

HAUSDORFF DIMENSION, ORTHOGONAL PROJECTIONS AND INTERSECTIONS WITH PLANES

PERTTI MATTILA

1. Introduction

Let E be a plane set which is measurable with respect to the s -dimensional Hausdorff measure \mathcal{H}^s with $0 < \mathcal{H}^s(E) < \infty$. In [10] Marstrand proved among other things the following three theorems:

(1) If $1 < s \leq 2$, then the Lebesgue measure $\mathcal{L}^1[p(E)]$ is positive for almost all orthogonal projections $p : R^2 \rightarrow R^1$.

(2) If $0 \leq s \leq 1$, then the Hausdorff dimension of $p(E)$ is equal to s for almost all orthogonal projections $p : R^2 \rightarrow R^1$.

(3) If $1 < s \leq 2$, then at \mathcal{H}^s almost all points $x \in E$ the following is true: For almost all straight lines l through x , $\mathcal{H}^{s-1}(E \cap l) < \infty$ and the Hausdorff dimension of $E \cap l$ is equal $s-1$.

Results similar to (1) and (2) were also obtained by Kaufman [7] with potential-theoretic methods. Kaufman was also able to show that the exceptional set of projections in (2) has \mathcal{H}^s measure zero.

In this paper we generalize the results (1)–(3) by replacing the dimensions 2 and 1 by arbitrary dimensions n and m , $0 < m < n$. We use the method of Marstrand in (1) and (3) and that of Kaufman in (2). Certainly Marstrand's ideas could also be used to generalize (2), but Kaufman's method is shorter and gives a result on capacities and on the Hausdorff measures of the exceptional set. On the other hand, Marstrand's method allows us to improve (1) in two respects. Firstly, the conclusion $\mathcal{L}^m[p(E)] > 0$ can be replaced by the stronger conclusion $\int N(p|E, y) d\mathcal{L}^m y = \infty$, where $N(p|E, y)$ is the number of points in the set $E \cap p^{-1}\{y\}$. Secondly, the hypothesis that the Hausdorff dimension of E is greater than m can be replaced by the weaker hypothesis $C_m(E) > 0$, where C_m is the capacity corresponding to the Riesz kernel $|x-y|^{-m}$. As a corollary it follows that if the integral-geometric measure $\mathcal{I}_1^m(E)$ is finite, then $C_m(E) = 0$.

When we consider a set E with $\mathcal{H}^s(E) > 0$, two assumptions will often be equally convenient; namely, that E is a Suslin set or that E is \mathcal{H}^s measurable with $\mathcal{H}^s(E) < \infty$, because in both cases E contains a compact set K with $0 < \mathcal{H}^s(K) < \infty$. The case where E is \mathcal{H}^s measurable and $\mathcal{H}^s(E) = \infty$ is entirely different. In fact, assuming the continuum hypothesis Besicovitch [1] has constructed an \mathcal{H}^1 measurable plane set G such that $\mathcal{L}^2(G) = 1$ and that the only subsets of G of zero \mathcal{L}^2 measure are countable sets.

Several instructive examples and remarks related to the present subject can be found in [10].

Most of the terminology and notation will be as in [5].

2. The measures $\vartheta_{n,m}^*$ and $\gamma_{n,m}$

Let $O(n)$ be the orthogonal group of the Euclidean n -space R^n and let ϑ_n be the unique Haar measure on $O(n)$ such that $\vartheta_n[O(n)] = 1$. If m is an integer such that $0 \leq m \leq n$, then $O^*(n, m)$ is the space of all orthogonal projections from R^n onto R^m , and $G(n, m)$ is the space of all m -dimensional linear subspaces of R^n . We shall use the $O(n)$ invariant measures $\vartheta_{n,m}^*$ on $O^*(n, m)$ and $\gamma_{n,m}$ on $G(n, m)$ (see [5, 2.7.16(6)]). If $p \in O^*(n, m)$ and $V \in G(n, m)$, $A \subset O^*(n, m)$ and $B \subset G(n, m)$, then by definition

$$(2.1) \quad \vartheta_{n,m}^*(A) = \vartheta_n\{g \in O(n) : p \circ g \in A\},$$

$$(2.2) \quad \gamma_{n,m}(B) = \vartheta_n\{g \in O(n) : g(V) \in B\}.$$

The measures $\vartheta_{n,m}^*$ and $\gamma_{n,n-m}$ are related by

$$(2.3) \quad \gamma_{n,n-m}(A) = \vartheta_{n,m}^*\{p \in O^*(n, m) : \ker p \in A\}$$

for $A \subset G(n, n-m)$.

If $0 \leq s \leq n$, we let \mathcal{H}^s be the s -dimensional Hausdorff measure over R^n as defined in [5, 2.10.2(1)]. Then $\mathcal{H}^n = \mathcal{L}^n$, the Lebesgue measure in R^n . We denote by S^{n-1} the unit sphere $\{x \in R^n : |x| = 1\}$ and, when $a \in R^n$ and $r > 0$, by $B^n(a, r)$ or $B(a, r)$ the closed ball $\{x \in R^n : |x-a| \leq r\}$.

Throughout the rest of the paper m and n will be integers such that $0 < m < n$.

2.4. Lemma. *There is a constant c_1 depending only on n and m such that*

$$\mathcal{H}^{n-1}\{x \in S^{n-1} : |p(x)| \leq \delta\} \leq c_1 \delta^m$$

for all $p \in O^*(n, m)$ and $\delta > 0$.

Proof. It is sufficient to consider $0 < \delta < 1/2$, and we may assume that p is given by

$$p(x_1, \dots, x_n) = (x_1, \dots, x_m).$$

Define $f: B^m(0, 1/2) \times S^{n-m-1} \rightarrow S^{n-1} \subset R^m \times R^{n-m}$ by

$$f(y, z) = (y, (1 - |y|^2)^{1/2} z).$$

Then f is Lipschitzian and maps $B^m(0, \delta) \times S^{n-m-1}$ onto $\{x \in S^{n-1} : |p(x)| \leq \delta\}$ for $0 < \delta < 1/2$. Denoting the Lipschitz constant of f by K and using [5, 3.2.23], we find that

$$\begin{aligned} & \mathcal{H}^{n-1}\{x \in S^{n-1} : |p(x)| \leq \delta\} \leq K^{n-1} \mathcal{H}^{n-1}[B^m(0, \delta) \times S^{n-m-1}] \\ & = K^{n-1} \mathcal{L}^m[B^m(0, \delta)] \mathcal{H}^{n-m-1}(S^{n-m-1}) = K^{n-1} \alpha(m) \mathcal{H}^{n-m-1}(S^{n-m-1}) \delta^m. \end{aligned}$$

2.5. Lemma. *There is a constant c_2 depending only on n and m such that*

$$\vartheta_{n,m}^*\{p \in O^*(n, m) : |p(x)| \leq \delta\} \leq c_2 \delta^m |x|^{-m}$$

for all $x \in R^n$ and $\delta > 0$.

