Annales Academiæ Scientiarum Fennicæ
Series A. I. Mathematica
Volumen 1, 1975, 285–296

ON CERTAIN IRREDUCIBLE MODULES OF THE LIE ALGEBRA $\mathfrak{gl}(4, \mathbf{C})$

JOUKO MICKELSSON

Introduction

Let $\mathfrak{g}$ be a complex Lie algebra and $\mathfrak{k}$ a reductive subalgebra in $\mathfrak{g}$. We study irreducible $\mathfrak{k}$-finite $\mathfrak{g}$-modules by means of the step algebra $S(\mathfrak{g}, \mathfrak{k})$ of the pair $(\mathfrak{g}, \mathfrak{k})$. If V is a $\mathfrak{g}$-module and V_α is the sum of all irreducible finite-dimensional $\mathfrak{k}$-submodules of V with maximal weight α, we denote by $A_{\beta,\alpha}$ the subspace of the enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ such that $A_{\beta,\alpha} V_\alpha^+ \subset V_\beta^+$ for any $\mathfrak{g}$-module V; by definition, V_α^+ consists of all maximal vectors in V_α. Let $M_\alpha = \sum_{\beta<\alpha} A_{\beta,\alpha}$ and let D be the zero-step algebra, $D V_\alpha^+ \subset V_\alpha^+$ for any $\mathfrak{g}$-module V. It is shown that the equivalence classes $[V]$ of irreducible $\mathfrak{g}$-modules V, such that $V_\alpha \neq 0$ and $V_\beta = 0$ for $\beta < \alpha$, are in natural $1-1$ correspondence with the equivalence classes of irreducible $D / D \cap U(\mathfrak{g}) M_\alpha$ -modules. Using this result, the case $\mathfrak{k} = \mathfrak{gl}(2, \mathbf{C}) \oplus \mathfrak{gl}(2, \mathbf{C})$, $\mathfrak{g} = \mathfrak{gl}(4, \mathbf{C})$ is studied in detail.

1. Preliminaries

Let $\mathfrak{g}$ be a complex Lie algebra, $\mathfrak{k}$ a non-commutative reductive subalgebra in $\mathfrak{g}$ and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{k}$. Let $\mathfrak{h}^*$ be the (complex) dual of $\mathfrak{h}$, $\Delta \subset \mathfrak{h}^*$ a set of simple roots and Λ the set of dominant integral elements in $\mathfrak{h}^*$. By definition, an element $\alpha \in \mathfrak{h}^*$ is dominant integral if the restriction of α, to the subalgebra of $\mathfrak{h}$ which belongs to the semi-simple part of $\mathfrak{k}$, is dominant integral with respect to the choice Δ of simple roots.

Next we choose a basis $\{ h_1, ..., h_l \}$ for $\mathfrak{h}$, such that $h_1, ..., h_p$ are in the semi-simple part of $\mathfrak{k}$ and $h_{p+1}, ..., h_l$ commute with $\mathfrak{k}$ $(p \leq l)$. We introduce a partial ordering "$<$" on Λ by putting $\lambda < \mu$ if $\lambda(h_i) \neq \mu(h_i)$ for some $i \leq p$ and the first non-zero member in the sequence

doi:10.5186/aasfm.1975.0112

$\lambda(h_1) - \mu(h_1)$, $\lambda(h_2) - \mu(h_2)$,... is negative. We assume furthermore that the choice $\{ h_1, ..., h_l \}$ is such that the ordering "$<$" is compatible with the strong partial ordering defined by the choice of the simple roots. We split

$$\mathfrak{k} = \mathfrak{k}_+ \oplus \mathfrak{h} \oplus \mathfrak{k}_-$$

where $\mathfrak{k}_+$ corresponds to positive roots and $\mathfrak{k}_-$ to negative roots.

We denote by $U(\mathfrak{a})$ the enveloping algebra of an arbitrary Lie algebra $\mathfrak{a}$. We define

$$S(\mathfrak{g} , \mathfrak{k}) = \{ u \in U(\mathfrak{g}) \mid \mathfrak{k}_+ u \subset U(\mathfrak{g}) \mathfrak{k}_+ \} .$$

Let $\mathfrak{g}'$ be an ad$\mathfrak{k}$-invariant complement of $\mathfrak{k}$ in $\mathfrak{g}$ and $\{ t_1, ..., t_n \}$ a basis of $\mathfrak{g}'$ such that t_i has weight λ_i , $\lambda_i \geq \lambda_j$ when $i > j$, with respect to $\mathfrak{h}$.

If $(i) = (i_1, ..., i_n)$ is a sequence of non-negative integers we put $t(i) = t_1^{i_1} ... t_n^{i_n}$ and

$$U_1 = \sum_{(i)} t(i) \, U(\mathfrak{h}) .$$

We can now write

$$U(\mathfrak{g}) = U_1 \oplus U(\mathfrak{g}) \mathfrak{k}_+ \oplus U_1 \, U(\mathfrak{k}_-) \, \mathfrak{k}_- .$$

Let $P' : U(\mathfrak{g}) \to U_1$ be the projection on the first summand and let P be the restriction of P' to $S(\mathfrak{g} , \mathfrak{k})$.

