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COVERING PROPERTIES OF HARMONIC BL-MAPPINGS III

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1. Introduction

This article is devoted to the study of harmonic Bl-mappings between harmonic spaces satisfying the general axiomatics of Constantinescu and Cornea (see [5], p. 30). Actually we shall present an improvement of the contents of Section 2 in our earlier article [8].

Suppose $\mathscr{F}(X)$ (resp. $\mathscr{F}(X')$) is a mapping which associates a family $\mathscr{F}_U(X)$ (resp. $\mathscr{F}_{U'}(X')$) of numerical functions for all open sets U (resp. U') in a harmonic space X (resp. X'). We say that a continuous mapping $\varphi : X \to X'$ between two harmonic spaces X and X' inversely preserves the above collections of families of functions, if $f' \circ \varphi \in \mathscr{F}_{\varphi^{-1}(U')}(X)$, whenever $U' \subseteq X'$ is an open set such that $\varphi^{-1}(U') \neq \emptyset$ and $f' \in \mathscr{F}_{U'}(X')$. In this notation, the earlier presentations mainly define a harmonic mapping as a continuous mapping which inversely preserves harmonic functions (see e.g. [3], [8]–[11]). These articles have been devoted to harmonic mappings either between Brelot spaces ([3], [8], [9]) or between Bauer spaces satisfying the convergence axiom of Doob ([10], [11]).

In the general axiomatics the inverse preservation of the sheaf of harmonic functions is too weak a condition to give a sufficiently interesting class of continuous mappings. Therefore we define in this article harmonic mappings as continuous mappings which inversely preserve the sheaf of hyperharmonic functions. In a certain sense the idea for this definition, which reduces back to the usual one whenever X' is a Bauer space (in the sense of [5]), goes back to Sibony ([11], p. 91 and Définition 19). The same idea has been used also by Fuglede to define finely harmonic mappings [6]. Our definition of a harmonic mapping and its basic consequences are introduced in the second section. The third and fourth section contain some results on covering properties of Bl-mappings similar to those presented in Section 2 in [8]. However, the definition of a normal map-

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ping is now slightly more general. It appears, roughly speaking, that the exceptional covering set, which was a polar set in [8], is now either polar or else possesses a non-empty interior. As soon as the range space X' is elliptic, the exceptional set reduces to a polar set. In the fifth section similar results will be proved for open Bl-mappings.

2. The definition of harmonic mappings

Lemma 2.1. (See also [2], Satz 2.1.1.) *Let $\mathcal{V}$ be a base for the neighbourhoods of a point $x \in X$ such that $\mathcal{V}$ contains only relatively compact resolutive neighbourhoods of x and let $s : V \to \overline{\mathbf{R}}$ be a hyperharmonic function on a neighbourhood V of x. Then*

$$s(x) = \sup_{W \in \mathcal{V}} \mu^W s(x) .$$

Proof. We note that

$$s(x) \geqq \sup_{W \in \mathcal{V}} \mu^W s(x)$$

trivially. To prove the converse inequality, let $\alpha < s(x)$. We may assume that there is a strictly positive harmonic function h on $W \in \mathcal{V}$, cl $W \subseteq V$, such that $h(x) = 1$. Thus we get $\alpha\, h(x) < s(x)$. We may also assume that $\alpha\, h < s$ on cl W. Then

$$\alpha = \alpha\, h(x) = \alpha\, \mu^W h(x) = \mu^W(\alpha\, h)(x) \leqq \mu^W s(x)$$

and so

$$\alpha \leqq \sup_{W \in \mathcal{V}} \mu^W s(x) .$$

The lemma follows.

Definition 2.2. *A continuous mapping $\varphi : X \to X'$ inversely preserving the sheaf of hyperharmonic functions is a harmonic mapping.*

Remark 2.3. Obviously a harmonic mapping inversely preserves also the sheaf of harmonic functions. The converse result is not true in general. In fact, let us consider two harmonic spaces $\mathbf{R}_1 = (\mathbf{R}, \mathcal{U}_1)$ and $\mathbf{R}_2 = (\mathbf{R}, \mathcal{U}_2)$ over the real axis $\mathbf{R}$ where the hyperharmonic sheaf $\mathcal{U}_1$ (resp. $\mathcal{U}_2$) is formed by lower semi-continuous, lower finite functions $u : U \to \overline{\mathbf{R}}$ on the open sets $U \subseteq \mathbf{R}$ which are increasing (resp. decreasing) on the components of U (see [5], Theorem 2.1.2). We denote by $g : \mathbf{R}_1 \to \mathbf{R}_2$ the ordinary exponential function. Obviously g inversely preserves the harmonic sheaf, but it does not inversely preserve non-constant hyperharmonic functions.

T h e o r e m 2.4. *A continuous mapping* $\varphi : X \to X'$ *into a Bauer space* X' *inversely preserving the harmonic sheaf is a harmonic mapping.*

Proof. This proof follows the well-known idea used in [3] to prove the corresponding Theorem 3.1.

We may consider the inverse preserving property for positive continuous superharmonic functions s' on a $\mathscr{P}$-set $U' \subseteq X'$ only. In fact, let s' be hyperharmonic on U' and let $\{ V'_\alpha \subseteq U' \mid \alpha \in I \}$ be an open cover of U' by $\mathscr{P}$-sets relatively compact in U' such that on any $V'_{\alpha'}$, $\alpha \in I$, there is a bounded harmonic function h' which satisfies $\inf h'_\alpha(V'_\alpha) > 0$. Hence there is $b'_\alpha \in \mathbf{R}$ such that $u' = s' + b'_\alpha h'_\alpha \geqq 0$ on V'_α. Since φ inversely preserves harmonic functions, then

$$s' \circ \varphi = (s' + b'_\alpha h'_\alpha) \circ \varphi - b'_\alpha h'_\alpha \circ \varphi = u' \circ \varphi - b'_\alpha h'_\alpha \circ \varphi$$

is hyperharmonic on $\varphi^{-1}(V'_\alpha) \neq \emptyset$ as soon as this is true for positive hyperharmonic functions. Moreover, we get by [5], Corollary 2.3.1,

$$u' \circ \varphi = (\sup_{\beta \in J} u'_\beta) \circ \varphi = \sup_{\beta \in J} (u'_\beta \circ \varphi)$$

on $\varphi^{-1}(V'_\alpha)$, where $\{ u'_\beta \mid \beta \in J \}$ is the family of positive continuous superharmonic functions dominated by u' on V'_α. Hence the theorem follows.

