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MULTIPLIER PRESERVING ISOMORPHISMS BETWEEN MÖBIUS GROUPS

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Let us consider the case that we are given two groups G and G' of conformal mappings acting in the unit disk $D = \{ z \in \mathbf{C} : |z| < 1 \}$ such that D / G and D / G' are Riemann surfaces. Let $\varphi : G \to G'$ be an isomorphism. We would want to know under what circumstances φ is a conjugation in a group F containing at least all conformal self-mappings of D, i.e.

$$\varphi(T) = f T f^{-1}, \qquad T \in G, \quad f \in F .$$

If the group F is the group of all homeomorphisms of F, then D / G and D / G' are homeomorphic (and φ is induced by this homeomorphism); if F is the group of all quasiconformal mappings of D, then φ defines a point in the Teichmüller space of G; and finally if F is the group of all Möbius transformations of D, then φ defines the same point as $\mathrm{id} : G \to G$ in the Teichmüller space of G.

The above problem in the theory of Riemann surfaces and Teichmüller spaces is our starting point in this paper. We generalize it in such a way that instead of the group of conformal mappings of the unit disk we consider the group of Möbius transformations of the n-sphere S^n. It also turns out that we need not demand that our groups act discontinuously; in fact a much weaker condition (concerning the existence of loxodromic mappings in the group) suffices. Under these conditions we prove that an isomorphism between two groups of Möbius transformations of S^n is a conjugation by a Möbius transformation if and only if it has the property of "preserving multipliers" (Theorem 2). We also show that if such an isomorphism is a conjugation in the group of all quasiconformal mappings of S^n, then it cannot deform these "multipliers" arbitrarily: one can show that there is an upper limit to the deformation of multipliers (Theorem 1).

A Möbius transformation of S^n, the n-dimensional sphere, is a conformal or anticonformal self-map of S^n. Such mappings can be classified

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as *loxodromic, parabolic* or *elliptic* (if distinct from the identity) (see Martio-Srebro [2], where a Möbius transformation was assumed to be orientation preserving, but the anticonformal case is not essentially different from that). If S is a loxodromic Möbius transformation, it is conjugate in the group of all Möbius transformations of S^n to a transformation of the form

$$(1) \qquad T(z) = O(\lambda z) = \lambda O(z), \quad z \in \mathbf{R}^n, \quad 1 < \lambda \in \mathbf{R}, \\ O \in \mathrm{O}(n) \quad (T(\infty) = \infty),$$

where $\mathrm{O}(n)$ is the group of orthogonal transformations of $\mathbf{R}^n$. We denote by $\mathrm{SO}(n)$ the group of orthogonal transformations of $\mathbf{R}^n$ with determinant 1. Then T is orientation preserving if and only if $O \in \mathrm{SO}(n)$. The element $O \in \mathrm{O}(n)$ is not unique: only its conjugacy class depends on S ($= R\,T\,R^{-1}$ for some Möbius transformation R). In contrast, λ is unique, supposing that ∞ is the attracting fixed point of T. This is equivalent to the fact that $\lambda > 1$.

Henceforth we shall denote the attracting fixed point of a loxodromic transformation S by $P(S)$; the repelling fixed point is $N(S)$. The group of all conformal and anticonformal Möbius transformations is denoted by $\mathrm{GM}(n)$; the subgroup consisting of all orientation preserving Möbius transformations is $\mathrm{SGM}(n)$. This does not seem to be the common usage: as a rule $\mathrm{GM}(n)$ is what we denote by $\mathrm{SGM}(n)$.

The real number λ specified by (1) is called the *multiplier* of S ($= R\,T\,R^{-1}$) and denoted $\mathrm{mul}\, S$. We denote by $\mathrm{rot}\, S$ the conjugacy class of O in $\mathrm{O}(n)$. If S is already in the form (1), i.e. it fixes 0 and ∞, we denote the element O by $\mathrm{rot}\, S$. If $\mathrm{rot}\, S = \mathrm{id} \in \mathrm{O}(n)$ we say that S is *hyperbolic.*

If a Möbius transformation T of S^n is not loxodromic it is *elliptic* or *parabolic* (if not the identity). It is elliptic if it (or its extension to S^{n+1}) can be put into the form (1) with $\lambda = 1$ by conjugation. It is parabolic if it can be put into the form

$$T(z) = O(z) + a, \quad \text{for } z \in \mathbf{R}^n \quad (T(\infty) = \infty),$$

where $a \in \mathbf{R}^n \setminus \{0\}$, $O \in \mathrm{O}(n)$, $O(a) = a$. If $T \in \mathrm{GM}(n)$ is not loxodromic we set $\mathrm{mul}\, T = 1$. If T is parabolic we consider that the attracting and repelling fixed points of T are defined and set $P(T) = N(T) =$ the fixed point of T. If T is elliptic or the identity, $P(T)$ and $N(T)$ are not defined.

Let G and G' be two subgroups of $\mathrm{GM}(n)$ and let

$$\varphi : G \to G'$$

be an isomorphism with

$$\operatorname{mul} T = \operatorname{mul} \varphi(T)$$

for every $T \in G$. Then we say that φ is a *multiplier preserving isomorphism.*

Closely related to the concept of a multiplier preserving isomorphism is that of the *dilatation* of an isomorphism (cf. Sorvali [5]). Suppose that there is a real number $k \geq 1$ such that, given an isomorphism $\varphi : G \to G'$,

$$(\operatorname{mul} T)^{1/k} \leq \operatorname{mul} \varphi(T) \leq (\operatorname{mul} T)^k ,$$

for every $T \in G$. Then we say that the dilatation of φ is less than or equal to k. The dilatation of φ is the smallest number k for which these inequalities are valid. Of course, the dilatation of an isomorphism need not be finite. It is seen that to say that φ is multiplier preserving amounts to the same as to say that the dilatation of φ is 1. If φ is conjugation in the group of conformal and anticonformal Möbius transformations, then the dilatation of φ is 1. We shall show that if trivial cases are excluded, then the converse is also true.