It is shown in [2, 4] that the mapping $P : S(\mathfrak{g} , \mathfrak{k}) \to U_1$ is injective modulo $U(\mathfrak{g}) \mathfrak{k}_+$. Furthermore, for each t_i there exists $s_i \in S(\mathfrak{g} , \mathfrak{k})$ such that $P(s_i) = t_i u_i$, where $u_i \in U(\mathfrak{h})$ is such that $u_i(\lambda) \neq 0$ if $\lambda + \lambda_i \in \Lambda$; any $u \in U(\mathfrak{h})$ can be identified with a polynomial on $\mathfrak{h}^*$.

We define here the step algebra $S_0(\mathfrak{g} , \mathfrak{k})$ with unit $\mathbf{1}$ as the subalgebra of $S(\mathfrak{g}, \mathfrak{k})$ generated by $s_1, ..., s_n$ and $U(\mathfrak{g}) \mathfrak{k}_+$. We denote by D the centralizer of $\mathfrak{h}$ in $S_0(\mathfrak{g} , \mathfrak{k})$.

2. Description of irreducible $\mathfrak{g}$-modules by the action of D

Let V be a $\mathfrak{g}$-module. We say that it is $\mathfrak{k}$-finite if it is a direct sum of irreducible finite-dimensional $\mathfrak{k}$-modules, when V is considered as a $\mathfrak{k}$-module through the restriction of $\mathfrak{g}$ to $\mathfrak{k}$. If α is any dominant integral element of $\mathfrak{h}^*$, we denote by V_α the sum of all irreducible $\mathfrak{k}$-modules with maximal weight α , contained in V . It is known that V is $\mathfrak{k}$-finite if it is generated by some V_α , $\alpha \in \Lambda$, [1]. We define

$$V^+ = \{ x \in V \mid \mathfrak{k}_+ x = 0 \} , \qquad V_\alpha^+ = V^+ \cap V_\alpha .$$

It is shown in [5] that $(U(\mathfrak{g}) x)^+ = S_0(\mathfrak{g} , \mathfrak{k}) x$ for any $\mathfrak{k}$-finite $\mathfrak{g}$-module and for any $x \in V^+$. In particular,

$$V^+ = S_0(\mathfrak{g}, \mathfrak{k})\, x$$

if V is irreducible and $0 \neq x \in V^+$. We set

$$A_{\beta,\alpha} = \{ u \in U(\mathfrak{g}) \mid u\, V_\alpha^+ \subset V_\beta^+ \text{ for any } \mathfrak{g}\text{-module } V \}$$

and

$$M_\alpha = \sum_{\beta < \alpha} A_{\beta,\alpha}.$$

We denote by I_α the annihilator in $U(\mathfrak{k})$ of the vector of maximal weight in an irreducible $\mathfrak{k}$-module with maximal weight α; it is clear that $U(g)\, I_\alpha \subset M_\alpha$ for any α and

$$D \subset \bigcap_\alpha A_{\alpha,\alpha}, \qquad C \subset \bigcap_\alpha A_{\alpha,\alpha}$$

where C is the centralizer of $\mathfrak{k}$ in $U(\mathfrak{g})$. It follows that we can consider any V_α^+ as a D-module or as a C-module. In the case $V_\beta = 0$ for $\beta < \alpha$, the D-module V_α^+ is in a natural way also a D_α-module, where

$$D_\alpha = D \,/\, D \cap U(\mathfrak{g})\, M_\alpha.$$

We say that V_α is a minimal component of V if $V_\alpha \neq 0$ and $V_\beta = 0$ for $\beta < \alpha$. It follows from our choice of ordering in Λ that any $\mathfrak{k}$-finite $\mathfrak{g}$-module has at least one minimal component and the minimal component is unique if $\mathfrak{k}$ is semi-simple.

Theorem 1. *Assume that* $D_\alpha \neq 0$. *Then the mapping* $V \mapsto V_\alpha^+$ *induces a bijection between the set* $\hat{G}_\alpha$ *of equivalence classes* $[V]$ *of irreducible* $\mathfrak{g}$-*modules with minimal component* V_α *and between the set* $\hat{D}_\alpha$ *of equivalence classes of irreducible non-zero* D_α-*modules.*

Proof. Let us first consider the $\mathfrak{g}$-module $W = U(\mathfrak{g}) \,/\, U(\mathfrak{g})\, M_\alpha$. It is clear that W_α is the minimal component of W. Now W is generated by the vector $0 \neq x = \mathbf{1} + U(\mathfrak{g})\, M_\alpha \in W_\alpha$; thus it is $\mathfrak{k}$-finite and

$$W^+ = S_0(\mathfrak{g}, \mathfrak{k})\, (\mathbf{1} + U(\mathfrak{g})\, M_\alpha).$$

Since $C(\mathbf{1} + U(\mathfrak{g})\, M_\alpha) \subset W_\alpha^+$ we have

$$C \subset D + U(\mathfrak{g})\, M_\alpha.$$

In [1, 3] it is shown that a $\mathfrak{g}$-module V with minimal component V_α is determined (up to equivalence) by the $C \,/\, C \cap U(\mathfrak{g})\, M_\alpha$-module V_α^+; it then follows from the inclusion above that the mapping $\hat{G}_\alpha \to \hat{D}_\alpha$, $[V] \to [V_\alpha^+]$, is injective.