We now proceed under the additional assumptions. Let $U \neq \emptyset$ be a relatively compact set in $\varphi^{-1}(U') \neq \emptyset$, hence $\varphi(\mathrm{cl}\, U) \subseteq U'$ is a compact non-empty set. Let us consider the collection $\{ \mathscr{V}' \}$ of all finite open covers $\mathscr{V}'$ of $\varphi(\mathrm{cl}\, U)$ by regular sets which are relatively compact in U'. Let $\mathscr{V}'$ be such an open cover. If $V' \in \mathscr{V}'$ and $y' \in \partial V'$, then

$$\liminf_{V' \ni x' \to y'} \mu^{V'} s'(x') \geqq s'(y'),$$

since V' is regular. Therefore the Poisson modification $s'_{V'}$ of s' defined in U', which is hyperharmonic by [5], Corollary 2.1.2, takes the form

$$x' \mapsto s'_{V'}(x') = \begin{cases} s'(x') & \text{for } x' \in U' \setminus V' \\ \mu^{V'} s'(x') & \text{for } x' \in V'. \end{cases}$$

Let us define a hyperharmonic function $s'_{\mathscr{V}'}$ on U' by

$$x' \mapsto s'_{\mathscr{V}'}(x') = \inf_{V' \in \mathscr{V}'} s'_{V'}(x') = \begin{cases} s'(x') & \text{for } x' \in U' \setminus \bigcup_{V' \in \mathscr{V}'} V' \\ \inf_{x' \in V' \in \mathscr{V}'} \mu^{V'} s'(x') & \text{for } x' \in \bigcup_{V' \in \mathscr{V}'} V'. \end{cases}$$

Obviously $s'_{\mathscr{V}'} \circ \varphi$ is lower semi-continuous on U. To prove its hyperharmonicity, let $x \in U$ and let V be a relatively compact resolutive set such that

$$x \in V, \qquad \mathrm{cl}\, V \subseteq \bigcap_{\varphi(x) \in V' \in \mathscr{V}'} \varphi^{-1}(V') \cap U .$$

We then define $t: \bigcap_{\varphi(x) \in V' \in \mathscr{V}'} \varphi^{-1}(V') \to \overline{\mathbf{R}}$ as follows:

$$z \mapsto t(z) = \inf_{\varphi(x) \in V' \in \mathscr{V}'} s'_{V'}(\varphi(z)) = \inf_{\varphi(x) \in V' \in \mathscr{V}'} \mu^{V'} s'(\varphi(z)) .$$

Since s' is superharmonic on U', $(\mu^{V'} s') \circ \varphi$ is harmonic on $\varphi^{-1}(U')$, hence t is superharmonic. Moreover, $t \geqq s'_{\mathscr{V}'} \circ \varphi$ and $t(x) = s'_{\mathscr{V}'}(\varphi(x))$. Therefore

$$\mu^V (s'_{\mathscr{V}'} \circ \varphi)(x) \leqq \mu^V t(x) \leqq t(x) = s'_{\mathscr{V}'}(\varphi(x)) ,$$

hence $s'_{\mathscr{V}'} \circ \varphi$ is hyperharmonic on U by [5], Corollary 2.3.4.

The family $\{ s'_{\mathscr{V}'} \}$ restricted into $\varphi(\mathrm{cl}\, U)$ is upper directed. In fact, since U' is a $\mathscr{P}$-set, the intersection of any two regular sets in U' is again regular ([5], Corollary 6.3.8). Therefore, if $\mathscr{V}'_1$ and $\mathscr{V}'_2$ are two open covers in the collection $\{ \mathscr{V}' \}$, then

$$\mathscr{W}' = \{ V'_1 \cap V'_2 \neq \emptyset \mid V'_1 \in \mathscr{V}'_1 ,\ V'_2 \in \mathscr{V}'_2 \}$$

belongs to the same collection. If we define, for $W' = V'_1 \cap V'_2 \in \mathscr{W}'$,

$$x' \mapsto s'_0(x') = \begin{cases} s'(x') & \text{for } x' \in \partial W' \cap \partial V'_1 \\ \mu^{V_1'} s'(x') & \text{for } x' \in \partial W' \cap V'_1 , \end{cases}$$

then, by [5], Proposition 2.4.4, we get

$$\mu^{V_1'} s' = H^{V_1'}_{s'} = \mathrm{cl}\, H^{W'}_{s_0'} \leqq H^{W'}_{s'} = \mu^{W'} s'$$

on W', hence obviously

$$s'_{\mathscr{W}'} \geqq \max (s'_{\mathscr{V}'_1} , s'_{\mathscr{V}'_2})$$

on $\varphi(\mathrm{cl}\, U)$.

Finally we get

$$s' = \sup_{\mathscr{V}'} s'_{\mathscr{V}'} ,$$

on $\varphi(\mathrm{cl}\, U)$. In fact, if $x' \in \varphi(\mathrm{cl}\, U)$ and $\alpha < s'(x')$, then we may take a regular neighbourhood W' of x' relatively compact in U' such that $\mu^{W'} s'(x') > \alpha$ by Lemma 2.1. Obviously we may construct a finite open cover $\mathscr{V}'$ of $\varphi(\mathrm{cl}\, U)$ in the collection $\{ \mathscr{V}' \}$ such that if $x' \in \mathrm{cl}\, V'$ and $V' \in \mathscr{V}'$, then $V' = W'$. Then we get

$$\mu^{W'} s'(x') = s'_{\mathscr{V}'}(x') .$$

Therefore

$$s' \circ \varphi = (\sup_{\mathscr{V}'} s'_{\mathscr{V}'}) \circ \varphi = \sup_{\mathscr{V}'} (s'_{\mathscr{V}'} \circ \varphi)$$

is hyperharmonic on U, hence by the sheaf property of hyperharmonic functions on $\varphi^{-1}(U')$.

Theorem 2.5. *If $\varphi : X \to X'$ is a homeomorphic harmonic mapping into a Bauer space X', then $\varphi^{-1} : X' \to X$ is a harmonic mapping.*

Proof. (See [3], Theorem 3.4.) Let $U \subseteq X$ be an open set and let $s : U \to \overline{\mathbf{R}}$ be a hyperharmonic function. Then $s \circ \varphi^{-1}$ is hyperharmonic on $\varphi(U)$. To prove this, we may assume that s is a finite continuous superharmonic function. In fact, by [5], Corollary 2.3.1, we may take an open cover of U by relatively compact $\mathscr{P}$-sets $W_\alpha \subseteq U$, $\alpha \in I$, such that $s \mid W_\alpha$ can be represented as the supremum of its continuous superharmonic minorants $\{ s_\beta \mid \beta \in J \}$. Hence, if $s_\beta \circ \varphi^{-1}$ is hyperharmonic on $\varphi(W_\alpha)$ for all $\beta \in J$, then

$$s \circ \varphi^{-1} = (\sup_{\beta \in J} s_\beta) \circ \varphi^{-1} = \sup_{\beta \in J} (s_\beta \circ \varphi^{-1})$$

is hyperharmonic on $\varphi(W_\alpha)$ and by the sheaf property of hyperharmonic functions on the whole $\varphi(U)$.

