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THE DOMAINS OF NORMALITY OF AN ENTIRE FUNCTION

I. N. BAKER

1. Introduction

If f is a rational or entire function of the complex variable z its natural iterates f_n are defined by $f_1(z) = f(z)$, $f_{n+1}(z) = f(f_n(z))$, $n = 1, 2, \ldots$. The theory developed by Fatou [7,8] and Julia [11] deals with the set $\mathfrak{C} = \mathfrak{C}(f)$ of points of the complex plane in whose neighbourhood $\{f_n(z)\}$ is a normal family. It is convenient to express many results in terms of the complement $\mathfrak{F}(f)$ of $\mathfrak{C}$, i.e. the set of non-normality. We shall assume throughout that f is not a rational function of order 0 or 1. Then $\mathfrak{F} = \mathfrak{F}(f)$ has the following properties (see [7] and [8]).

I. *$\mathfrak{F}(f)$ is a non-empty perfect set.*

II. *$\mathfrak{F}(f)$ and $\mathfrak{C}(f)$ are completely invariant under the mapping $z \to f(z)$.*

In general a set S is called completely invariant under $z \to f(z)$ if $\alpha \in S$ implies that $f(\alpha) \in S$ and that $\beta \in S$ for every solution β of $f(\beta) = \alpha$.

The components G_i of $\mathfrak{C}(f)$ are maximal domains of normality for $\{f_n\}$. The theory considers the various ways in which $\mathfrak{F}$ may separate these components and the limit functions which arise from those subsequences of $\{f_n\}$ which are locally uniformly convergent in G_i.

It may happen for rational f that $\mathfrak{F}$ is totally disconnected (a 'discontinuum') so that $\mathfrak{C}$ consists of a single domain. This occurs for $f(z) = z^2 - p$, where $p > 2$ is a constant, in which case $\mathfrak{F}(f)$ is a bounded, totally disconnected subset of the real axis (Myrberg [12]). At the end of [8] Fatou raises the question as to whether there are transcendental entire functions f for which $\mathfrak{F}(f)$ is totally disconnected.

Concerning the set $\mathfrak{C}(f)$ H. Töpfer [15] has shown:

III. *If f is transcendental and entire and if $\mathfrak{C}(f)$ has an unbounded component G, then every other component of $\mathfrak{C}(f)$ is simply-connected. If in addition G is multiply connected, then G is completely invariant under the mapping $z \to f(z)$.*

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In this note we shall prove the

Theorem 1. *If f is transcendental and entire, then $\mathfrak{C}(f)$ has no unbounded multiply-connected component.*

Since the total discontinuity of $\mathfrak{F}(f)$ implies that $\mathfrak{C}(f)$ is an unbounded connected domain Fatou's question is answered by the

Corollary. *For transcendental entire f the set $\mathfrak{F}(f)$ must contain non-degenerate continua.*

Various authors e.g. Brolin [6], Garber [9], Oba and Pitcher [13] have investigated the metric properties of $\mathfrak{F}(f)$, giving estimates of Hausdorff dimensions, capacities, and so on. The only significant lower estimate in the transcendental entire case was given in [9], where it was shown that the logarithmic capacity of $\mathfrak{F}$ is strictly positive. Our corollary strengthens this result considerably. We remark also that the set $\mathfrak{F}$ can even fill out the whole plane in some cases ([4]).

Returning to the components of $\mathfrak{C}(f)$ in the theorem: it is indeed possible that multiply-connected components exist for transcendental entire f, as shown by an example in [1]. In this case the multiply-connected domains are of course bounded.

If f is a transcendental entire function, any completely invariant component of $\mathfrak{C}(f)$ is unbounded and hence, by our theorem, simply-connected. It was shown in [3] that there can be at most one such completely invariant component. P. Bhattacharyya [5] deduced from this that the number of components of $\mathfrak{C}(f)$ is either 1 or infinite. He also showed that for $g(z) = e^{a+z} - e^a$, $a < 0$, $\mathfrak{C}(g)$ consists of a single (completely invariant) component. It is not clear whether the existence of a completely invariant component of $\mathfrak{C}$ precludes the existence of other components or not. We can prove

Theorem 2. *If f is a transcendental entire function such that $\mathfrak{C}(f)$ has a completely invariant component G, then in every other component of $\mathfrak{C}(f)$ f is univalent.*

Corollary. *A function f which satisfies the conditions of Theorem 2 can have at most one attractive fixpoint.*

An attractive fixpoint α is a point for which $f(\alpha) = \alpha$, $|f(\alpha)| < 1$. Two different attractive fixpoints belong to different components of $\mathfrak{C}(f)$ and (c.f. [7,8]) f is not univalent in these components. The corollary follows. The example $g(z) = e^{a+z} - e^a$, $a < 0$, shows that one attractive fixpoint is possible.

2. Lemmas needed in the Proofs

Additional results about $\mathfrak{F}(f)$ are proved in [7] for rational f and in [8] for entire f, except where mentioned below.