To prove the surjectivity of our mapping one can use a similar argument as in [1, 3]: Let W be an irreducible D_α-module. Let L be the annihilator in D of a non-zero vector $x \in W$ (note that W is a D-module through the quotient map $D \to D_\alpha$) so that $W \cong D / L$ as D-modules. Put

$$N = \{ u \in U(\mathfrak{g}) \mid U(\mathfrak{g})\, u \cap D \subset L \}$$

and consider the irreducible $\mathfrak{g}$-module $U(\mathfrak{g}) / N = V$. We have to show that $V_\alpha^+ \cong W$ as D-modules. Now $D \cap U(\mathfrak{g})\, M_\alpha \subset L$ and therefore $U(\mathfrak{g})\, M_\alpha \subset N$; it follows that the vector $x = \mathbf{1} + N \in V$ is annihilated by $I_\alpha \subset U(\mathfrak{g})\, M_\alpha$ i.e. $x \in V_\alpha^+$. Then

$$V_\alpha^+ = D\,x = D + N .$$

The mapping $\varphi : V_\alpha^+ \to D / L \cong W$, $\varphi(d + N) = d + L$ (where $d \in D$) is a D-linear isomorphism; the injectivity of φ follows from the fact that $D \cap N = L$. If $\beta < \alpha$ then $V_\beta^+ = A_{\beta,\alpha}\, V_\alpha^+ = 0$ because of

$$A_{\beta,\alpha} \subset U(\mathfrak{g})\, M_\alpha \subset N \qquad (\text{when } \beta < \alpha) .$$

We conclude that V_α is the minimal component of V.

If $D_\alpha = 0$ it is not difficult to see that $\hat{G}_\alpha = \emptyset$.

3. Step algebra $S_0(\mathfrak{gl}(4), \mathfrak{gl}(2) \oplus \mathfrak{gl}(2))$

Let $\mathfrak{g} = \mathfrak{gl}(4, \mathbf{C})$ be the complex reductive Lie algebra consisting of 4×4 -complex matrices with the basis

$$\{e_{ij}\}_{i,j=1}^4 ; \qquad (e_{ij})_{kl} = \delta_{ik}\, \delta_{jl}$$

and commutation relations

$$[e_{ij}, e_{kl}] = \delta_{jk}\, e_{il} - \delta_{il}\, e_{kj} .$$

The subalgebra $\mathfrak{k} = \mathfrak{gl}(2, \mathbf{C}) \oplus \mathfrak{gl}(2, \mathbf{C})$ with the basis

$$\{ e_{12}, e_{21}, h_1, h_3 \} \cup \{ e_{34}, e_{43}, h_2, h_4 \}$$

is reductive in $\mathfrak{g}$. The elements $h_1 = e_{11} - e_{22}$, $h_2 = e_{33} - e_{44}$, $h_3 = e_{11} + e_{22}$, $h_4 = e_{33} + e_{44}$ span a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{k}$. Note that $\mathfrak{h}$ is also a Cartan subalgebra in $\mathfrak{g}$. If $\alpha \in \Lambda$ we set $\alpha_i = \alpha(h_i)$ and $\alpha < \beta$ if $\alpha_1 < \beta_1$ or $\alpha_1 = \beta_1$ and $\alpha_2 < \beta_2$. An element $\alpha \in \mathfrak{h}^*$ is dominant integral, $\alpha \in \Lambda$, if α_1 and α_2 are non-negative integers and α_3, α_4 arbitrary complex numbers.

A $\mathfrak{k}$-invariant complement $\mathfrak{g}'$ of $\mathfrak{k}$ in $\mathfrak{g}$ has now the ordered basis

$$(*) \qquad \{ e_{41}, e_{23}, e_{31}, e_{24}, e_{13}, e_{42}, e_{14}, e_{32} \} .$$

As described in Section 1, we associate with each $e_{ij} \in \mathfrak{g}'$ a step s_{ij} :

$$\begin{cases} s_{41} &= e_{41} h_1 h_2 + e_{42} e_{21} h_2 - e_{31} e_{43} h_1 - e_{32} e_{43} e_{21} \\ s_{31} &= e_{31} h_1 + e_{32} e_{21} \\ s_{24} &= e_{24} h_1 - e_{14} e_{21} \\ s_{23} &= e_{23} h_1 h_2 - e_{13} e_{21} h_2 + e_{24} e_{43} h_1 - e_{14} e_{43} e_{21} \end{cases}$$

$$\begin{cases} s_{14} &= e_{14} \\ s_{13} &= e_{13} h_2 + e_{14} e_{43} \\ s_{32} &= e_{32} \\ s_{42} &= e_{42} h_2 - e_{32} e_{43} \end{cases}$$

The set R_- of first four elements correspond to negative roots under the adjoint action of $\mathfrak{h}$ and the set R_+ of last four elements correspond to positive roots.

Let $S_0(\mathfrak{g}, \mathfrak{k})$ be the algebra generated by the s_{ij}'s and by $U(\mathfrak{g})\,\mathfrak{k}_+$, where $\mathfrak{k}_+ = \mathbf{C} \cdot e_{12} + \mathbf{C} \cdot e_{34}$.