Let now $V' \subseteq \varphi(U)$ be an open relatively compact resolutive regular set such that $\varphi^{-1}(V')$ is a relatively compact MP-set. Since $s \circ \varphi^{-1}$ is continuous on $\partial V'$, then $f^* : \operatorname{cl} V' \to \mathbf{R}$ defined as

$$x' \mapsto f^*(x') = \begin{cases} \mu_{x'}^{V'}(s \circ \varphi^{-1}) & \text{for } x' \in V' \\ s \circ \varphi^{-1}(x') & \text{for } x' \in \partial V' \end{cases}$$

is its continuous extension into $\operatorname{cl} V'$ which is harmonic on V'. Therefore

$$s(y) = s(\varphi^{-1}(\varphi(y))) = f^*(\varphi(y)) = \lim_{V' \ni x' \to \varphi(y)} f^*(x') = \lim_{\varphi^{-1}(V') \ni x \to y} (f^* \circ \varphi)(x)$$

and so

$$\liminf_{\varphi^{-1}(V') \ni x \to y} (s - f^* \circ \varphi)(x) \geqq s(y) - \lim_{\varphi^{-1}(V') \ni x \to y} (f^* \circ \varphi)(x) = 0$$

for all $y \in \partial \varphi^{-1}(V')$. Since $\varphi^{-1}(V')$ is, by assumption, a relatively compact MP-set, we get $s - f^* \circ \varphi \geqq 0$ on $\varphi^{-1}(V')$. Hence

$$\mu^{V'}(s \circ \varphi^{-1}) = f^* \leqq s \circ \varphi^{-1} \text{ on } V'.$$

Since the open sets V' satisfying the conditions described above form a base for the topology of $\varphi(U)$, $s \circ \varphi^{-1}$ is hyperharmonic on $\varphi(U)$ by [5], Corollary 2.3.4. The theorem follows.

3. Covering properties of harmonic Bl-mappings

Definition 3.1. *A harmonic mapping* $\varphi : X \to X'$ *inversely preserving locally bounded potentials is a Bl-mapping.*

Theorem 3.2. *If* $\varphi : X \to X'$ *is a harmonic Bl-mapping, if* $U' \subseteq X'$ *is a* $\mathscr{P}$*-domain such that* $\varphi^{-1}(U') \neq \emptyset$, *if* V *is the union of a non-empty subfamily of the components of* $\varphi^{-1}(U')$ *and if* $\varphi(V)$ *is an open set, then either* $U' \setminus \varphi(V)$ *is a polar set in* U' *or else* $\mathrm{cl}\,(\varphi(V))$ *is a non-trivial absorbent set in* U' .

Proof. The notation φ in this proof means the restricted mapping $\varphi : V \to U'$. We denote by $\mathscr{P}_0'$ the family of all locally bounded potentials on U' not vanishing identically on $E_0' = U' \setminus \varphi(V)$. Let us assume that E_0' is a non-polar set in U' . By assumption E_0' is closed in U' . If there is $x' \in U' \setminus E_0'$ and $p' \in \mathscr{P}_0'$ such that

$$q'(x') = (\hat{R}_{p'}^{E_0'})_{U'}(x') > 0 ,$$

then $q' \circ \varphi$ is a potential on V not vanishing identically, since $x' \in \varphi(V)$. On the other hand, q' is harmonic on $U' \setminus E_0'$ by [5], Proposition 5.3.1, and therefore $q' \circ \varphi$ is harmonic on V , a contradiction.

Hence we must have $q'(x') = 0$ for any point $x' \in U' \setminus E_0'$ and any $p' \in \mathscr{P}_0'$. Actually, we shall prove that

$$(\hat{R}_{p'}^{E_0'})_{U'} = 0 \quad \text{on } U' \setminus \mathrm{int}\, E_0' = U' \setminus \mathrm{int}_f\, E_0' \subset U'$$

for all $p' \in \mathscr{P}_0'$. This implies, by [5], Proposition 5.3.1,

$$(\hat{R}_{p'}^{E_0'})_{U'} = \begin{cases} p' & \text{on } \mathrm{int}\, E_0' \\ 0 & \text{on } U' \setminus \mathrm{int}\, E_0' , \end{cases}$$

hence there is an absorbent set $F'(p') \supseteq U' \setminus \mathrm{int}\, E_0'$. Since U' is a $\mathscr{P}$-set,

$$\mathrm{cl}\,(\varphi(V)) = \mathrm{cl}\,(U' \setminus E_0') = U' \setminus \mathrm{int}\, E_0' = \bigcap_{p' \in \mathscr{P}_0'} F'(p')$$

is a non-trivial absorbent set in U' ([5], Proposition 6.1.2). To complete the proof we observe, by [5], Proposition 7.1.2 and Proposition 7.1.5, that there is $p' \in \mathscr{P}_0'$ such that

$$(\hat{R}_{p'}^{E_0'})_{U'} \not\equiv 0 ,$$

since E_0' is non-polar. Further, $\mathrm{int}\, E_0' \neq \emptyset$, since otherwise $E_0' = \partial E_0'$ and there is $x' \in U' \setminus E_0'$ such that

$$(\hat{R}_{p'}^{E_0'})_{U'}(x') > 0 .$$

If now

$$(\hat{R}_{p'}^{E_0'})_{U'}(y') > 0$$

for some $y' \in \partial E_0'$ and some $p' \in \mathscr{P}_0'$, then also there is $x' \in U' \setminus E_0'$ such that

$$(\hat{R}_{p'}^{E_0'})_{U'}(x') > 0 .$$

Therefore we get, for any $p' \in \mathscr{P}_0'$,

$$(\hat{R}_{p'}^{E_0'})_{U'} = 0 \quad \text{on } \partial E_0' ,$$

hence

$$(\hat{R}_{p'}^{E_0'})_{U'} = 0 \quad \text{on } U' \setminus \operatorname{int} E_0' .$$

If $y' \in \partial E_0'$, then there is $p' \in \mathscr{P}_0'$ such that

$$p'(y') > (\hat{R}_{p'}^{E_0'})_{U'}(y') = 0 ,$$

since U' is a $\mathscr{P}$-set. Hence, by [5], Proposition 5.3.1, $\partial E_0' \subseteq E_0' \setminus \operatorname{int}_f E_0'$, which implies $\operatorname{int} E_0' = \operatorname{int}_f E_0'$. The theorem follows.

Theorem 3.3. (See also [3], Corollary 3.7.) *If $\varphi : X \to X'$ is a harmonic Bl-mapping, if $U' \subseteq X'$ is an elliptic $\mathscr{P}$-domain such that $\varphi^{-1}(U') \neq \emptyset$ and if V is the union of a non-empty subfamily of the components of $\varphi^{-1}(U')$, then every closed set $F' \subseteq U' \setminus \varphi(V)$ is a polar set in U'.*

Proof. Using the notations in the proof of the preceding theorem we note that every potential $p' \in \mathscr{P}_0'$ is strictly positive. Let us assume that there is a non-polar closed set $F' \subseteq U' \setminus \varphi(V)$. We may assume that F' is compact ([4], Theorem 8 and [5], Exercise 6.2.6). There is a potential $p' \in \mathscr{P}_0'$ such that

$$q' = (\hat{R}_{p'}^{F'})_{U'}$$

is a strictly positive potential on U' ([5], Proposition 7.1.2 and Proposition 7.1.5), hence $q' \circ \varphi$ is a strictly positive potential on V. This is a contradiction, since q' is harmonic on $U' \setminus F'$ and so $q' \circ \varphi$ is harmonic on V.