Proof. Since $p(x) = |x|p(x/|x|)$ for $x \neq 0$, we may assume that $x \in S^{n-1}$. Fix $q \in O^*(n, m)$ and define $\Phi: O(n) \rightarrow S^{n-1}$ by $\Phi(g) = g(x)$ for $g \in O(n)$. Applying statement (3) in [5, 3.2.47] with $r = n-1$, $r + \mu = n(n-1)/2$ and observing that $\mathcal{H}^{r+\mu}$ is then a constant multiple of ϑ_n because of [5, 3.2.28(5)], we find a constant c depending only on n such that

$$\vartheta_n[\Phi^{-1}(E)] = c \mathcal{H}^{n-1}(E)$$

for $E \subset S^{n-1}$. Hence by (2.1) and Lemma 2.4,

$$\begin{aligned} & \vartheta_{n,m}^*\{p \in O^*(n, m) : |p(x)| \leq \delta\} \\ & = \vartheta_n\{g \in O(n) : |(q \circ g)(x)| \leq \delta\} \\ & = \vartheta_n(\Phi^{-1}\{y \in S^{n-1} : |q(y)| \leq \delta\}) \\ & = c \mathcal{H}^{n-1}\{y \in S^{n-1} : |q(y)| \leq \delta\} \leq c c_1 \delta^m. \end{aligned}$$

The corresponding inequality for the measure $\gamma_{n,m}$ follows from Lemma 2.5 and (2.3):

2.6. Lemma. *There is a constant c_3 depending only on n and m such that*

$$\gamma_{n,m}\{V \in G(n, m) : \text{dist}(x, V) \leq \delta\} \leq c_3 \delta^{n-m} |x|^{m-n}$$

for all $x \in R^n$ and $\delta > 0$.

3. Integral inequalities

In this section we assume that

(3.1) M is a separable metric space,

(3.2) $F: R^n \times M \rightarrow R^1$ is a non-negative Borel function such that $F(x, a) \leq |x|$ for all $(x, a) \in R^n \times M$,

(3.3) α and b are positive numbers and φ is a Borel regular measure over M such that

$$\varphi\{a \in M : F(x, a) \leq \delta\} \leq b \delta^\alpha |x|^{-\alpha}$$

for all $x \in R^n$ and $\delta > 0$.

The separability of M makes the subsequent applications of Fubini's theorem legitimate. The results of this section will be applied with $M = O^*(n, m)$, $F: (x, p) \mapsto |p(x)|$ and $M = G(n, m)$, $F: (x, V) \mapsto \text{dist}(x, V)$. Lemmas 2.5 and 2.6 guarantee that $\vartheta_{n,m}^*$ and $\gamma_{n,m}$ satisfy the condition (3.3) with $\alpha = m$ and $\alpha = n - m$, respectively. In Sections 5 and 6 we shall replace φ by other measures over $O^*(n, m)$ and $G(n, m)$.

3.4. If μ measures a set X (i.e. μ is an (outer) measure over X), $\mu(X) < \infty$, a and t are real numbers, $a \geq 0$ and $f: X \rightarrow \{x: 0 \leq x \leq a\}$ is μ measurable, then

$$\int_X f^t d\mu = \int_0^a r^t d\mu\{x \in X : f(x) \leq r\},$$

where the right hand side is a Riemann-Stieltjes integral. In the cases where $t \geq 0$ or $\inf f(X) > 0$, this follows immediately from the fact that $\int_X f^t d\mu$ lies between the sums $\sum_{i=1}^k r_i^t \mu\{x : r_{i-1} < f(x) \leq r_i\}$ and $\sum_{i=1}^k r_{i-1}^t \mu\{x : r_{i-1} < f(x) \leq r_i\}$ whenever $0 = r_0 < r_1 < \dots < r_k = a$. In the general case one can approximate by functions $f_n = \sup(f, 1/n)$, $n = 1, 2, \dots$.

3.5. Lemma. If $0 < t < \alpha$ and $x \in R^n$, then

$$\int F(x, a)^{-t} d\varphi a \leq c |x|^{-t},$$

where $c = b [1 + t(\alpha - t)^{-1}]$.

Proof. Apply 3.4, integrate by parts, and use (3.2) and (3.3) to deduce

$$\int F(x, a)^{-t} d\varphi a = \int_0^{|x|} r^{-t} d\varphi\{a : F(x, a) \leq r\}$$

$$\begin{aligned} &\leq |x|^{-t} \varphi\{a : F(x, a) \leq |x|\} + t \int_0^{|x|} r^{-t-1} \varphi\{a : F(x, a) \leq r\} d\mathcal{L}^1 r \\ &\leq b |x|^{-t} + b t |x|^{-\alpha} \int_0^{|x|} r^{\alpha-t-1} d\mathcal{L}^1 r = b [1 + t(\alpha - t)^{-1}] |x|^{-t}. \end{aligned}$$

3.6. Lemma. If μ is a Borel regular measure over R^n , $\mu(R^n) < \infty$, $x \in R^n$ and $\delta > 0$, then

$$\int \mu\{y \in R^n : F(x-y, a) \leq \delta\} d\varphi a \leq b \delta^\alpha \int |x-y|^{-\alpha} d\mu y.$$

Proof. Observing that $\{(y, a) \in R^n \times M : F(x-y, a) \leq \delta\}$ is a Borel set and applying Fubini's theorem and (3.3), we obtain

$$\begin{aligned} &\int \mu\{y \in R^n : F(x-y, a) \leq \delta\} d\varphi a \\ &= \int \varphi\{a \in M : F(x-y, a) \leq \delta\} d\mu y \leq b \delta^\alpha \int |x-y|^{-\alpha} d\mu y. \end{aligned}$$

3.7. If μ measures R^n , $x \in R^n$, $a \in M$ and $r > 0$, we define

$$\begin{aligned} \psi_r^\alpha(\mu, x, a) &= \liminf_{\delta \downarrow 0} \delta^{-\alpha} \mu\{y \in B(x, r) : F(x-y, a) \leq \delta\}, \\ \psi^\alpha(\mu, x, a) &= \lim_{r \downarrow 0} \psi_r^\alpha(\mu, x, a). \end{aligned}$$

The set $\{(y, x, a) : y \in B(x, r), F(x-y, a) \leq \delta\}$ is a Borel set in $R^n \times R^n \times M$. Therefore $\mu\{y \in B(x, r) : F(x-y, a) \leq \delta\}$ is a Borel function with respect to (x, a) according to Fubini's theorem, and it follows by standard methods (cf. [4, 3.4]) that both ψ_r^α and ψ^α are Borel functions with respect to (x, a) .