Note that the projection P acting on an element s_{ij} gives the first term in s_{ij}. Let $Z(\mathfrak{g})$ be the center of $U(\mathfrak{g})$. It is known that $Z(\mathfrak{g})$ is generated by the Gelfand-elements

$$\begin{aligned} z_1 &= \sum e_{ii} , \\ z_2 &= \sum e_{ij} e_{ji} , \\ z_3 &= \sum e_{ij} e_{jk} e_{ki} , \\ z_4 &= \sum e_{ij} e_{jk} e_{kl} e_{li} \end{aligned}$$

with a summation over repeated indecis. The projection of a z_i on U_1 is easily calculated from the formulas above when one remembers that $\mathfrak{k}_\pm$ on the right gives zero (when the elements (*) appear in correct order). For example,

$$\begin{aligned} P(z_1) &= h_3 + h_4 \\ P(z_2) &= 2\,(e_{41} e_{14} + e_{31} e_{13} + e_{23} e_{32} + e_{24} e_{42}) + \tfrac{1}{2} \sum_{i=1}^{4} h_i^2 + 3\,h_1 + h_2 . \end{aligned}$$

We denote by J_α the left-ideal in $U(\mathfrak{g})$ generated by $\mathfrak{k}_+$ and by the set $\{ h - \alpha(h) \cdot 1 \mid h \in \mathfrak{h} \}$. Let Z_0 be the subalgebra of $Z(\mathfrak{g})$ generated by $\mathbf{1}$, z_2 and z_3.

Lemma 1. *Let* $s \in R_-$, $s' \in R_+$ *and* $\alpha \in \Lambda$.

(1) *If* $\alpha_1 \geq 1$ *then* $s\,s' \in \mathbf{C} \cdot R_+ R_- + \mathbf{C} \cdot \mathbf{1} + J_\alpha$

(2) *If* $\alpha_1 = 0$ *then* $s\,s' \in \mathbf{C} \cdot R_+ R_- + Z_0 + J_\alpha$ *provided that* $[h_2 + h_3 , s\,s'] = \varepsilon \cdot s\,s'$ *with* $\varepsilon = 0 , \pm 4$.

(3) *If* $\alpha_1 = 0$ *then* $s_{31}\, s_{42} \equiv -\dfrac{\alpha_2}{\alpha_2 + 2}\, s_{41}\, s_{32}$, $s_{24}\, s_{13} \equiv$ $-\dfrac{\alpha_2}{(\alpha_2 + 2)}\, s_{23}\, s_{14}$, $s_{31}\, s_{14} \equiv s_{24}\, s_{32}$, $s_{23}\, s_{42} \equiv s_{41}\, s_{13}$ $\mod J_\alpha$.

Proof. Since the mapping $P : S_0(\mathfrak{g}, \mathfrak{k}) \to U_1$ is injective modulo $U(\mathfrak{g})\, \mathfrak{k}_+$, it is sufficient to consider the projections $P(s_{ij}\, s_{kl})$ when proving the relations (1) – (3); this is a great simplification in the computations. By a direct calculation,

$$s_{41}\, s_{42} = s_{42}\, s_{41} , \qquad s_{31}\, s_{32} = s_{32}\, s_{31} ,$$
$$s_{23}\, s_{13} = s_{13}\, s_{23} , \qquad s_{24}\, s_{14} = s_{14}\, s_{24} ,$$

so that we are left with twelve pairs from the total sixteen. We consider as an example the pair $s_{41}\, s_{14}$. We have three different cases:

a) $\alpha_1 \geq 1$. After brute calculations one gets

$$s_{41}\, s_{14} \equiv \frac{1}{\alpha_1} \cdot s_{42}\, s_{24} + \frac{\alpha_1 + 1}{\alpha_1\, (\alpha_2 + 1)}\, s_{13}\, s_{31} + \frac{(\alpha_1 + 1)\, (\alpha_2 + 2)}{\alpha_1 (\alpha_2 + 1)} \cdot s_{14}\, s_{41} + a \cdot \mathbf{1} \qquad \mod J_\alpha$$

where $a \in \mathbf{C}$ depends on α ; we have been too lazy to compute it (it is not necessary to know the value of a here).

b) $\alpha_1 = 0$, $\alpha_2 \geq 1$. Comparing $P(s_{41}\, s_{14})$ with $P(z_2)$ and $P(z_3)$ one sees that

$$s_{41}\, s_{14} \equiv (1/4)\, (\alpha_2 + \alpha_3 + \alpha_4 + 14/3)\, z_2 + (1/3)\, z_3 + a \cdot \mathbf{1} \qquad \mod J_\alpha$$

for some $a \in \mathbf{C}$ depending on α .

c) $\alpha_1 = \alpha_2 = 0$. In the same way as above one gets

$$s_{41}\, s_{14} \equiv (1/2)\, z_2 + a \cdot \mathbf{1} \qquad \mod J_\alpha .$$

The remaining terms $s\, s'$ $(s \in R_-\, ,\ s' \in R_+)$ are treated in a similar way.