Theorem 3.4. *If $\varphi : X \to X'$ is a harmonic Bl-mapping, if $U' \subseteq X'$ is a one-dimensional orientated $\mathscr{P}$-Bauer domain such that $\varphi^{-1}(U') \neq \emptyset$, if V is the union of a non-empty subfamily of the components of $\varphi^{-1}(U')$ and if $\varphi(V)$ is an open set, then $U' \setminus \varphi(V) = \emptyset$ and $\varphi : V \to U'$ is a surjective mapping.*

Proof. By [7], Theorem 2.2, there is an open dense subset V' of U' such that V' is an elliptic Bauer space, hence all of its components V_α', $\alpha \in I$, are elliptic $\mathscr{P}$-domains. Let again φ denote the restricted mapping $\varphi : V \to U'$. By Theorem 3.3, for the mappings $\varphi : \varphi^{-1}(V_\alpha') \to V_\alpha'$, $\alpha \in I$, any closed set in $V_\alpha' \setminus \varphi(\varphi^{-1}(V_\alpha'))$ is polar in V_α', respectively. Since V_α' is a one-dimensional manifold, all of its points are non-polar, hence $\varphi(\varphi^{-1}(V'_\alpha)) = V'_\alpha$ for all $\alpha \in I$. Therefore we get $\varphi(\varphi^{-1}(V')) = V'$. Since V' is dense in U',

$$U' = \mathrm{cl}\, V' = \mathrm{cl}\,(\varphi(\varphi^{-1}(V'))) \subseteq \mathrm{cl}\,(\varphi(V)) \subseteq U' .$$

Therefore $U' \setminus \varphi(V)$ is a polar set in U' by Theorem 3.2. Since U' is a one-dimensional manifold, $\varphi(V) = U'$.

Theorem 3.5. *If $\varphi : X \to X'$ is a harmonic Bl-mapping, if $U' \subseteq X'$, $\varphi^{-1}(U') \neq \emptyset$, is a domain possessing a pseudoexhaustion $\{ U'_i \mid i \in \mathbf{N} \}$ by elliptic $\mathscr{P}$-domains (for the definition of a pseudo-exhaustion, see* [4], *p.* 382*) and if V is the union of a non-empty subfamily of the components of $\varphi^{-1}(U')$, then every closed set $F' \subseteq U' \setminus \varphi(V)$ is a polar set in U'.*

Proof. Since $\{ U'_i \mid i \in \mathbf{N} \}$ is a pseudoexhaustion,

$$U' \setminus V' = U' \setminus \bigcup_{i=1}^{\infty} U'_i$$

is a polar set in U'. Let φ denote the restricted mapping $\varphi : V \to U'$ and denote $A'_i = U'_i \setminus \varphi(\varphi^{-1}(U'_i))$ for every $i \in \mathbf{N}$. If $F' \subseteq U' \setminus \varphi(V)$ is a closed set, then $F' \cap U'_i \subseteq A'_i$ is a closed set in U'_i. Since U'_i is an elliptic $\mathscr{P}$-domain, $F' \cap U'_i$ is a polar set in U'_i by Theorem 3.3. By [5], p. 142, there is an open cover $\mathscr{W}'_i$ of U'_i such that for every $W' \in \mathscr{W}'_i$ we get $W' \subseteq U'_i$ and

$$(\hat{R}_\infty^{(F' \cap U_i') \cap W'})_{W'} = 0 .$$

The family of sets $\mathscr{W} = \cup \{ \mathscr{W}'_i \mid i \in \mathbf{N} \}$ is an open cover of V'. If $W' \in \mathscr{W}$, then $W' \in \mathscr{W}'_i$ for some $i \in \mathbf{N}$, hence

$$(\hat{R}_\infty^{(F' \cap V') \cap W'})_{W'} = (\hat{R}_\infty^{(F' \cap U_i') \cap W'})_{W'} = 0 .$$

Therefore $F' \cap V'$ is a polar set in V'. By the definition of a pseudo-exhaustion, V' is a K_σ-set, hence $F' \cap V'$ is polar in U' ([5], Exercise 6.2.2). Since $F' \setminus V'$ is polar in U', the theorem follows.

Remark 3.6. In the above Theorem 3.2, both of the given possibilities can actually appear. In fact, let the real axis $\mathbf{R}$ be endowed with the decreasing hyperharmonic sheaf introduced in Remark 2.3. Then the ordinary exponential function $g : \mathbf{R} \to \mathbf{R}$ is an open harmonic Bl-mapping such that $\mathrm{cl}\,(g(\mathbf{R})) = \mathbf{R}_+ \cup \{0\}$ is a non-trivial absorbent set in $\mathbf{R}$. This example also demonstrates that Theorem 3.4 does not hold for general one-dimensional harmonic spaces. We also note that $\mathrm{cl}\,(g(\mathbf{R}))$ is not an absorbent set in $\mathbf{R}$, if $\mathbf{R}$ is endowed with the corresponding increasing hyperharmonic sheaf. This is explained immediately by the fact that g is not a Bl-mapping in this case.

T h e o r e m 3.7. *Let, under the assumptions of Theorem* 3.2, $F' = \mathrm{cl}\,(\varphi(V))$ *be a non-trivial absorbent set and let* F' *be endowed with the induced hyperharmonic sheaf (see* [1], *p.* 893 *and* [5], *Exercise* 6.1.8*). Then* $\varphi: V \to F'$ *is a harmonic mapping. If* F' *is connected, then* $F' \setminus \varphi(V)$ *is a polar set in the harmonic space* F' *if and only if* $\varphi: V \to F'$ *is a Bl-mapping.*

Proof. In this proof we shall speak about F'-hyperharmonic functions, U'-potentials etc., the meaning of these notions being self-evident.

If $W' \subseteq F'$ is open in F', then $W' = V' \cap F'$ for some open set V' in U'. Hence W' is closed and finely open in V'. If h' is F'-hyperharmonic on W', then $(R_{h'}^{W'})_{V'}$ is U'-hyperharmonic on V' by [5], Corollary 5.1.3. Thus $(R_{h'}^{W'})_{V'} \circ \varphi = h' \circ \varphi$ is hyperharmonic on $\varphi^{-1}(W') = \varphi^{-1}(V')$, hence $\varphi: V \to F'$ is a harmonic mapping.