3.8. Lemma. If μ is a Borel regular measure over R^n , $\mu(R^n) < \infty$, $x \in R^n$ and $r > 0$, then

$$\int \psi_r^\alpha(\mu, x, a) d\varphi a \leq b \int_{B(x, r)} |x-y|^{-\alpha} d\mu y.$$

This follows directly from Fatou's lemma and Lemma 3.6.

4. Projections into lower dimensional space

Here we adopt the notation of the previous section with $M = O^*(n, m)$, $F : (x, p) \mapsto |p(x)|$, $\alpha = m$ and $\varphi = \vartheta_{n, m}^*$. Then

$$\psi_r^m(\mu, x, p) = \liminf_{\delta \downarrow 0} \delta^{-m} \mu\{y \in B(x, r) : |p(x-y)| \leq \delta\}.$$

For any function f , $N(f, y)$ is the number of points (possibly 0 or ∞) in the set $f^{-1}\{y\}$. If A is a Borel set of R^n , then the function $y \mapsto N(p | A, y)$ is \mathcal{L}^m measurable for all $p \in O^*(n, m)$ (see [5, 2.10.10]) and the function $(y, p) \mapsto N(p | A, y)$ is $\mathcal{L}^n \times \vartheta_{n,m}^*$ measurable (see [5, 2.10.16]). Actually, it can be seen as in 6.1 that these functions are measurable whenever A is a Suslin set, but we shall not make use of this fact here.

If μ measures a set X and $A \subset X$, then $\mu \llcorner A$ is the restriction measure defined by $(\mu \llcorner A)(B) = \mu(A \cap B)$ for $B \subset X$.

4.1. Lemma. *If μ is a Borel regular measure over R^n , E is a Borel set of R^n with $\mu(E) < \infty$, $p \in O^*(n, m)$ and $h > 0$, then*

$$\mu\{x \in E : \psi^m(\mu \llcorner E, x, p) \leq h\} \leq 2^m m^{m/2} h \int N(p | E, y) d\mathcal{L}^m y.$$

Proof. It is sufficient to prove that

$$\mu\{x \in E : \psi^m(\mu \llcorner E, x, p) < h\} \leq 2^m m^{m/2} h \int N(p | E, y) d\mathcal{L}^m y.$$

To simplify the notation, we assume that p is given by

$$p(x_1, \dots, x_n) = (x_1, \dots, x_m).$$

Let r and d be positive numbers such that $d < r/n$. We identify $R^n = R^m \times R^{n-m}$ and express R^{n-m} as

$$R^{n-m} = \bigcup_{i=1}^{\infty} Q_i,$$

where Q_i 's are closed cubes parallel to the coordinate axis with mutually disjoint interiors and with sidelength d . We may assume that $\int N(p | E, y) d\mathcal{L}^m y < \infty$. Then we can choose the cubes Q_i such that $\mathcal{L}^m(p[E \cap (R^m \times \partial Q_i)]) = 0$, because $\mathcal{L}^m[p(E \cap V)]$ can be positive only for countably many disjoint $(n-1)$ -dimensional affine subspaces V of R^n .

If $x \in R^n$, $\delta > 0$ and $\varrho > 0$, we denote

$$I(x, \delta, \varrho) = \{y \in R^n : |x_i - y_i| \leq \delta, \\ i = 1, \dots, m, |x_i - y_i| \leq \varrho, i = m+1, \dots, n\}.$$

Let

$$f(x) = \liminf_{\delta \downarrow 0} \delta^{-m} \mu[E \cap I(x, \delta, d)] \text{ for } x \in R^n,$$

and

$$E_i = E \cap \{x \in R^m \times Q_i : f(x) < h\} \text{ for } i = 1, 2, \dots$$

By standard arguments [4, 3.4] one can see that f is a Borel function. Let $\varepsilon > 0$ and fix i . By [5, 2.2.2] there is a closed set $F_i \subset E_i$ such that

$$\mu(E_i) \leq (1 + \varepsilon) \mu(F_i).$$

Next we select closed cubes $P_j \subset R^m$ parallel to the coordinate axis such that

$$p(F_i) \subset \bigcup_{j=1}^{\infty} P_j, \quad \sum_{j=1}^{\infty} \mathcal{L}^m(P_j) \leq \mathcal{L}^m[p(F_i)] + \varepsilon.$$

Then $F_i \subset \bigcup_{j=1}^{\infty} P_j \times Q_i$ and

$$\mu(F_i) \leq \sum_{j=1}^{\infty} \mu[F_i \cap (P_j \times Q_i)].$$

We shall show that

$$(1) \quad \mu[F_i \cap (P_j \times Q_i)] \leq 2^m h \mathcal{L}^m(P_j) \text{ for } j = 1, 2, \dots$$

Suppose that (1) is false for some j . Then dividing P_j repeatedly into 2^m congruent subcubes, we get a decreasing sequence (R_k) of subcubes of P_j such that

$$(2) \quad \text{diam}(R_k) = 2^{-k} \text{diam}(P_j),$$

$$(3) \quad \mu[F_i \cap (R_k \times Q_i)] > 2^m h \mathcal{L}^m(R_k).$$

Since the compact non-empty sets $F_i \cap (R_k \times Q_i)$ form a decreasing sequence, there is a point $x \in \bigcap_{k=1}^{\infty} F_i \cap (R_k \times Q_i)$. Let δ_k be the sidelength of R_k . Then

$$F_i \cap (R_k \times Q_i) \subset E \cap I(x, \delta_k, d),$$

and it follows from (3) that

$$(4) \quad \liminf_{k \rightarrow \infty} \delta_k^{-m} \mu[E \cap I(x, \delta_k, d)] \geq 2^m h.$$