4. Non-singular $\mathfrak{k}$-finite $\mathfrak{g}$-modules

For $\mathfrak{g} = \mathfrak{gl}(4, \mathbf{C})$, $\mathfrak{k} = \mathfrak{gl}(2, \mathbf{C}) \oplus \mathfrak{gl}(2, \mathbf{C})$ let again

$$A_{\beta,\alpha} = \{ u \in U(\mathfrak{g}) \mid u\, V_\alpha^+ \subset V_\beta^+ \text{ for any } \mathfrak{g}\text{-module } V \} ,$$
$$M_\alpha = \sum_{\beta < \alpha} A_{\beta,\alpha} , \qquad D_\alpha = D \,/\, D \cap U(\mathfrak{g})\, M_\alpha$$

where D is the centralizer of $\mathfrak{h}$ in $S_0(\mathfrak{g}, \mathfrak{k})$.

T h e o r e m 2. *If* $\alpha \in \Lambda$ *such that* $\alpha_1 \geq 1$ *then* $D_\alpha \cong \mathbf{C}$.

Proof. Consider an arbitrary monomial $s = s_{ij} s_{kl} \dots s_{nm}$ in D . If the last factor on the right belongs to R_- then $s \equiv 0 \mod M_\alpha$. Thus let us assume that the last factor belongs to R_+ . Since s commutes with $\mathfrak{h}$, not all factors can be in R_+ ; let s_{ba} be the last one in R_+ , when reading from right to left, and let s_{dc} be the first one in R_- ,

$$s = s_1 s_{dc} s_{ba} s_2$$

where s_2 contains only elements from R_+ . Using Lemma 1 we can write

$$s = \sum_{t \in R_+} \sum_{t' \in R_-} a_{tt'} \cdot s_1 t t' s_2 + \text{terms of lower degree} \qquad \mod M_\alpha$$

where $a_{tt'} \in \mathbf{C}$. Using induction on the degree of s and s_2 one sees that the elements from R_- can be shifted to the left giving zero $\mod M_\alpha$. It follows that any element of D is in $\mathbf{C} \cdot \mathbf{1} \mod M_\alpha$. On the other hand, it is easy to see that $\mathbf{1} \notin D \cap U(\mathfrak{g}) M_\alpha$.

C o r o l l a r y . *For each* $\alpha \in \Lambda$ *such that* $\alpha_1 \geq 1$ *there exists a unique equivalence class* $[V]$ *of irreducible* $\mathfrak{k}$*-finite* $\mathfrak{g}$*-modules* V *such that* V_α *is minimal in* V . *Furthermore, the minimal component* V_α *is uniquely determined i.e.* $V_\beta = 0$ *if* $\alpha_1 = \beta_1$, $\alpha_2 = \beta_2$ *but* $\alpha \neq \beta$.

Proof. The first part follows directly from Theorems 1 and 2. As was proven in [5], any element in V_β^+ can be written as a linear combination of elements $s_{ij} s_{kl} \dots s_{nm} x = s x$ where $s_{ij}, \dots, s_{nm} \in R_- \cup R_+$ and $0 \neq x \in V_\alpha^+$. If now $\alpha_1 = \beta_1$, $\alpha_2 = \beta_2$, the element s has to commute with h_1 and h_2 ; it follows that there are as many factors from R_- as from R_+ in s . The reduction modulo M_α used in the proof of Theorem 2 can be applied and it follows that $s x = a x$ for some $a \in \mathbf{C}$ i.e. $s x = 0$ if $\alpha \neq \beta$.

Let V be a finite-dimensional irreducible $\mathfrak{g}$-module. Using the well-known results about reducing $\mathfrak{gl}(4, \mathbf{C})$ -modules with respect to $\mathfrak{gl}(2, \mathbf{C}) \oplus \mathfrak{gl}(2, \mathbf{C})$ it is seen that there always exists $V_\alpha \subset V$ with $\alpha_1 = 0$. Thus all $\mathfrak{g}$-modules described in Corollary of Theorem 2 are infinite-dimensional.

We call the modules of the Corollary above non-singular because they are completely labelled by the minimal component V_α .

5. The 1-singular case

In this section we classify all the irreducible $\mathfrak{k}$-finite $\mathfrak{g}$-modules with minimal component V_α such that $\alpha_1 = 0$, $\alpha_2 \geq 2$.

T h e o r e m 3. *If* $\alpha \in \Lambda$ *such that* $\alpha_1 = 0$, $\alpha_2 \geq 2$ *then* $D_\alpha \cong Z_0$.

Proof. 1) Using the same argument as in the proof of Theorem 2 it is seen that any element in $D_\alpha = D \,/\, D \cap U(\mathfrak{g})\, M_\alpha$ can be written as a linear combination of terms of the type $u_1 \dots u_n + D \cap U(\mathfrak{g})\, M_\alpha$ where $u_1, \dots, u_n \in R_- R_+$. We denote

$$u_+ = s_{31}\, s_{14}\,, \qquad u_- = s_{23}\, s_{42}\,,$$
$$u_0 = s_{41}\, s_{32}\,, \qquad u_0' = s_{23}\, s_{14}\,.$$

Using (2) and (3) in Lemma 1 it is shown that it is sufficient to study only products of the elements $u_\pm$, u_0 and u_0' modulo M_α.