Let us assume now that F' is connected. If $\varphi: V \to F'$ is a Bl-mapping, then the assertion follows by Theorem 3.2, since F' is a $\mathscr{P}$-domain. If, on the other hand, $F' \setminus \varphi(V)$ is a polar set in the connected space F', then let p' be a locally bounded F'-potential on $W' = V' \cap F'$ open in F'. Let us denote $q' = p' \mid W' \cap \varphi(V) = p' \mid V' \cap \varphi(V)$ and let $h' \geqq 0$ be a F'-harmonic minorant of q'. Then h' possesses a F'-harmonic extension into the whole F' by [5], Corollary 6.2.5, the extension being obviously majorized by p' ([5], Theorem 6.2.1). Hence h' vanishes identically and q' is a locally bounded F'-potential on $W' \cap \varphi(V)$. Obviously q' is also a locally bounded U'-potential on $W' \cap \varphi(V) = V' \cap \varphi(V)$. Therefore $p' \circ \varphi = q' \circ \varphi$ is a potential on $\varphi^{-1}(W') = \varphi^{-1}(V' \cap \varphi(V))$, hence $\varphi: V \to F'$ is a Bl-mapping.

4. Covering properties of normal mappings

The following Definition 4.3 for normal mappings is more general than our earlier one ([8], Definition 2.1.5). Also we allow less restricted harmonic spaces than in [8].

L e m m a 4.1. *Let* U *be a* $\mathscr{P}$*-domain, let* F *be a closed polar set in* U *and let* $B \subseteq U \setminus F$ *be non-polar in* U. *Then* B *is non-polar in* $U \setminus F$.

Proof. If B is polar in $U \setminus F$, then $(\hat{R}_p^B)_{U \setminus F} = 0$ for all potentials p on $U \setminus F$. Let now p be any finite continuous potential on U. Let $h_0 \geqq 0$ be a harmonic minorant of $p \mid (U \setminus F)$. Then

$$\limsup_{U \setminus F \ni x \to y} h_0(x) \leqq \limsup_{U \setminus F \ni x \to y} p(x) < +\infty$$

for all $y \in F$. By [5], Corollary 6.2.5, h_0 can be extended to a harmonic function h on U. Obviously $0 \leqq h \leqq p$, hence $h = 0$. Therefore

$p \mid (U \setminus F)$ is a potential on $U \setminus F$ and $(\hat{R}_p^B)_{U \setminus F} = 0$. According to [5], Theorem 6.2.1, we easily verify that $(\hat{R}_p^B)_U \mid (U \setminus F) = 0$ and, by lower semi-continuity, $(\hat{R}_p^B)_U = 0$. By [5], Proposition 7.1.2, $\mu^B = 0$ for every measure $\mu \in \Lambda(U)$ ([5], p. 159). By [5], Proposition 7.1.5, B is a polar set in U, a contradiction.

Lemma 4.2. *If $A \subseteq U$ is a closed non-polar set, then there is $x \in A$ such that $G \cap A$ is non-polar in G for any $G \in \mathcal{G}(x)$, where $\mathcal{G}(x)$ is the family of open connected relatively compact neighbourhoods of x. Any point $x \in A$ satisfying the assertion of this lemma is called a strong point of A in U.*

Proof. Otherwise let, for all $x \in A$, $G_x \in \mathcal{G}(x)$ be selected such that $G_x \cap A$ is polar in G_x. Let then $\mathcal{W}_x = \{ W_\alpha \subseteq G_x \mid \alpha \in I_x \}$ be an open cover of G_x such that $(R_\infty^{A \cap W_\alpha})_{W_\alpha} = 0$ for all $\alpha \in I_x$. Then

$$\{ U \setminus A , W_\alpha \mid W_\alpha \in \mathcal{W}_x , \ x \in A \}$$

is an open cover of U which implies the polarity of A in U.

Definition 4.3. *A harmonic Bl-mapping $\varphi : X \to X'$ is a normal mapping, if $D'_\varphi = \{ x' \in X' \mid \exists\, x \in \varphi^{-1}(x') \ \text{with} \ n(\varphi , x) > 1 \}$ is nowhere dense and if the subset of its polar points is a polar set in X' and if moreover $n(\varphi , x' , V)$ is lower semi-continuous (see* [8], *Remark* 2.1.4*) for any $\mathcal{P}$-domain $U' \subseteq X'$ and for the union of any non-empty subfamily of the components of $\varphi^{-1}(U')$.*

Theorem 4.4. *If $\varphi : X \to X'$ is a normal mapping, if $U' \subseteq X'$ is a $\mathcal{P}$-domain such that $\varphi^{-1}(U') \neq \emptyset$ and if V is the union of a non-empty subfamily of the components of $\varphi^{-1}(U')$, then*

$$F' = \{ x' \in U' \mid n(\varphi , x' , V) < \sup_{z' \in U'} n(\varphi , z' , V) = N \}$$

is either a polar set in U' or else $\operatorname{int} F' \neq \emptyset$.

Proof. The restricted mapping $\varphi : V \to U'$ is again denoted by φ in this proof. Further we denote $E'_p = \{ x' \in U' \mid n(\varphi , x' , V) = p \}$ for $p = 0, \dots, N$ and $F'_r = E'_0 \cup \cdots \cup E'_r$ for $r = 0, \dots, N$. By normality, F'_r is closed in U' for any $r < +\infty$. If F' is non-polar, then we may take the smallest k such that F'_k is non-polar in U'. If $k = 0$, then $\operatorname{int} F' \supseteq \operatorname{int} F'_0 = \operatorname{int} E'_0 = U' \setminus \operatorname{cl}(U' \setminus E'_0) \neq \emptyset$ by Theorem 3.2. Hence we may assume that $1 \leq k < N$.

Let us suppose first that at least one point $y' \in \partial F'_k$ is non-polar. Obviously $y' \in E'_k$. Let us denote $\varphi^{-1}(y') = \{ z_1, \dots, z_s \}$, where $s \leq k$. Let further $G' \in \mathcal{G}(y')$ be constructed by [8], Lemma 1.3.4 and let $V_1, \dots, V_s$ be the components of $\varphi^{-1}(G')$ corresponding respectively to the points $z_1, \dots, z_s$. Since $y' \in \partial F'_k$, then $n(\varphi , z' , V) \geq k + 1$ in an open set $W' \subseteq G'$. Since D'_φ is nowhere dense, there is $z' \in W' \setminus D'_\varphi$. Since V_i

is a minimal neighbourhood of z_i, $i = 1, \dots, s$ ([8], p. 23), we get $\bar{n}(\varphi, z', V_1 \cup \dots \cup V_s) \leqq k$, hence there are some components V_α of $\varphi^{-1}(G')$ which are distinct from $V_1, \dots, V_s$. Let W denote their union. The normal mapping $\varphi : W \to G'$ omits the non-polar point y'. By Theorem 3.2 $\mathrm{cl}\,(\varphi(W))$ is a non-trivial absorbent set in G', hence $G_0' = G' \setminus \mathrm{cl}\,(\varphi(W))$ is a non-empty open set. Let us take an arbitrary $x' \in G_0'$. Since D_φ' is nowhere dense, x' is a cluster point of $G_0' \setminus D_\varphi'$. If $w' \in G_0' \setminus D_\varphi'$, then

$$n(\varphi, w', V) = n(\varphi, w', \bigcup_{i=1}^{s} V_i) = \bar{n}(\varphi, w', \bigcup_{i=1}^{s} V_i) \leqq k,$$

hence $w' \in F_k'$. Since F_k' is closed, $x' \in F_k'$. Thus $G_0' \subseteq F_k'$ and so $\mathrm{int}\, F' \supseteq \mathrm{int}\, F_k' \neq \emptyset$.