For any $0 < \delta < \delta_1$ there is k such that $\delta_k \leq \delta < 2\delta_k$ because of (2). Then (4) yields $f(x) \geq h$. But this contradicts with the fact that $x \in E_i$. Hence (1) holds, and we get

$$\begin{aligned} \mu(E_i) &\leq (1 + \varepsilon) \mu(F_i) \leq (1 + \varepsilon) \sum_{j=1}^{\infty} \mu[F_i \cap (P_j \times Q_i)] \\ &\leq (1 + \varepsilon) 2^m h \sum_{j=1}^{\infty} \mathcal{L}^m(P_j) \leq (1 + \varepsilon) 2^m h (\mathcal{L}^m[p(F_i)] + \varepsilon) \\ &\leq (1 + \varepsilon) 2^m h (\mathcal{L}^m[p(E_i)] + \varepsilon). \end{aligned}$$

Letting $\varepsilon \downarrow 0$, we have

$$\mu(E_i) \leq 2^m h \mathcal{L}^m[p(E_i)].$$

Summing with respect to i and recalling that $\mathcal{L}^m[p(E_i \cap E_{i'})] = 0$ for $i \neq i'$, we obtain

$$\begin{aligned} \mu\{x \in E : f(x) < h\} &\leq \sum_{i=1}^{\infty} \mu(E_i) \\ &\leq 2^m h \sum_{i=1}^{\infty} \mathcal{L}^m[p(E_i)] \leq 2^m h \int N(p | E, y) d\mathcal{L}^m y. \end{aligned}$$

Since $d < r/n$, we have

$$I(x, \delta, d) \subset \{y \in B(x, r) : |p(x-y)| \leq m^{1/2} \delta\}$$

for sufficiently small $\delta > 0$. Hence

$$\mu\{x \in E : \psi_r^m(\mu \llcorner E, x, p) < h\} \leq 2^m m^{m/2} h \int N(p | E, y) d\mathcal{L}^m y.$$

Letting $r \downarrow 0$ completes the proof.

For the definition of the integralgeometric measure \mathcal{I}_1^m see [5, 2.10.5].

4.2. Theorem. *Let μ be a Borel regular measure over R^n and E a μ measurable subset of R^n with $0 < \mu(E) < \infty$. If*

$$\int_E |x-y|^{-m} d\mu y < \infty$$

for μ almost all $x \in E$, then

$$\int^* N(p | E, y) d\mathcal{L}^m y = \infty$$

for $\vartheta_{n,m}^*$ almost all $p \in O^*(n, m)$, and $\mathcal{I}_1^m(E) = \infty$.

Proof. Since E contains a closed set of positive μ measure [5, 2.2.2–3], it is sufficient to prove the theorem in the case where E is closed. Let P be the set of those $p \in O^*(n, m)$ for which $\int N(p | E, y) d\mathcal{L}^m y < \infty$. Then it follows from Lemma 4.1 that $p \in P$ implies $\psi^m(\mu \llcorner E, x, p) > 0$ for μ almost all $x \in E$, whence $\int_E \psi^m(\mu \llcorner E, x, p) d\mu x > 0$. On the other hand, under the present assumptions Lemma 3.8 combined with Fatou's lemma shows that $\int \psi^m(\mu \llcorner E, x, p) d\vartheta_{n,m}^* p = 0$ for μ almost all $x \in E$. Thus by Fubini's theorem,

$$\int_E \int \psi^m(\mu \llcorner E, x, p) d\mu x d\vartheta_{n,m}^* p = \int_E \int \psi^m(\mu \llcorner E, x, p) d\vartheta_{n,m}^* p d\mu x = 0.$$

Hence $\vartheta_{n,m}^*(P) = 0$. This proves the first assertion. The second assertion, $\mathcal{I}_1^m(E) = \infty$, follows from the first and [5, 2.10.15].

4.3. *Capacity.* The inner capacity C_s , $0 < s < n$, is defined as follows: If K is a compact subset of R^n , then

$$C_s(K) = \sup \mu(R^n),$$

where the supremum is taken over all (non-negative) Radon measures μ such that the support of μ , $\text{spt } \mu$, is contained in K and $\int |x-y|^{-s} d\mu y \leq 1$ for all $x \in \text{spt } \mu$. For arbitrary $E \subset R^n$, we set

$$C_s(E) = \sup \{ C_s(K) : K \subset E, K \text{ is compact} \}.$$

The corresponding outer capacity C_s^* is defined for $E \subset R^n$ by

$$C_s^*(E) = \inf \{ C_s(G) : E \subset G, G \text{ is open} \}.$$

Then $C_s^*(E) = C_s(E)$ for all Suslin sets E of R^n (see [9, Theorem 2.8, p. 156]).

If μ is a Radon measure and $\int |x-y|^{-s} d\mu y \leq 1$ for all $x \in \text{spt } \mu$, it follows easily (see [9, Theorem 1.5, p. 66]) that $\int |x-y|^{-s} d\mu y \leq 2^s$ for all $x \in R^n$. Hence $C_s(E) > 0$ if and only if there exists a Radon measure μ such that $\text{spt } \mu \subset E$, $0 < \mu(E) < \infty$ and $\int |x-y|^{-s} d\mu y \leq 1$ for all $x \in R^n$. Thus we obtain from Theorem 4.2:

4.4. *Theorem.* If $E \subset R^n$ is measurable with respect to every Radon measure and if $C_m(E) > 0$, then $\int^* N(p | E, y) d\mathcal{L}^m y = \infty$ for $\vartheta_{n,m}^*$ almost all $p \in O^*(n, m)$.

If $E \subset R^n$, we denote by $\dim E$ the Hausdorff dimension of E i.e. $\dim E = \inf \{ s : \mathcal{H}^s(E) = 0 \}$. It is well-known that

$$\dim E = \inf \{ s : C_s^*(E) = 0 \} = \sup \{ s : C_s^*(E) > 0 \}$$

(see e.g. [6, Section 2] or [9, Theorem 3.13, p. 196 and Theorem 3.14, p. 200]). Consequently,

$$(4.5) \quad \dim E = \inf \{ s : C_s(E) = 0 \} = \sup \{ s : C_s(E) > 0 \},$$

if E is a Suslin set. Therefore

4.6. *Theorem.* If E is a Suslin set of R^n and $\dim E > m$, then $\int^* N(p | E, y) d\mathcal{L}^m y = \infty$ for $\vartheta_{n,m}^*$ almost all $p \in O^*(n, m)$.

This theorem could also be deduced directly from Theorem 4.2, because for $s > m$ the use of Riemann-Stieltjes integration shows that $\int_E |x-y|^{-m} d\mathcal{H}^s y < \infty$ whenever the upper density $\Theta^{*s}(\mathcal{H}^s \llcorner E, x)$ is finite.