IV. *For any integer* $n \geq 1$ *we have* $\mathfrak{F}(f_n) = \mathfrak{F}(f)$.

V. *For every* $\alpha \in \mathfrak{F}(f)$ *and for every complex* β *(excluding at most two exceptional* β*-values) there exist a sequence of positive integers* $\{n_k\}$ *and a sequence of complex numbers* $\{\alpha_k\}$ *such that*

$$f_{n_k}(\alpha_k) = \beta, \quad \lim \alpha_k = \alpha .$$

A fixpoint α of order n of f is a solution of $f_n(\alpha) = \alpha$; α is said to have exact order n if $f_k(\alpha) \neq \alpha$ for $1 \leq k < n$ and in this case the multiplier of α is the number $f_n'(\alpha)$. If $|f_n'(\alpha)| > 1$ the fixpoint α is called repulsive and belongs to $\mathfrak{F}(f)$. Moreover one has

VI. $\mathfrak{F}(f)$ *is the derivative of the set of fixpoints of all orders or* f. *It is even true that the repulsive fixpoints are dense in* $\mathfrak{F}$ *(shown in* [2] *for entire* f).

In addition we need

L e m m a 1. (Pólya [14]). *Let* e, g *and* h *be entire functions satisfying*

$$e(z) = g(h(z)), \tag{1}$$

$$h(0) = 0. \tag{2}$$

There is a constant $c > 0$ *(in fact* $c = 1/8$) *independent of* e, g, h *such that*

$$M(e, r) > M(g, c\, M(h, r/2)), \tag{3}$$

where $M(e, r)$ *denotes* $\max\limits_{|z|=r} |e(z)|$.

L e m m a 2. (Schottky's theorem, see e.g. [10]). *There exists an absolute constant* C *such that for every function* $f(z)$ *which is regular and satisfies* $f(z) \neq 0, 1$ *in* $|z| < 1$ *we have for*

$$M(f, r) = \max_{|z|=r} |f(z)| < \exp\left[\frac{1}{1-r}((1 + r) \log \max (1, |f(0)|) + 2\, C\, r)\right].$$

3. Proof of Theorem 1

Suppose that f is a transcendental entire function and that G is an unbounded, multiply connected component of $\mathfrak{C}(f)$. Property VI shows that there are in $\mathfrak{F}$ two repulsive fixpoints z_1, z_2 of order say p and q respectively, which may be taken to be different from the exceptional values in V. Both are repulsive fixpoints of f_{pq} and IV shows we can replace f_{pq} by f and assume z_1, z_2 are repulsive fixpoints of f. Replacing $f(z)$

by $(f(a + b\,z) - a)/b$, a , b constant, merely subjects $\mathfrak{F}$ and $\mathfrak{C}$ to a linear transformation, so we may without loss of generality assume that $z_1 = 0$ and $z_2 = 1$ are first order repulsive fixpoints of f, i.e. $0, 1 \in \mathfrak{F}$, and that 0 is not an exceptional point in the sense of V.

Now if any of the locally-convergent subsequences of $\{f_n\}$ in G has a finite and hence regular limit it follows that the convergence remains uniform in the interior of any Jordan curve in G , so that G is not multiply-connected. Thus $f_n(z)$ must converge locally uniformly to ∞ in G .

The multiply-connected domain G must contain a Jordan curve γ in whose interior lie points of $\mathfrak{F}$, and so by V-points of the form $f_{-n}(0)$ for some arbitrarily large n . Thus for sufficiently large n the set $\gamma_1 = f_n(\gamma)$ is (by III) a curve in G which winds round 0 at least once and whose minimum distance r from 0 is as large as we please. We choose n so large that

(4) $$(1/8)\, M(f, t/4) > t \quad \text{for} \quad t \geq r .$$

We next choose an m such that $\gamma_2 = f_m(\gamma)$ is a curve in G which winds round 0 and which has a minimum distance s from 0 satisfying

(5) $$s > M(f_2, 2R) ,$$

where R is the greatest distance of γ_1 from 0 . Join γ_1 to γ_2 by a path γ_3 in G and denote by K the union of γ_1 , γ_2 and γ_3 .

Denote by $4\,\delta$ the distance of the compact set K from $\mathfrak{F}$. Then $\delta > 0$. There is a finite collection C of say N discs of radius δ whose centres lie on K and whose union covers K . Since K is connected, there is for any pair t_1 , t_2 in K a chain of $p \leq N$ points $t_1 = w_1, w_2, \ldots, w_p = t_2$ in K such that w_i , w_{i+1} lie in a common disc of C . Thus $|w_{i+1} - w_i| < 2\,\delta$.

Suppose that in a $(3\,\delta)$-neighbourhood L of K the function g is regular, satisfies $|g(z)| > 1$ and omits the values 0 and 1 . The disc $|w - w_i| < 3\,\delta$ lies in L and contains w_{i+1} . Applying Lemma 2 to the function $g(w_i + 3\,\delta\, z)$ in the unit disc we see that there is an absolute constant $A > 1$ such that

$$|g(w_{i+1})| < A\, |g(w_i)|^5 .$$

Hence for t_1 , t_2 as above

(6) $$|g(t_2)| < B\, |g(t_1)|^C$$

where the constants $C = 5^N$, $B = A^{1+5+\ldots+5^N}$ are independent of g or of the choice of t_1 , t_2 in K .