From now on we may assume that all points in $\partial F_k'$ are polar in U'. Obviously $\partial F_k'$ is non-polar in U' ([5], Proposition 6.2.5), hence

$$B' = \partial F_k' \setminus D_\varphi' \cup F_{k-1}' = \partial F_k' \setminus (D_\varphi' \cap E_k') \cup F_{k-1}'$$

is non-polar in U'. The set $D_\varphi' \cap E_k'$ is closed in the open set $U' \setminus F_{k-1}'$. In fact, if z' is a cluster point of $D_\varphi' \cap E_k'$ in $U' \setminus F_{k-1}'$, then obviously $z' \in E_k'$. Let $G' \in \mathscr{G}(z')$, $G' \subseteq U' \setminus F_{k-1}'$, be constructed by [8], Lemma 1.3.4. If $z' \notin D_\varphi'$, then all points in the components $V_1, \dots, V_k$ of $\varphi^{-1}(G')$ corresponding to the points $z_1, \dots, z_k$, respectively, are of simple local multiplicity ([8], Lemma 1.3.6). By normality,

$$\varphi(V_1) \cap \dots \cap \varphi(V_k) \neq \emptyset$$

is open and $n(\varphi, y', V_1 \cup \dots \cup V_k) = k$ for all $y' \in \varphi(V_1) \cap \dots \cap \varphi(V_k)$. For all $y' \in E_k' \cap \varphi(V_1) \cap \dots \cap \varphi(V_k)$ we get $\varphi^{-1}(y') \subseteq V_1 \cup \dots \cup V_k$, hence $y' \notin D_\varphi'$, a contradiction. Therefore $z' \in D_\varphi'$.

Since $D_\varphi' \cap E_k'$ is closed in $U' \setminus F_{k-1}'$, then

$$U_0' = U' \setminus (D_\varphi' \cap E_k') \cup F_{k-1}'$$

is an open set and $B' = \partial F_k' \cap U_0'$ is closed in U_0'. The set B' is non-polar in U_0' by Lemma 4.1 and by Lemma 4.2 there is a strong point y' of B' in U_0'. Let us fix $G' \in \mathscr{G}(y')$ according to [8], Lemma 1.3.4 and let $V_1, \dots, V_k$ be the components of $\varphi^{-1}(G')$ corresponding respectively to the points of $\varphi^{-1}(y')$. Let W' denote the component containing y' of the non-empty open set $\varphi(V_1) \cap \dots \cap \varphi(V_k)$. Obviously $W' \cap (U' \setminus F_k') \neq \emptyset$. Since all points in $V_1 \cup \dots \cup V_k$ are of simple local multiplicity, there are some components V_α of $\varphi^{-1}(G')$ which are distinct from $V_1, \dots, V_k$. Let W denote their union. The normal mapping $\varphi : W \to G'$ omits the set $B' \cap W'$ which is non-polar in G'. By Theorem 3.2 $\mathrm{cl}\,(\varphi(W))$ is a non-trivial absorbent set in G', hence $G' \setminus \mathrm{cl}\,(\varphi(W))$

is a non-empty open set. Exactly as to above we see that $G' \setminus \mathrm{cl}\,(\varphi(W)) \subseteq F'_k$, hence $\mathrm{int}\, F' \neq \emptyset$.

Corollary 4.5. *If* U' *is an elliptic* $\mathcal{P}$*-domain, then* F' *is a polar set in* U'.

Corollary 4.6. *If* U' *is a one-dimensional orientated* $\mathcal{P}$*-Bauer domain, then* $F' = \emptyset$ *and* $n(\varphi, x', V) = N$ *for all* $x' \in U'$.

Proof. Let us suppose $F' \neq \emptyset$. Since $\varphi : V \to U'$ is surjective by Theorem 3.4, we may take the smallest $k \geq 1$ such that $F'_k \neq \emptyset$. Let us take $y' \in \partial F'_k$. Obviously y' is non-polar and $y' \in E'_k$. Let $G' \in \mathcal{G}(y')$ be constructed by [8], Lemma 1.3.4 and let $V_1, \ldots, V_s$ be the components of $\varphi^{-1}(G')$ corresponding respectively to the points of $\varphi^{-1}(y') = \{z_1, \ldots, z_s\}$, $s \leq k$. Obviously $n(\varphi, z', V) \geq k + 1$ in an open set $W' \subseteq G'$. Since D'_φ is nowhere dense, there is $z' \in W' \setminus D'_\varphi$. Since V_i is a minimal neighbourhood of z_i, $i = 1, \ldots, s$, we get

$$\bar{n}(\varphi, z', V_1 \cup \cdots \cup V_s) \leq k,$$

hence there is at least one component V_{s+1} of $\varphi^{-1}(G')$ which is distinct from $V_1, \ldots, V_s$. The normal mapping $\varphi : V_{s+1} \to G'$ omits y', a contradiction to Theorem 3.4.

Corollary 4.7. *If* $\varphi : X \to X'$ *is a normal mapping, if* $U' \subset X'$, $\varphi^{-1}(U') \neq \emptyset$, *is a domain possessing a pseudoexhaustion* $\{U'_i \mid i \in \mathbf{N}\}$ *by elliptic* $\mathcal{P}$*-domains and if* V *is the union of a non-empty subfamily of the components of* $\varphi^{-1}(U')$, *then* F' *is a polar set in* U'.

Proof. Let us denote $V' = \cup \{U'_i \mid i \in \mathbf{N}\}$. Suppose F' is non-polar in U' and let us take the smallest k such that F'_k is non-polar in U'. By Theorem 3.5 we may assume that $1 \leq k < N$. Obviously it is $V' \cap F'_k \neq \emptyset$ and $V' \cap (U' \setminus F'_k) \neq \emptyset$, hence $V' \cap \partial F'_k \neq \emptyset$. Since $V' \cap \partial F'_k$ is non-polar in V', there is a strong point y' of $V' \cap \partial F'_k$ in V'. Let $G' \in \mathcal{G}(y')$ be constructed by [8], Lemma 1.3.4 and let $V_1, \ldots, V_s$ be the components of $\varphi^{-1}(G')$ corresponding respectively to the points of $\varphi^{-1}(y') = \{z_1, \ldots, z_s\}$, $s \leq k$. We may assume that G' is an elliptic $\mathcal{P}$-domain. By Corollary 4.5 the normal mapping

$$\varphi : V_1 \cup \cdots \cup V_s \to G'$$

satisfies $n(\varphi, x', V_1 \cup \cdots \cup V_s) = k$ for all points x' outside of a polar set E' in G'. Therefore there is at least one component V_{s+1} of $\varphi^{-1}(G')$ which is distinct from $V_1, \ldots, V_s$. The normal mapping $\varphi : V_{s+1} \to G'$ omits obviously the non-polar set $(G' \setminus E') \cap F'_k$. Since G' is an elliptic $\mathcal{P}$-domain, we get a contradiction to Theorem 3.3.