In case E is a Borel set, the conclusion in 4.4 and 4.6 implies that $\mathcal{I}_1^m(E) = \infty$. But since \mathcal{I}_1^m is Borel regular [5, 2.10.1], we have more generally

4.7. *Theorem.* If $E \subset R^n$ and $\mathcal{I}_1^m(E) < \infty$, then $C_m(E) = 0$ and $\dim E \leq m$.

5. Projections into higher dimensional space

In the first three lemmas of this section, we assume that $0 < \alpha \leq m$, $b > 0$ and φ is a Borel regular measure over $O^*(n, m)$ such that

$$\varphi\{p \in O^*(n, m) : |p(x)| \leq \delta\} \leq b \delta^\alpha |x|^{-\alpha}$$

for all $x \in R^n$ and $\delta > 0$.

If $0 < s < m$ and μ is a Radon measure over R^n , we denote the s -energy of μ by $I_s(\mu)$ i.e.

$$I_s(\mu) = \int \int |x-y|^{-s} d\mu y d\mu x.$$

Then for compact sets $K \subset R^n$,

$$C_s(K)^{-1} = \inf I_s(\mu),$$

where the infimum is taken over all Radon measures μ such that $\text{spt } \mu \subset K$ and $\mu(K) = 1$ (see [9, p. 139]).

5.1. Lemma. *If $0 < s < \alpha$ and K is a compact set of R^n , then*

$$\int C_s[p(K)]^{-1} d\varphi p \leq c C_s(K)^{-1},$$

where $c = b [1 + s(\alpha-s)^{-1}]$.

Proof. The integrand is measurable; in fact, the function $p \mapsto C_s[p(K)]$ is upper semicontinuous in $O^*(n, m)$. To see this, let $p \in O^*(n, m)$ and $\varepsilon > 0$. Then there is an open set $G \supset p(K)$ such that $C_s(G) \leq C_s[p(K)] + \varepsilon$. When q is sufficiently close to p , then $q(K) \subset G$, whence $C_s[q(K)] \leq C_s(G) \leq C_s[p(K)] + \varepsilon$.

Let μ be a Radon measure such that $\text{spt } \mu \subset K$ and $\mu(K) = 1$. If $p \in O^*(n, m)$, then the measure $p_{\#}\mu$, defined by $p_{\#}\mu(A) = \mu[p^{-1}(A)]$ for $A \subset R^m$, is a Radon measure over R^m [5, 2.2.17], $\text{spt } p_{\#}\mu \subset p(K)$ and $p_{\#}\mu[p(K)] = 1$. By [5, 2.4.18(1)], we have

$$I_s(p_{\#}\mu) = \int \int |p(x-y)|^{-s} d\mu y d\mu x.$$

Fubini's theorem and Lemma 3.5 yield

$$\int I_s(p_{\#}\mu) d\varphi p = \int \int \int |p(x-y)|^{-s} d\varphi p d\mu y d\mu x \leq c I_s(\mu),$$

from which the lemma follows.

5.2. Lemma. *If $0 < s < \alpha$, $E \subset R^n$ and $C_s(E) > 0$, then there is a Borel set $P \subset O^*(n, m)$, independent of φ , such that $\varphi(P) = 0$ and $C_s[p(E)] > 0$ for all $p \in O^*(n, m) \setminus P$.*

Proof. Choose a compact set $K \subset E$ such that $C_s(K) > 0$. Then the set

$$P = \{ p \in O^*(n, m) : C_s[p(K)] = 0 \}$$

has the required properties by Lemma 5.1 and its proof.

5.3. **L e m m a.** *If E is a Suslin set of R^n and $\dim E \leq \alpha$, then there is a Borel set $P \subset O^*(n, m)$, independent of φ , such that $\varphi(P) = 0$ and $\dim p(E) = \dim E$ for all $p \in O^*(n, m) \setminus P$.*

This follows from (4.5) and 5.2 if we choose $P = \cup_{j=1}^\infty P_j$, where P_j is the Borel set of Lemma 5.2 corresponding to $s = \dim E - 1/j$.

By Lemma 2.5, the above results hold with $\alpha = m$ and $\varphi = \vartheta_{n,m}^*$. We now proceed to obtain more exact information about the Hausdorff measures of the exceptional sets of Lemmas 5.2 and 5.3. Recall from [5, 3.2.28] that $O^*(n, m)$ is an $m(2n - m - 1)/2$ dimensional submanifold of class ∞ of the euclidean space $\text{Hom}(R^n, R^m)$ with

$$0 < \mathcal{H}^{m(2n-m-1)/2}[O^*(n, m)] < \infty.$$

We use the norm $\|f\| = \sup \{ |f(x)| : |x| = 1 \}$ on $\text{Hom}(R^n, R^m)$. Then $B(f, r) = \{ g \in \text{Hom}(R^n, R^m) : \|f - g\| \leq r \}$ for $f \in \text{Hom}(R^n, R^m)$ and $r > 0$.

We shall apply the following theorem, which is due to O. Frostman. A proof is given in [2, pp. 7-8].

5.4. **T h e o r e m.** *Let E be a Suslin set of R^n and $s \geq 0$. Then $\mathcal{H}^s(E) > 0$ if and only if there exists a Borel regular measure φ such that $0 < \varphi(E) < \infty$ and $\varphi[B(x, r)] \leq r^s$ for all $x \in R^n$ and $r > 0$.*

To be able to combine the above theorem with the lemmas of this section, we prove the following

5.5. **L e m m a.** *If φ is a Borel regular measure over $O^*(n, m)$ with $\varphi[O^*(n, m)] < \infty$, $t \geq \nu = m(2n - m - 3)/2$ and*

$$\varphi[O^*(n, m) \cap B(p, \delta)] \leq \delta^t$$

for all $p \in O^*(n, m)$ and $\delta > 0$, then

$$\varphi\{ p \in O^*(n, m) : |p(x)| \leq \delta \} \leq b \delta^{t-\nu} |x|^{\nu-t}$$

for all $x \in R^n$ and $\delta > 0$, where b is a constant depending only on n, m and t .