2) By a direct calculation one can show that

$$u_0\, u_+ \equiv \frac{\alpha_2 + 4}{\alpha_2 + 2}\, u_+\, u_0 \tag{1}$$

$$u_0\, u_- \equiv \frac{\alpha_2}{\alpha_2 + 2}\, u_-\, u_0 \tag{2}$$

$$u_0'\, u_+ \equiv \frac{\alpha_2 + 4}{\alpha_2 + 2}\, u_+\, u_0' \tag{3}$$

$$u_0'\, u_- \equiv \frac{\alpha_2}{\alpha_2 + 2}\, u_-\, u_0' \tag{4}$$

(5) $\alpha_2^2\,(\alpha_2 - 1)\, u_0\, u_0' \equiv -(\alpha_2 + 2)^2\, \alpha_2\, u_+\, u_- - (1/9)\,(\alpha_2 - 1)\,(\alpha_2 + 2)^2\, [x_3 - (1/2)\,(3\,\alpha_0 + 4)\, x_2 - 3\,\alpha_2\,(\alpha_0 - \alpha_4 + 2)]\, [x_3 - (1/2)\,(3\,\alpha_0 - 3\,\alpha_2 + 4)\, x_2]$

(6) $\alpha_2^2\,(\alpha_2 - 1)\, u_0'\, u_0 \equiv -(\alpha_2 + 2)^2\, \alpha_2\, u_+\, u_- - (1/9)\,(\alpha_2 - 1)\,(\alpha_2 + 2)^2\, [x_3 - (1/2)\,(3\,\alpha_0 - 3\,\alpha_2 + 4)\, x_2 + 3\,\alpha_2\,(\alpha_0 - \alpha_2 - \alpha_4)]\, [x_3 - (1/2)\,(3\,\alpha_0 + 4)\, x_2 - 6\,\alpha_2\,(\alpha_0 - \alpha_4 + 1)]$

(7) $\alpha_2\,(\alpha_2 - 1)\, u_-\, u_+ \equiv (\alpha_2^2 + 3\,\alpha_2 + 4)\, u_+\, u_- + z(\alpha) \quad \mathrm{mod}\, J_\alpha$

where $z(\alpha)$ is some element in Z_0 and

$$\alpha_0 = (1/2)\,(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)$$
$$x_2 = z_2 - \gamma(z_2)\,, \qquad x_3 = z_3 - \gamma(z_3)$$

and $\gamma : Z(\mathfrak{g}) \to U(\mathfrak{h})$ is the Harish-Chandra homomorphism.

From the first four equations it follows that any product of the elements $u_\pm$, u_0 and u_0' can be written in the form $a \cdot v\, w \ \mathrm{mod}\, J_\alpha$, where v contains only factors u_0, u_0' and w consists of $u_\pm$; a is a complex number. Using the commutation argument in the proof of Theorem 2 it follows from equation (7) that $w \in Z_0 + D \cap U(\mathfrak{g})\, M_\alpha$; note that the number of factors u_+ in w is equal to the number of the u_-'s because of $[h_2, u_\pm] = \pm 2 u_\pm$ and $0 = [h_2, v w] = v[h_2, w]$.

3) We have shown that any element of D is a polynomial of u_0, u_0', z_2 and z_3 modulo M_α. Let us consider a product v of the elements u_0 and u_0'. Now $[h_3, u_0] = 2 u_0$ and $[h_3, u_0'] = -2 u_0'$ so that h_3 commutes with v (and $v \in D$) only when the number of u_0's contained in v is equal to the number of the elements u_0'. Using equations (5) and (6) and the fact that $[h_2, u_-] = -2 u_- \in M_\alpha$, we conclude that $v \in Z_0$ mod M_α. Thus $D_\alpha \cong Z_0 / Z_0 \cap U(\mathfrak{g}) M_\alpha$.

4) Finally, we have to show that $Z_0 \cap U(\mathfrak{g}) M_\alpha = 0$. Sketch of the proof: Let $z = \sum u_i v_i$ be an element of $Z_0 \cap U(\mathfrak{g}) M_\alpha$ where the v_i's belong to M_α and the u_i's are elements of $U(\mathfrak{g})$. From the fact that $(U(\mathfrak{g}) x)^+ = S_0(\mathfrak{g}, \mathfrak{k}) x$ for any $\mathfrak{k}$-finite $\mathfrak{g}$-module V and for any $x \in V^+$ it follows that the u_i's and v_i's can be assumed to be in $S_0(\mathfrak{g}, \mathfrak{k})$. Let I_α be the annihilator in $U(\mathfrak{k})$ of a maximal vector in an irreducible $\mathfrak{k}$-module with maximal weight α; clearly $J_\alpha \subset U(\mathfrak{g}) I_\alpha$. If now $v \in M_\alpha$ such that $[h_1, v] = q \cdot v$ with $q < 0$, then $v \equiv 0 \mod U(\mathfrak{g}) I_\alpha$ because $\alpha_1 = 0$. Using this fact together with Lemma 1 and with the formulas (1)–(7) one can show that z must be of the form