5. Covering properties of open Bl-mappings

Preliminary versions (for mappings between Brelot spaces) of the results to be presented in this section appear in [8], Theorem 2.2.11 and in [9], Section 3.2. The following results are similar to those ones of the preceding section. Therefore the following proofs are not presented in full detail.

Lemma 5.1. *Let $\varphi : X \to X'$ be an open harmonic mapping, $U' \subseteq X'$ a domain such that $\varphi^{-1}(U') \neq \emptyset$ and V the union of a non-empty subfamily of the components of $\varphi^{-1}(U')$. Then*

$$SF'_p = \{ x' \in U' \mid \bar{n}(\varphi, x', V) \leqq p \}$$

is closed in U' for any $p \in \mathbf{N}_0$.

Theorem 5.2. *Let $\varphi : X \to X'$ be an open harmonic Bl-mapping such that D'_φ is a polar set. If $U' \subseteq X'$ is a $\mathscr{P}$-domain such that $\varphi^{-1}(U') \neq \emptyset$ and if V is the union of a non-empty subfamily of the components of $\varphi^{-1}(U')$, then*

$$SF' = \{ x' \in U' \mid n(\varphi, x', V) < \sup_{z' \in U'} \bar{n}(\varphi, z', V) = \bar{N} \}$$

is a polar set in U' or else $\operatorname{int} SF' \neq \emptyset$.

Proof. If SF' is non-polar in U', then let us take the smallest k such that

$$SF'_k = \bigcup_{i=0}^{k} SE'_i = \bigcup_{i=0}^{k} \{ x' \in U' \mid \bar{n}(\varphi, x', V) = i \}$$

is non-polar in U'. If $k = 0$, then $\operatorname{int} SF' \supseteq \operatorname{int} SF'_0 = \operatorname{int} E'_0 \neq \emptyset$ by Theorem 3.2. Therefore we may assume that $1 \leqq k < \bar{N}$.

We shall construct a point $y' \in \partial SF'_k \cap SE'_k \cap (U' \setminus D'_\varphi)$ and a set $B' \subseteq SE'_k$ such that $B' \cap G'$ is non-polar in G'_0 for all $G' \subseteq G'_0$, whenever $G' \in \mathscr{G}(y')$ and where $G'_0 \in \mathscr{G}(y')$ is a fixed neighbourhood of y' constructed by [8], Lemma 1.3.4, for the points of $\varphi^{-1}(y') = \{ z_1, \ldots, z_k \}$. To proceed, let $V_1, \ldots, V_k$ be the minimal components of $\varphi^{-1}(G'_0)$ corresponding respectively to the points $z_1, \ldots, z_k$ and let W' be the component containing y' of the open set $\varphi(V_1) \cap \cdots \cap \varphi(V_k) \neq \emptyset$. Obviously $W' \cap (U' \setminus SF'_k) \neq \emptyset$, hence there are some components V_α of $\varphi^{-1}(G'_0)$ which are distinct from $V_1, \ldots, V_k$. Let W denote their union. The open Bl-mapping $\varphi : W \to G'_0$ omits $B' \cap W'$ which is non-polar in G'_0. By Theorem 3.2 $\operatorname{cl} \varphi(W)$ is a non-trivial absorbent set in G'_0, hence the non-empty set $G'_0 \setminus \operatorname{cl} \varphi(W)$ is open. Obviously $G'_0 \setminus \operatorname{cl} \varphi(W) \subseteq SF'_k$, hence $\operatorname{int} SF' \supseteq \operatorname{int} SF'_k \neq \emptyset$.

To construct y' and B' described above, suppose first that at least one point $y' \in \partial SF'_k$ is non-polar. Trivially $y' \in \partial SF'_k \cap SE'_k \cap (U' \setminus D'_\varphi)$ is the required point with $B' = SE'_k$.

Hence we may assume that all points in $\partial SF'_k$ are polar in U'. Then

$$B' = \partial SF'_k \setminus D'_\varphi \cup SF'_{k-1} = \partial SF'_k \setminus (D'_\varphi \cap SE'_k) \cup SF'_{k-1}$$

is non-polar in U'. The set $D'_\varphi \cap SE'_k$ is closed in $U' \setminus SF'_{k-1}$. In fact, if z' is a cluster point of $D'_\varphi \cap SE'_k$ in $U' \setminus SF'_{k-1}$, then $z' \in SE'_k$. Let $G'_0 \in \mathscr{G}(z')$ be constructed by [8], Lemma 1.3.4. We may assume that $G'_0 \subseteq U' \setminus SF'_{k-1}$. If $z' \notin D'_\varphi$, then $\bar{n}(\varphi, y', V_1 \cup \cdots \cup V_k) = k$ for all $y' \in \varphi(V_1) \cap \cdots \cap \varphi(V_k) \neq \emptyset$. Hence we get $\varphi^{-1}(y') \subseteq V_1 \cup \cdots \cup V_k$ for all $y' \in SE'_k \cap \varphi(V_1) \cap \cdots \cap \varphi(V_k)$. Hence $y' \notin D'_\varphi$, a contradiction. Exactly as in the proof of Theorem 4.4 we find now the required point $y' \in \partial SF'_k \cap SE'_k \cap (U' \setminus D'_\varphi)$, the required set being

$$B' = \partial SF'_k \setminus D'_\varphi \cup SF'_{k-1} \subseteq SE'_k .$$

The theorem follows.

Corollary 5.3. *If U' is an elliptic $\mathscr{P}$-domain, then SF' is a polar set in U'.*

Corollary 5.4. *If U' is a one-dimensional orientated $\mathscr{P}$-Bauer domain, then $SF' = F' = \emptyset$.*

Proof. The proof of this corollary proceeds exactly as the proof of Corollary 4.6, if we only note that $U' \cap D'_\varphi = \emptyset$. By the same argument $SF' = F'$.