Proof. We may assume that $x \in S^{n-1}$ and $\delta < 1$. Put

$$A_\delta = \{ p \in O^*(n, m) : |p(x)| \leq \delta \},$$

$$B = \{ p \in O^*(n, m) : p(x) = 0 \}.$$

Let $p \in A_\delta$. Then there is $a \in S^{n-1} \cap \ker p$ such that $|x - a| \leq 2\delta$. Choose $g \in O(n)$ so that $g(x) = a$ and $\|g - \mathbf{1}_{R^n}\| = |x - a|$. One can

find such a g by writing $R^n = V \oplus V^\perp$, where $\{x, a\} \subset V \in G(n, 2)$ and V^\perp is the orthogonal complement of V , letting f be a rotation in V for which $f(x) = a$ and taking $g = f \oplus \mathbf{1}_{V^\perp}$. Then $\|p - p \circ g\| \leq 2\delta$ and $p \circ g \in B$. This shows that

$$(1) \quad A_\delta \subset \{p \in O^*(n, m) : \text{dist}(p, B) \leq 2\delta\}.$$

For $p \in O^*(n, m)$, we denote by p^* the adjoint of p [5, 1.7.4] and let $O(n, m) = \{p^* : p \in O^*(n, m)\}$ be the set of all orthogonal injections $R^m \rightarrow R^n$. If $p \in B$, the image $\text{im } p^*$ is contained in $\{y \in R^n : y \cdot x = 0\}$. Choosing a linear isometry $f : \{y \in R^n : y \cdot x = 0\} \rightarrow R^{n-1}$, we can define an isometry $F : B \rightarrow O(n-1, m)$ by $F(p) = f \circ p^*$. Thus it follows from [5, 3.2.28(5)] that $0 < \mathcal{H}^v(B) < \infty$ and that B is a v -dimensional manifold of class ∞ . Hence the upper density $\Theta^{*v}(\mathcal{H}^v \llcorner B, p)$ is positive for all $p \in B$. Moreover, $\mathcal{H}^v[B \cap B(p, r)]$ is independent of p for all $r > 0$. Consequently, there is a positive number c depending only on n and m such that

$$\mathcal{H}^v[B \cap B(p, r)] \geq c r^v$$

for all $p \in B$ and $0 < r < 1$. From (1) it follows that

$$\mathcal{H}^v[B \cap B(p, 3\delta)] \geq c \delta^v$$

for $p \in A_\delta$. We define a Borel function $f : A_\delta \times B \rightarrow R^1$ setting $f(p, q) = 1$, if $p \in B(q, 3\delta)$, and $f(p, q) = 0$ otherwise. Then by Fubini's theorem,

$$\begin{aligned} c \delta^v \varphi(A_\delta) &\leq \int_{A_\delta} \mathcal{H}^v[B \cap B(p, 3\delta)] d\varphi p = \int_{A_\delta} \int_B f(p, q) d\mathcal{H}^v q d\varphi p \\ &= \int_B \int_{A_\delta} f(p, q) d\varphi p d\mathcal{H}^v q = \int_B \varphi[A_\delta \cap B(q, 3\delta)] d\mathcal{H}^v q \leq 3^t \delta^t \mathcal{H}^v(B), \end{aligned}$$

and $\varphi(A_\delta) \leq b \delta^{t-v}$ with $b = 3^t c^{-1} \mathcal{H}^v(B)$. This completes the proof.

Combining 5.2–5.5, we can state the results of this section as follows:

5.6. **Theorem.** *Let $E \subset R^n$.*

(a) *If $0 < s < m$ and $C_s(E) > 0$, then $C_s[p(E)] > 0$ for $\vartheta_{n,m}^*$ almost all $p \in O^*(n, m)$. More precisely, the Hausdorff dimension of the exceptional set is at most $s + m(2n - m - 3)/2$.*

(b) *If E is a Suslin set and $s = \dim E \leq m$, then $\dim p(E) = \dim E$ for $\vartheta_{n,m}^*$, more precisely $\mathcal{H}^{s+m(2n-m-3)/2}$, almost all $p \in O^*(n, m)$.*

5.7. **Remarks.** (1) If $V \in G(n, m)$, let P_V be the orthogonal projection from R^n onto V . By similar methods as above, one can prove that

$$\dim \{V \in G(n, m) : C_s[P_V(E)] = 0\} \leq s + m(n - m - 1)$$

with the assumptions of 5.6(a), and

$$\mathcal{H}^{s+m(n-m-1)}\{V \in G(n, m) : \dim P_V(E) < \dim E\} = 0$$

with the assumptions of 5.6(b).

(2) In [8] Kaufman has given an example of a compact plane set D such that $\dim D \leq 1$ and that the set of those $p \in O^*(2, 1)$ for which $\dim p(D) < \dim D$ is of positive Hausdorff dimension. Later it has been shown (unpublished) first by Kaufman in the case $n = 2$ and then by Kaufman and the present author in co-operation in the general case that $s+m(2n-m-3)/2$ is the best possible upper bound for the Hausdorff dimension of the exceptional set of Theorem 5.6.

(3) It follows from [9, Theorem 2.9, p. 158] that $C_s^*[p(E)] \leq C_s^*(E)$ for all $E \subset R^n$ and $p \in O^*(n, m)$.

6. Intersections with planes

Here we use the methods of Marstrand [10, Section 6] to examine the intersections of an s -dimensional set with m -dimensional affine subspaces of R^n . If $x \in R^n$ and $A \subset R^n$, we denote by $\tau_x A$ the x -translate $\{y \in R^n : y-x \in A\}$ of A .

6.1. Lemma. *If E is a Suslin set of R^n and $t \geq 0$, then the function*

$$(x, V) \mapsto \mathcal{H}^t(E \cap \tau_x V)$$

is a Suslin function in $R^n \times G(n, m)$, i.e. the inverse image of $\{v : v > u\}$ is a Suslin set in $R^n \times G(n, m)$ for all $u \geq 0$.

Proof. We shall apply the following statement, which is a special case of a result of Dellacherie [3, Chapitre VI, Corollaire 21]:

If A is a Suslin set in $R^n \times R^n \times G(n, m)$ and $A(x, V) = \{y \in R^n : (y, x, V) \in A\}$ for $(x, V) \in R^n \times G(n, m)$, then the function $(x, V) \mapsto \mathcal{H}^t[A(x, V)]$ is a Suslin function in $R^n \times G(n, m)$.

We choose

$$\begin{aligned} A &= \{(y, x, V) : y \in E \cap \tau_x V\} \\ &= [E \times R^n \times G(n, m)] \cap \{(y, x, V) : x-y \in V\}. \end{aligned}$$

Then A is a Suslin set as an intersection of a Suslin set and a closed set, $A(x, V) = E \cap \tau_x V$, and the lemma follows.

