

Then

$$M(f\Gamma) \leq M(\Gamma') = \omega_{n-1}\left(\log\left(\frac{r}{rm^{-\mu}}\right)\right)^{1-n} = \omega_{n-1}(\mu \log m)^{1-n} .$$

Use this upper bound of $M(f\Gamma)$ in (3.3) and the lemma follows.

3.4. Theorem. *Suppose that* $f: G \to R^n$ *is* K*-quasiregular and* $x \in G$. *Then for every bounded open cone* $\Delta = \text{Cone}(x, e; \alpha) \cap B^n(x, s)$ *with* $e \in S^{n-1}$, $\alpha \in (0, \pi)$ *and* $s > 0$,

$$N(f, \Delta) \geq \frac{b_n(\alpha)}{\omega_{n-1} K_I(f) K_O(f)} i(x, f) ,$$

where $b_n(\alpha)$ *is as in* (2.2).

Proof. Let $r_x > 0$ be as in Lemma 2.3 and choose $t > 0$ so that $0 < t < \min\{ s, l^*(x, f, r_x)\}$. Set $A = \Delta \cap B^n(x, t)$. Then Lemma 3.1 yields for every $m > 1$

$$i(x, f) \leq \omega_{n-1} K_I(f) K_O(f) \frac{N(f, A)}{M(\Gamma) (\log m)^{n-1}} , \tag{3.5}$$

where $\Gamma = \Delta(S^{n-1}(x, t/(mC^{*2})), S^{n-1}(x, t); A)$. By [4, 7.7]

$$M(\Gamma) = b_n(\alpha) [\log (m C^{*2})]^{1-n} .$$

Substitute this in (3.5) and let $m \to \infty$. The theorem follows because $N(f, A) \leq N(f, \Delta)$.

3.6. *Remark.* In the plane Theorem 3.4 gives the best possible lower bound for $N(f, \Delta)$. If k and m are positive integers, we define, using the complex notation, $f: C \to C$ by $f(z) = z^{mk}$ for all $z \in C$, and set

$$\Delta = \{ r e^{i\varphi} \mid 0 < r < 1, 0 < \varphi < 2\pi/k \} .$$

Then $K(f) = 1$, $i(0, f) = m k$ and $N(f, \Delta) = m$. On the other hand, Theorem 3.4 yields with $\alpha = \pi/k$,

$$N(f, \Delta) \geq \frac{b_2(\pi/k)}{\omega_1 K(f)^2} i(0, f) = \frac{2\pi/k}{2\pi \cdot 1} m k = m .$$

3.7. *Open question.* Theorem 3.4 implies that there exists $z_0 \in \Delta$ such that f assumes the value $f(z_0)$ at least $b_n(\alpha)(\omega_{n-1} K^2)^{-1} i(x, f)$ times in Δ. Is this typical of all $z \in \Delta$ (which are sufficiently close to x)? The question could be formulated explicitly as follows: If f and Δ are as in Theorem 3.4 with $m \geq 1$ such that there are points $y \in \Delta$ arbitrarily close to x with $N(y, f, \Delta) \leq m$, is $i(x, f)$ then bounded by some constant depending only on n, K, α and m?

4. On the local structure of B_f

4.1. In this section we want to show that the cone theorem [3, 4.4] can be derived from Theorem 3.4, too. Thus Theorem 3.4 can be considered as a generalization of [3, 4.4]. First we need the following lemma.

4.2. Lemma. *Suppose that* $f: G \to R^n$ *is* K*-quasiregular and* $x \in G$. *Let* $r_x > 0$ *and* $C^* = C^*(n, K)$ *be as in Lemma* 2.3. *If* $A \subset U(x, f, r_x)$, *then*

$$N(f, A) = \max_{t>0} N(f, A \cap [\mathrm{cl}\, B^n(x, t) \setminus B^n(x, t/C^*)]) .$$

Proof. Choose any $y \in A \subset U(x, f, r_x)$. By Lemma 2.3 for $s = |f(x) - f(y)|$, $\partial U(x, f, s) = U(x, f, r_x) \cap f^{-1} S^{n-1}(f(x), s)$ and $L^*(x, f, s) \leq C^* l^*(x, f, s)$. Then for $t = L^*(x, f, s)$ we have

$$A \cap f^{-1}(f(y)) \subset A \cap \partial U(x, f, s) \subset A \cap [\mathrm{cl}\, B^n(x, t) \setminus B^n(x, t/C^*)] .$$

The lemma follows.

4.3. Theorem [3, 4.4]. *Suppose that* $f: G \to R^n$ *is* K*-quasiregular,* $n \geq 3$ *and* $x \in G$. *If*

$$\text{Cone}\,(x, e; \alpha) \cap B^n(x, t) \subset G \setminus B_f \tag{4.4}$$

with $e \in S^{n-1}$, $\alpha \in (0, \pi/2)$ *and* $t > 0$, *then* $i(x, f) \leq C$, *where* C *depends only on* n, K *and* α.

Proof. Let $C^* = C^*(n, K)$ and $r_x > 0$ be as in Lemma 2.3. Set $s_0 = \min\{l^*(x, f, r_x), t/2\}$. Let $\psi = \psi(n, K)$ be the constant in [3, 2.3]. Then by [3, 2.7] f is injective in every ball $B^n(x + s\,e, \gamma\, s)$ with

$$\gamma = \psi \sin \alpha, \quad 0 < s \leq s_0 .$$

Define balls $B_i = B^n(x_i, r_i)$, $i = 1, 2, \ldots$, by setting

$$x_1 = x + s_0\, e, \quad r_1 = \gamma\, |x_1 - x| \quad \text{and}$$

$$x_i = x_{i-1} - r_{i-1}\, e, \quad r_i = \gamma\, |x_i - x|, \quad i > 1 .$$

Then f is injective in every B_i and it is easy to see that $\cup_i B_i$ covers the set

$$\Delta_\beta = \text{Cone}\,(x, e; \beta) \cap B^n(x, s_0),$$

where $\beta \in (0, \pi/2)$ with $\tan \beta = \gamma/3^{1/2}$. So β depends only on n, K and α. We want to show that $N(f, \Delta_\beta)$ has an upper bound depending only on n, K and α, which will prove the theorem by Theorem 3.4.

To estimate $N(f, \Delta_\beta)$ we apply Lemma 4.2. Consider any $s \in (0, s_0]$. It is easy to see that to cover the set

$$\Delta_\beta^s = \Delta_\beta \cap [\mathrm{cl}\, B^n(x, s) \setminus B^n(x, s/C^*)]$$

with balls B_i we need at most m_s balls, where m_s is the smallest integer greater than $(s - s/C^*)/(\gamma\, s/C^*) + 1 = 1 + (C^* - 1)/\gamma$. Hence $N(f, \Delta_\beta^s) \le 2 + (C^* - 1)/\gamma$. Since $s \in (0, s_0]$ is arbitrary, Lemma 4.2 implies $N(f, \Delta_\beta) \leqq 2 + (C^* - 1)/\gamma$. This upper bound of $N(f, \Delta_\beta)$ depends only on n, K and α. The theorem is proved.

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