Since $f_n \to \infty$ locally uniformly in G , while $f_n(G) \subset G$ so $f_n(z) \neq 0, 1 \in \mathfrak{F}$ for $z \in G$, we see that for all sufficiently large n the functions f_n satisfy $|f_n(z)| > 1$, $f_n(z) \neq 0, 1$ in L. Thus by (6) if t_1 is any point of γ_1 and if t_2 is the point of γ_2 at which $|f_n|$ is a maximum, we have

$$(7) \qquad |f_n(t_2)| < B\,|f(t_1)|^C, \qquad n \geq n_0 .$$

However by the choice of s in (5)

$$\begin{aligned} |f_n(t_2)| &\geq M(f_n, s) \\ &\geq M(f_n, M(f_2, 2R)) \\ &\geq M(f_{n+2}, 2R) \\ &\geq M(f, (1/8)\, M(f_{n+1}, R)) \end{aligned}$$

by Lemma 1. But on γ_1 we have $f_n(z) \to \infty$ and so $M(f_{n+1}, R) \to \infty$ as $n \to \infty$. Thus the last expression above is, for all sufficiently large n, greater than

$$\begin{aligned} B\,(M(f_{n+1}, R))^C &> B\,(M(f_n, (1/8)\, M(f, R/2)))^C \\ &\geq B\,(M(f_n, R))^C \\ &\geq B\,|f_n(t_1)|^C \end{aligned}$$

by (4). Thus we have a contradiction with (7). The theorem is proved.

Proof of Theorem 2

Suppose the transcendental entire function f has a completely invariant component G of $\mathfrak{C}(f)$. Then G is necessarily unbounded and simply connected. All other components of $\mathfrak{C}$ are simply connected. Suppose that there is a component $H \neq G$ of $\mathfrak{C}(f)$ in which f is not univalent. Now by II $f(H)$ lies in some component $K \neq G$ of $\mathfrak{C}(f)$.

Take a value $k = f(p) = f(q)$ where $p \in H$, $q \in H$, $p \neq q$, $f'(p) \neq 0$, $f'(q) \neq 0$. Thus there are branches $z = P(w)$ and $z = Q(w)$ of the inverse f^{-1} of $w = f(z)$ which are regular at $w = k \in K$ and satisfy $p = P(k)$, $q = P(k)$.

By Gross' star theorem we may continue $P(w)$, $Q(w)$ regularly to ∞ along almost any ray starting at k, in particular along some ray L which meets G. Denote by γ the segment of L from k to a certain point $g \in G$. Then $P(\gamma)$, $Q(\gamma)$ are disjoint curves joining $p \in H$ to $p' = P(g) \in G$ and $q \in H$ to $q' = Q(g) \in G$, respectively.

Join p to q by a simple arc β in H, and p' to q' by a simple arc $\beta' \in G$. Let $\bar{p}$ be the last intersection of β with $P(\gamma)$, $\bar{q}$ the first intersection with $Q(\gamma)$. Let $\bar{\beta}$ be the subarc of β which joins $\bar{p}$ to $\bar{q}$. Similarly define $\bar{p}'$ as the last intersection of β' with $P(\gamma)$, $\bar{q}'$ the first intersection with $Q(\gamma)$ and $\bar{\beta}'$ as the subarc $\bar{p}'\bar{q}'$ of β' . Denote by π the subarc $\bar{p}\bar{p}'$ of $P(\gamma)$, by $\varkappa$ the subarc $\bar{q}\,\bar{q}'$ of $Q(\gamma)$. Then $\pi\,\bar{\beta}'(\varkappa)^{-1}(\beta')^{-1}$ is a Jordan curve C whose interior D maps under $z \to f(z)$ into a bounded region $f(D)$ whose boundary is contained in $f(C) \subset f(\beta) \cup f(\beta') \cup \gamma$.

The $f(\beta)$ and $f(\beta')$ are closed bounded and disjoint curves. The unbounded component M of their complement contains $\mathfrak{F}(f)$. Thus M meets γ since $\mathfrak{F}(f)$ does. Now $f(\pi)$ is a segment of γ which joins $f(\beta)$ to $f(\beta')$. If t is the last point of interesction of γ with $f(\beta)$ and t' the first intersection with $f(\beta')$, then the segment tt' of γ is a crosscut of M whose ends belong to different components of the frontier. Thus tt' does not disconnect M . Since tt' belongs to $f(\pi)$ every point of tt' is a boundary value of $f(D)$. Thus $f(D)$ must contain the whole of $M - tt'$, i.e. an unbounded set. This contradicts the boundedness of D and the result is proved.

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Imperial College
Mathematics Department
London SW7 2AZ
England

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