We adopt the notation of Section 3 with $M = G(n, m)$ and $F(x, V) = \text{dist}(x, V)$ for $(x, V) \in R^n \times G(n, m)$. Then if μ measures R^n , $x \in R^n$, $V \in G(n, m)$ and $r > 0$, we have

$$\psi_r^{n-m}(\mu, x, V) = \liminf_{\delta \downarrow 0} \delta^{m-n} \mu\{y \in B(x, r) : \text{dist}(x-y, V) \leq \delta\}.$$

The following lemma can be proved by the same method as Lemma 16 in [10]. We omit the details.

6.2. Lemma. Suppose that E is an \mathcal{H}^s measurable subset of R^n with $0 < \mathcal{H}^s(E) < \infty$, $t > 0$ and $V \in G(n, m)$. If $\mathcal{H}^t(E \cap \tau_a V) = 0$ for all $a \in R^n$, then

$$\limsup_{r \downarrow 0} r^{-t} \psi_r^{n-m}(\mathcal{H}^s \llcorner E, x, V) = \infty$$

for \mathcal{H}^s almost all $x \in E$.

6.3. Lemma. If $0 < u < s$, $E \subset R^n$ and $\mathcal{H}^s(E) < \infty$, then

$$\limsup_{r \downarrow 0} r^{u-s} \int_{E \cap B(x, r)} |x-y|^{-u} d\mathcal{H}^s y \leq s(s-u)^{-1}$$

for \mathcal{H}^s almost all $x \in R^n$.

Proof. Let $x \in R^n$ and $r > 0$. Applying 3.4 and integrating by parts, we have

$$\begin{aligned} \int_{E \cap B(x, r)} |x-y|^{-u} d\mathcal{H}^s y &= \int_0^r \varrho^{-u} d\mathcal{H}^s[E \cap B(x, \varrho)] \\ &\leq r^{-u} \mathcal{H}^s[E \cap B(x, r)] + u \int_0^r \varrho^{-u-1} \mathcal{H}^s[E \cap B(x, \varrho)] d\mathcal{L}^1 \varrho. \end{aligned}$$

The assertion follows from the fact that the upper density $\Theta^{*s}(\mathcal{H}^s \llcorner E, x)$ is at most one for \mathcal{H}^s almost all $x \in R^n$ [5, 2.10.19(5)].

6.4. Lemma. If E is an \mathcal{H}^s measurable subset of R^n with $0 < \mathcal{H}^s(E) < \infty$, then

$$\dim E \cap \tau_x V \geq s+m-n$$

for $\mathcal{H}^s \times \gamma_{n,m}$ almost all $(x, V) \in E \times G(n, m)$.

Proof. It is sufficient to prove this for an arbitrary compact subset F of E and $s+m-n > 0$. Let $0 < t < s+m-n$. Denoting

$$f(x, V) = \limsup_{r \downarrow 0} r^{-t} \psi_r^{n-m}(\mathcal{H}^s \llcorner F, x, V)$$

for $(x, V) \in R^n \times G(n, m)$, we infer with the help of Lemmas 3.8, 6.3 and the obvious generalization of [10, Lemma 10] that $\int f(x, V) d\gamma_{n,m} V = 0$ for \mathcal{H}^s almost all $x \in F$, whence

$$(1) \quad \int_F \int f d\mathcal{H}^s d\gamma_{n,m} = 0.$$

Applying Lemma 6.2 to the set

$$E_V = \{ x \in F : \mathcal{H}^t(F \cap \tau_x V) = 0 \}$$

for $V \in G(n, m)$, we find that $f(x, V) = \infty$ for \mathcal{H}^s almost all $x \in E_V$. Hence (1) implies that $\mathcal{H}^s(E_V) = 0$ for $\gamma_{n,m}$ almost all $V \in G(n, m)$. The set $\{(x, V) : x \in E_V\}$ is $\mathcal{H}^s \times \gamma_{n,m}$ measurable by Lemma 6.1, and Fubini's theorem can be used to show that $\mathcal{H}^s \times \gamma_{n,m} \{(x, V) : x \in E_V\} = 0$. The lemma follows by the arbitrariness of t .

6.5. Lemma. Suppose that r and s are positive numbers, $x \in R^n$, $E \subset R^n$ and $\mathcal{H}^s(E) < \infty$. If $s \geq n - m$, then

$$\int^* \mathcal{H}^{s+m-n}([E \setminus B(x,r)] \cap \tau_x V) d\gamma_{n,m} V \leq c r^{m-n} \mathcal{H}^s[E \setminus B(x, r)],$$

where c is a constant depending only on n, m and s . If $s < n - m$, then

$$[E \setminus B(x,r)] \cap \tau_x V = \emptyset$$

for $\gamma_{n,m}$ almost all $V \in G(n, m)$.

Proof. For every positive integer k , choose balls $B_{k,i} \subset R^n$, $i = 1, 2, \dots$, such that

$$\begin{aligned} E \setminus B(x, r) &\subset \bigcup_{i=1}^{\infty} B_{k,i} \subset R^n \setminus B(x, r/2), \\ \sum_{i=1}^{\infty} \alpha(s) 4^{-s} (\text{diam } B_{k,i})^s &< \mathcal{H}^s[E \setminus B(x,r)] + k^{-1}, \\ \text{diam } B_{k,i} &< k^{-1}, \quad i = 1, 2, \dots \end{aligned}$$

Since $B_{k,i} \subset R^n \setminus B(x, r/2)$, we obtain from Lemma 2.6 that

$$\gamma_{n,m} \{ V \in G(n, m) : B_{k,i} \cap \tau_x V \neq \emptyset \} \leq c_3 2^{n-m} r^{m-n} (\text{diam } B_{k,i})^{n-m}.$$

If $s \geq n - m$, we get

$$\int [\text{diam } (B_{k,i} \cap \tau_x V)]^{s+m-n} d\gamma_{n,m} V \leq c_3 2^{n-m} r^{m-n} (\text{diam } B_{k,i})^s.$$

Here we agree that $\text{diam } \emptyset = 0$. The integrand is continuous, and therefore measurable. We sum over i and use Fatou's lemma to infer

$$\begin{aligned} &\int \liminf_{k \rightarrow \infty} \sum_{i=1}^{\infty} [\text{diam } (B_{k,i} \cap \tau_x V)]^{s+m-n} d\gamma_{n,m} V \\ &\leq \alpha(s)^{-1} 4^s c_3 2^{n-m} r^{m-n} \mathcal{H}^s[E \setminus B(x, r)]. \end{aligned}$$

But

$$\begin{aligned} & \mathcal{H}^{s+m-n}([E \setminus B(x, r)] \cap \tau_x V) \\ & \leq \liminf_{k \rightarrow \infty} \sum_{i=1}^{\infty} \alpha(s+m-n) 2^{n-m-s} [\text{diam}(B_{k,i} \cap \tau_x V)]^{s+m-n}, \end{aligned}$$

and the first assertion follows.