$$\sum_{k=1}^{n} y_k (u_+ u_-)^k \qquad \mod U(\mathfrak{g}) I_\alpha$$

where $y_1, \ldots, y_n \in Z_0$. By a direct calculation one sees that

$$u_+ u_- \equiv (1/4)(\alpha_2^2 + 3 \alpha_2 + 4) z_4 + y \qquad \mod J_\alpha$$

where $y \in Z_0$. It follows that $z = w_1 z_4 + w_2 \mod U(\mathfrak{g}) I_\alpha$ where w_1, $w_2 \in Z_0$ and $w_1 = 0$ only if $w_2 = 0$ (in the case $\alpha_2 \geq 2$). On the other hand, it is easy to see that $U(\mathfrak{g}) I_\alpha \cap Z(\mathfrak{g}) = 0$ (when $\alpha_2 \geq 2$). Since the generators z_i of $Z(\mathfrak{g})$ are independent, we conclude that $z \in Z_0$ only when $z = 0$.

Combining Theorems 1 and 3 we get:

Corollary. *For each $\alpha \in \Lambda$ such that $\alpha_1 = 0$, $\alpha_2 \geq 2$, and for each pair (c_2, c_3) of complex numbers there exists a unique equivalence class $[V]$ of irreducible $\mathfrak{k}$-finite $\mathfrak{g}$-modules V such that V_α is minimal in V and z_i is represented by the scalar c_i ($i = 2, 3$).*

However, unlike in the case $\alpha_1 \geq 1$, not all of the equivalence classes described above are distinct. We denote by $V[\alpha; c_2, c_3]$ an irreducible $\mathfrak{g}$-module of the Corollary. A module $V = V[\alpha'; c_2', c_3']$ can be isomorphic with $V' = V[\alpha'; c_2', c_3']$ only if $c_2' = c_2$, $c_3' = c_3$ and $V_{\alpha'}$ is a minimal component in V_α, according to Theorem 1. This is possible only when $\alpha_2' = \alpha_2$ and $\alpha_3' + \alpha_4' = \alpha_3 + \alpha_4$ since $\alpha_3 + \alpha_4$ is the value of the central element z_1 in V. Let $0 \neq v \in V_\alpha^+$. Then $V \cong V'$ iff there exists

$s \in S_0(\mathfrak{g}, \mathfrak{k})$ such that $s v \neq 0$ and $h_i s v = \alpha'_i s v$ for $i = 1, 2, 3, 4$. In this case $[h_i, s] = 0$ for $i = 1, 2$ and $[h_i, s] = (\alpha'_i - \alpha_i) s$ for $i = 3, 4$. From the commutation relations of the h_i's with the generators s_{ij} of $S_0(\mathfrak{g}, \mathfrak{k})$ it follows that $\alpha'_3 - \alpha_3 = -(\alpha'_4 - \alpha_4)$ is an even integer.

In the following we denote by $x_i(\beta)$ the value of $x_i = z_i - \gamma(z_i)$ when z_i takes the value c_i ($i = 2, 3$) and $\gamma(z_i) \in U(\mathfrak{h})$ is evaluated at $h_j = \beta_j$ ($j = 1, 2, 3, 4$).

Now we have a complete description of equivalences between different modules $V[\alpha; c_2, c_3]$:

Theorem 4. *The irreducible* $\mathfrak{g}$*-modules* $V[\alpha; c_2, c_3]$ *and* $V[\alpha'; c_2, c_3]$ *with* $\alpha'_1 = \alpha_1 = 0$, $\alpha'_2 = \alpha_2 \geq 2$ *and* $\alpha'_3 - \alpha_3 = -(\alpha'_4 - \alpha_4)$ *an even integer (for example, let* $\alpha'_3 - \alpha_3 \geq 0$*) are equivalent if and only if*

$$x_3(\beta) - (1/2)(3\alpha_0 - 3\alpha_2 + 4)\, x_2(\beta) \neq 0 \text{ and } -x_3(\beta) + (1/2)(3\alpha_0 + 4)\, x_2(\beta) + 3\alpha_2(\alpha_0 - \beta_4 + 2) \neq 0 \text{ for any } \beta \in \Lambda \text{ with } \beta_1 = 0,\ \beta_2 = \alpha_2,\ \beta_3 + \beta_4 = \alpha_3 + \alpha_4 \text{ and } \beta_3 = \alpha'_3 - 2, \alpha'_3 - 4, \ldots, \alpha_3.$$

Proof. Let $\beta_1 = \alpha_1 = 0$, $\beta_2 = \alpha_2$, $\beta_3 = \alpha_3 + 2k$, $\beta_4 = \alpha_4 - 2k$ where k is a non-negative integer. Let $0 \neq v \in V^+_\alpha[\alpha; c_2, c_3]$. From equations (4) and (5) on page 8 it follows that

$$(*) \qquad u_0^{k+1} u_0'^{k+1} v = [x_3(\beta) - (1/2)(3\alpha_0 - 3\alpha_2 + 4)\, x_2(\beta)] \times [-x_3(\beta) + (1/2)(3\alpha_0 + 4)\, x_2(\beta) + 3\alpha_2(\alpha_0 - \beta_4 + 2)] \times u_0^k u_0'^k v.$$

If the conditions of the Theorem are fulfilled then $u_0^n u_0'^n v = a \cdot v$ where $0 \neq a \in \mathbf{C}$ and $n = (1/2)(\alpha'_3 - \alpha_3)$. Thus $v' = u_0'^n \neq 0$ and $V[\alpha'; c_2, c_3] \cong V[\alpha; c_2, c_3]$.