Remark 5.5. The following Theorem 5.6 has the same form as Theorem 4.4 treating normal mappings. However, the situation in Theorem 5.6 is quite different as one can see by looking for Theorem 5.2. More precisely, in Theorem 5.2 one can not weaken the supposition D'_φ to be a polar set in U'. In fact, the Example 2.2.8 in [8] demonstrates that the assertion of Theorem 5.2 can fail, if D'_φ contains only one (non-polar) point. On the other hand, the example in Remark 4.2 in [8] states that the assertion of Theorem 5.2 can hold even if $D'_\varphi = U'$. We also note that under certain additional assumptions (see [8], Theorem 2.2.11) D'_φ is allowed to contain some non-polar points as soon as we consider F' instead of SF'.

Theorem 5.6. *Let $\varphi : X \to X'$ be an open harmonic Bl-mapping such that D'_φ is a polar set. If $U' \subseteq X'$ is a $\mathscr{P}$-domain such that $\varphi^{-1}(U') \neq \emptyset$ and if V is the union of a non-empty subfamily of the components of $\varphi^{-1}(U')$, then*

$$F' = \{ x' \in U' \mid n(\varphi, x', V) < \sup_{z' \in U'} n(\varphi, z', V) = N \}$$

is a polar set in U' or else $\operatorname{int} F' \neq \emptyset$.

Proof. Let us suppose that SF' is non-polar in U' and let k be the smallest integer such that SF'_k is non-polar in U'. By Theorem 3.2 we may assume that $1 \leqq k < N$. Furthermore $\operatorname{int} SF'_k \neq \emptyset$ by the proof of Theorem 5.2. By the same proof we observe that $D'_\varphi \cap SE'_k$ is closed in $U' \setminus SF'_{k-1}$. Hence

$$\begin{aligned}((\operatorname{int} SF'_k) \setminus SF'_{k-1}) &\cap ((U' \setminus SF'_{k-1}) \setminus (D'_\varphi \cap SE'_k)) \\ &= ((\operatorname{int} SF'_k) \setminus SF'_{k-1}) \setminus D'_\varphi \neq \emptyset\end{aligned}$$

is an open set in $U' \setminus SF'_{k-1}$, hence open in U'. Obviously

$$((\operatorname{int} SF'_k) \setminus SF'_{k-1}) \setminus D'_\varphi \subseteq SF'_k \setminus D'_\varphi \subseteq F'_k,$$

therefore

$$\operatorname{int} F' \supseteq \operatorname{int} F'_k \neq \emptyset .$$

If, on the other hand, SF' is polar in U' and if $\bar{N} = N$, then we get $F' \subseteq SF'$ trivially, hence F' is polar in U'. If $\bar{N} < N$, then $\bar{N} < +\infty$. Hence

$$SE'_{\bar{N}} = U' \setminus SF'_{\bar{N}-1} = U' \setminus SF' \neq \emptyset$$

is an open set in U'. Using the idea of the proof of Theorem 5.2 we see that $D'_\varphi \cap SE'_{\bar{N}}$ is closed in $U' \setminus SF'$, since $\bar{N} < +\infty$. Therefore

$$SE'_{\bar{N}} \setminus D'_\varphi = SE'_{\bar{N}} \setminus (D'_\varphi \cap SE'_{\bar{N}}) \neq \emptyset$$

is open, hence $\operatorname{int} F' \supseteq SE'_{\bar{N}} \setminus D'_\varphi \neq \emptyset$. $\_$

Remark 5.7. It remains open whether $\bar{N} = N$ in general. Anyhow, using Corollary 5.3 instead of Theorem 3.1.5 in [9], we can prove the following result repeating, word by word, the proof of Theorem 3.1.6 in [9], under the suppositions of Theorem 5.6.

Theorem 5.8. *If U' is an elliptic $\mathscr{P}$-domain, then $\bar{N} = N$ and F' is a polar set in U'.*

Theorem 5.9. *Let $\varphi: X \to X'$ be an open harmonic Bl-mapping such that D'_φ is a polar set. If $U' \subseteq X'$, $\varphi^{-1}(U') \neq \emptyset$, is a domain possessing a pseudoexhaustion $\{ U'_i \mid i \in \mathbf{N} \}$ by elliptic $\mathscr{P}$-domains and if V is the union of a non-empty subfamily of the components of $\varphi^{-1}(U')$, then F' (resp. SF') is a polar set in U'. Also in this case $\bar{N} = N$.*

Proof. Let us denote $V' = \cup \{ U'_i \mid i \in \mathbf{N} \}$. Suppose F' is non-polar in U' and let k be the smallest integer such that F'_k is non-polar in U'. By Theorem 3.5 we may assume that $1 \leqq k < N$. Obviously SF'_k is non-polar in U' and SF'_r is polar in U' for every $r < k$. Hence $V' \cap SF'_k \neq \emptyset$. The set $U' \setminus SF'_k$ is open and non-empty. In fact, otherwise $U' = SF'_k$ which implies immediately $U' = F'_k$, a contradiction. Therefore $V' \setminus SF'_k \neq \emptyset$ and $V' \cap \partial SF'_k \neq \emptyset$. Since $V' \cap \partial SF'_k$ is non-polar in V' by Lemma 4.1, the set A' of strong points of $V' \cap \partial SF'_k$

in V' is non-polar in V'. In fact, since $V' \cap \partial SF'_k = A' \cup B'$ is closed in V', there is an open set W' such that $V' \setminus W' \subseteq A'$ and that B' is a polar set in W' as one can see by using the definition of B'. If now A' is polar in V', then $V' \setminus W'$ is a closed polar set in V', hence B' is polar in V' by Lemma 4.1, a contradiction. Therefore we may take a strong point $y' \in A'$ such that $y' \in (V' \setminus D'_\varphi) \cap \partial SF'_k \cap SE'_k$. Let $G' \in \mathcal{G}(y')$ be constructed by [8], Lemma 1.3.4 and let $V_1, \ldots, V_k$ be the minimal components of $\varphi^{-1}(G')$ corresponding respectively to the points of $\varphi^{-1}(y') = \{ z_1, \ldots, z_k \}$. We may assume that $G' \subseteq V'$ is an elliptic $\mathcal{P}$-domain. Since φ is an open mapping, $\partial SF'_k \cap \varphi(V_1) \cap \cdots \cap \varphi(V_k)$ is a non-polar set in G'. Therefore $\bar{n}(\varphi, x', V_1 \cup \cdots \cup V_k) \leq k$ for all $x' \in G'$ by Corollary 5.3. Since $G' \setminus SF'_k \neq \emptyset$, there is at least one component V_{k+1} of $\varphi^{-1}(G')$ distinct from $V_1, \ldots, V_k$. The open harmonic Bl-mapping $\varphi : V_{k+1} \to G'$ omits the non-polar set $\partial SF'_k \cap \varphi(V_1) \cap \cdots \cap \varphi(V_k)$ in G', a contradiction to Theorem 3.3.

The assertion for SF' is contained in the above proof. If $\bar{N} < N$, then $\{ x' \in U' \mid n(\varphi, x', V) > \bar{N} \}$ is a non-polar set in U'. Hence D'_φ is a non-polar set in U', a contradiction.

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