If $s < n - m$, we conclude for all $k = 1, 2, \dots$ that

$$\begin{aligned} & \gamma_{n,m} \{ V \in G(n, m) : [E \setminus B(x, r)] \cap \tau_x V \neq \emptyset \} \\ & \leq \sum_{i=1}^{\infty} \gamma_{n,m} \{ V \in G(n, m) : B_{k,i} \cap \tau_x V \neq \emptyset \} \\ & \leq c_3 2^{n-m} r^{m-n} \sum_{i=1}^{\infty} (\text{diam } B_{k,i})^{n-m} \\ & \leq \alpha(s)^{-1} 4^s c_3 2^{n-m} r^{m-n} k^{m+s-n} (\mathcal{H}^s[E \setminus B(x, r)] + k^{-1}). \end{aligned}$$

Letting $k \rightarrow \infty$ gives the second assertion.

6.6. **Theorem.** *If $s > n - m$ and E is a Suslin set of R^n with $0 < \mathcal{H}^s(E) < \infty$, then*

$$\dim E \cap \tau_x V = s + m - n$$

and

$$\mathcal{H}^{s+m-n}(E \cap \tau_x V) < \infty$$

for $\mathcal{H}^s \times \gamma_{n,m}$ almost all $(x, V) \in E \times G(n, m)$.

Proof. Use Riemann-Stieltjes integration to deduce from Lemma 6.5 (as in [10, Lemma 19]) that

$$\begin{aligned} & \int^* \mathcal{H}^{s+m-n}(E \cap \tau_x V) d\gamma_{n,m} V \\ & \leq c r^{m-n} \mathcal{H}^s(E) + c \int_0^r \varrho^{m-n-1} \mathcal{H}^s[E \cap B(x, \varrho)] d\mathcal{L}^1 \varrho \end{aligned}$$

for $x \in R^n$ and $r > 0$. The right hand side is finite, whenever the upper density $\Theta^{*s}(\mathcal{H}^s \llcorner E, x)$ is finite. This happens for \mathcal{H}^s almost all $x \in E$ [5, 2.10.19(5)], whence

$$\gamma_{n,m} \{ V \in G(n, m) : \mathcal{H}^{s+m-n}(E \cap \tau_x V) = \infty \} = 0$$

for \mathcal{H}^s almost all $x \in E$. The assertions follow now from Lemmas 6.4, 6.1 and Fubini's theorem.

6.7. *Remark.* It seems to be an open question whether Theorem 6.6 holds for $s = n - m$. See remarks in [10, 7.3 and 8.1] and [5, 3.3.11].

6.8. **Theorem.** *If $s < n - m$, $x \in R^n$, $E \subset R^n$ and $\mathcal{H}^s(E) < \infty$, then*

$$(E \setminus \{x\}) \cap \tau_x V = \emptyset$$

for $\gamma_{n,m}$ almost all $V \in G(n, m)$.

This follows directly from Lemma 6.5.

In the last two theorems, we give some information about the Hausdorff measures of exceptional sets. To compare with Theorems 6.6 and 6.8, note that $\dim G(n, m) = m(n-m) = n-m+(m-1)(n-m)$.

6.9. **Theorem.** *If $0 < t < s$ and E is a Suslin set of R^n with $\mathcal{H}^s(E) < \infty$, then*

$$\mathcal{H}^t(E \cap \tau_x V) < \infty$$

for $\mathcal{H}^s \times \mathcal{H}^\alpha$ almost all $(x, V) \in E \times G(n, m)$, where

$$\alpha = s - t + (m - 1)(n - m).$$

6.10. **Theorem.** *If $s > 0$, $x \in R^n$ and E is a Suslin set of R^n with $\mathcal{H}^s(E) < \infty$, then*

$$\dim \{ V \in G(n, m) : (E \setminus \{x\}) \cap \tau_x V \neq \emptyset \} \leq s + (m - 1)(n - m).$$

Proofs. When $G(n, m)$ is identified with a subset of $\odot_2 \wedge_m R^n$ as in [5, 3.2.28(4)], the norm $\|\cdot\|$ on $\odot_2 \wedge_m R^n$ [5, 1.10.5] induces a metric d on $G(n, m)$. Let $\beta > \nu = (m - 1)(n - m)$ and let φ be a Borel regular measure over $G(n, m)$ such that $\varphi[B(V, \delta)] \leq \delta^\beta$ for all $V \in G(n, m)$ and $\delta > 0$. Then a proof similar to that of Lemma 5.5 shows that there is a constant b depending only on n, m and β such that

$$\varphi\{ V \in G(n, m) : \text{dist}(x, V) \leq \delta \} \leq b \delta^{\beta-\nu} |x|^{\nu-\beta}$$

for all $x \in R^n$ and $\delta > 0$. In fact, assume that $x \in S^{n-1}$, $\delta < 1$, and denote

$$\begin{aligned} A_\delta &= \{ V \in G(n, m) : \text{dist}(x, V) \leq \delta \}, \\ B &= \{ V \in G(n, m) : x \in V \}. \end{aligned}$$

Let $V \in A_\delta$ and choose $g \in O(n)$ such that $g(V) \in B$ and $\|g - \mathbf{1}_{R^n}\| \leq 2\delta$. If V is associated with a simple unit m -vector ζ [5, 1.6.1], then $g(V)$ is associated with $\wedge_m g(\zeta)$. Using [5, 3.2.28(4) and 1.7.5], one finds that $d[V, g(V)] \leq |\zeta - \wedge_m g(\zeta)| \leq 2m\delta$. Hence

$$A_\delta \subset \{ V : \text{dist}(V, B) \leq 2m\delta \}.$$

Moreover, B is isometric with $G(n-1, m-1)$. The rest of the argument reads as in 5.5.

The proof of Lemma 6.5 now holds with $\gamma_{n,m}$ replaced by φ and $n-m$ replaced by $\beta-\nu$. The exceptional sets in Theorems 6.9 and 6.10 are Suslin sets by Lemma 6.1. One can complete the proof of Theorem 6.9

choosing $\beta = \alpha$, arguing as in 6.6 and using Theorem 5.4. Theorem 6.10 follows with $\beta > s + \nu$ arbitrary, and with the help of the above mentioned generalization of Lemma 6.5 and Theorem 5.4.

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University of Helsinki
Department of Mathematics
SF-00100 Helsinki 10
Finland

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