Assume then that $V[\alpha; c_2, c_3] \cong V[\alpha'; c_2, c_3]$. It follows that there exists $s \in S_0(\mathfrak{g}, \mathfrak{k})$ such that $s v \neq 0$ and $h_i s v = \alpha'_i s v$ ($1 \leq i \leq 4$). Using the same method as in the proof of Theorem 3 it is shown that $s v = a \cdot u_0'^n v$, where $a \in \mathbf{C}$. We denote $v' = s v$. A similar argument shows that there exists $s' \in S_0(\mathfrak{g}, \mathfrak{k})$ such that $s' v' = v$ and $s' v' = b \cdot u_0^n v'$ for some $b \in \mathbf{C}$; thus $u_0^n u_0'^n v \neq 0$ and the rest follows from the equation (*) above.

Next we ask: Which of the modules $V[\alpha; c_2, c_3]$ are finite-dimensional?

Let $W[\lambda]$ be an irreducible finite-dimensional $\mathfrak{g}$-module with maximal weight $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$, where we have defined $\lambda_i = \lambda(e_{ii})$. As is well-known, $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$ and all the differences $\lambda_i - \lambda_j$ are integers. Using the branching rules for the reduction $\mathfrak{gl}(4) \downarrow \mathfrak{gl}(2) \oplus \mathfrak{gl}(2)$ of finite-dimensional $\mathfrak{gl}(4)$-modules one sees that the minimal component W_α of $W = W[\lambda]$ has the weight

$$\begin{aligned} \alpha_1 &= 0 , \\ \alpha_2 &= |\lambda_2 + \lambda_3 - \lambda_1 - \lambda_4| , \\ \alpha_3 &= 2\lambda_3 , \\ \alpha_4 &= \lambda_1 + \lambda_2 + \lambda_4 - \lambda_3 , \end{aligned}$$

where $\alpha_i = \alpha(h_i)$. The values c_i of the central elements z_i in W are obtained using the Harish-Chandra homomorphism γ; $c_i = \gamma(z_i)(\lambda)$. In the case $\alpha_2 = |\lambda_2 + \lambda_3 - \lambda_1 - \lambda_4| \geq 2$ we have therefore $W[\lambda] \cong V[\alpha; c_2, c_3]$, where α, c_2 and c_3 are obtained from λ by the recipe above.

6. The 2-singular case

In order to avoid tautology, we shall in this section give results without proofs.

We consider now the cases $\alpha_1 = 0$, $\alpha_2 = 0$ or $\alpha_2 = 1$. As before,

$$D_\alpha \cong Z / Z \cap U(\mathfrak{g}) M_\alpha .$$

a) *The case* $\alpha_1 = \alpha_2 = 0$. *The algebra* $Z / Z \cap U(\mathfrak{g}) M_\alpha$ *is generated by* z_2 *and* z_4;

$$z_3 = (1/2)(3\alpha_0 + 4) z_2 - \alpha_0^3 - 2\alpha_0 \alpha_3 \quad \mod U(\mathfrak{g}) I_\alpha .$$

For each two pairs (c_2, c_4) and (α_3, α_4) of complex numbers there exists a unique equivalence class of irreducible $\mathfrak{k}$-finite $\mathfrak{g}$-modules $V[\alpha_3, \alpha_4, c_2, c_4]$ such that the central element z_i takes the value c_i $(i = 2, 4)$ in V and V_α with $\alpha = (0, 0, \alpha_3, \alpha_4)$ is a minimal $\mathfrak{k}$-type in V.

For most values of α_i's and c_i's the module $V[\alpha_3, \alpha_4, c_2, c_4]$ is equivalent with $V[\alpha_3 + 2n, \alpha_4 - 2n, c_2, c_4]$ where n is a positive or negative integer; the computations are rather tedious and we will not present them here.

Note that the irreducible $\mathfrak{g}$-modules, corresponding to the principal series of unitary irreducible representations of the pseudo-unitary group $U(2, 2)$, are all contained in this class.

b) *The case* $\alpha_1 = 0$, $\alpha_2 = 1$. The algebra D_α is now isomorphic with the subalgebra of Z generated by z_2, z_3 and z_4. For each pair (α_3, α_4) and each triple (c_2, c_3, c_4) there exists a unique equivalence class of $\mathfrak{k}$-finite $\mathfrak{g}$-modules $V = V[\alpha_3, \alpha_4, c_2, c_3, c_4]$ such that z_i takes the value c_i $(i = 2, 3, 4)$ in V and V_α with $\alpha = (0, 1, \alpha_3, \alpha_4)$ is a minimal $\mathfrak{k}$-type in V.

Again, except for special values of α_i's and c_i's, the modules $V[\alpha_3 + 2n, \alpha_4 - 2n, c_2, c_3, c_4]$ are equivalent when n is an arbitrary integer.

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University of Jyväskylä
Department of Mathematics
SF-40100 Jyväskylä 10
Finland

Received 3 June 1975