Next we prove that if φ is a conjugation by a K-quasiconformal self-map of S^n, then the dilatation of φ is less than or equal to K. In view of the fact that 1-quasiconformal self-maps of S^n are just the Möbius transformations of S^n (see Mostow [3]) this generalizes the statement that conjugation by a Möbius transformation does not change multipliers.

Theorem 1. *Let G and G' be groups of Möbius transformations of S^n $(n > 1)$ and let $f: S^n \to S^n$ be K-quasiconformal. If $G' = f G f^{-1}$ and*

$$\varphi(T) = f \circ T \circ f^{-1} , \qquad \text{for } T \in G ,$$

then the dilatation of φ is less than or equal to K.

Proof. If $T \in G$ is loxodromic, $T' = \varphi(T)$ is also loxodromic. We may assume that both T and T' fix 0 and ∞ and are of the form

$$T(z) = O(\lambda z) , \quad z \in \mathbf{R}^n , \quad O \in \mathrm{O}(n) , \quad 1 < \lambda \in \mathbf{R} ,$$
$$T'(z) = O'(\lambda' z) , \quad z \in \mathbf{R}^n , \quad O' \in \mathrm{O}(n) , \quad 1 < \lambda' \in \mathbf{R} ,$$

and we must show that

$$(\log \lambda)/K \leq \log \lambda' \leq (\log \lambda) K .$$

Let D_n be the shell

$$D_n = \{ z \in \mathbf{R}^n : 1 \leq |z| \leq \lambda^n \}$$

and

$$D_n' = f(D_n) .$$

Since f is K-quasiconformal we have

$$K^{-1} \operatorname{mod} D_n \leq \operatorname{mod} D_n' \leq K \operatorname{mod} D_n ,$$

where $\operatorname{mod} D_n$ is defined by means of the conformal capacity of a shell (see Mostow [3] p. 80). We make use of the following facts concerning the modulus of a shell:

(a) If $D' \subset D$, then $\operatorname{mod} D' \leq \operatorname{mod} D$.

(b) If $D_{ab} = \{ z \in \mathbf{R}^n : a \leq |z| \leq b \}$, then

$$\operatorname{mod} D_{ab} = \log (b/a) .$$

Using (a) and (b) we have

$$K \log \lambda^n = K \operatorname{mod} D_n \geq \operatorname{mod} D_n' \geq \operatorname{mod} D_{M,\lambda'^n m} = \log (\lambda'^n m/\mathrm{M})$$

where

$$M = \max \{ |f(x)| : x \in S^{n-1} \} ,$$

$$m = \min \{ |f(x)| : x \in S^{n-1} \} ,$$

i.e.,

$$K \log \lambda \geq \log \lambda' + (1/n) \log (m/M) .$$

Since this is true for every $n \in \mathbf{N}$, we must have

$$K \log \lambda \geq \log \lambda' .$$

Similarly one shows

$$K^{-1} \log \lambda \leq \log \lambda' .$$

Remark. If $n = 1$ and f is k-quasisymmetric (this implies that $f(\infty) = \infty$ and that f is increasing) then one can show that the dilatation of φ is not greater than $\log 2/\log (1 + 1/k)$ (cf. Sorvali [5]). Sorvali's conditions are too strict; he assumes that G and G' are covering groups. In fact his proof is valid, without any change, also if this assumption is dropped.

Next we prove that if the dilatation of φ is 1 then it is a conjugation in the group of all (conformal or anticonformal) Möbius transformations. For this we need the following lemmata.

L e m m a 1. *Let* $Q \in \mathrm{SO}(n)$ *be fixed. Then there is a* K*-quasiconformal mapping* $f: \mathbf{R}^n \to \mathbf{R}^n$ *such that*

$$f(z) = z , \qquad |z| \leq 1 ,$$

$$f(z) = Q(z) , \qquad |z| \geq r ,$$

where $1 < r$ *and where* K *depends on* r *in such a way that there is a constant* c_Q *(depends on* Q*) such that, beginning from some fixed* r,

$$K \leq 1 + c_Q/\log r .$$

Moreover, we have

$$c = \sup\{ c_Q : Q \in \mathrm{SO}(n) \} < \infty \quad \textit{and} \quad \lim_{Q \to \mathrm{id}} c_Q = 0 .$$

Proof. The linear space $\mathbf{R}^n$ has a representation as a direct sum (see Greub [1]),

$$\mathbf{R}^n = E_1 \oplus \dots \oplus E_k ,$$

where each E_i is a one- or two-dimensional linear subspace of $\mathbf{R}^n$, each E_i is invariant under Q, the mapping $Q \mid E_i$ is the identity if E_i is one-dimensional, and $Q \mid E_i$ is an orientation preserving rotation of E_i if E_i is two-dimensional. Let $E_1, \dots, E_l$ be one-dimensional, $E_{l+1}, \dots, E_k$ be two-dimensional and let $Q \mid E_i$, $l < i \leq k$, be the rotation through the angle θ_i.

We define the mapping f_0 as follows. For $1 \leq |z| \leq r_0$, $r_0 > 1$ let

$$f_0(z) = O(z)z , \tag{2}$$

where $O(z) \in \mathrm{SO}(n)$ is the orthogonal mapping for which

$$\begin{aligned} O(z) \mid E_i &= \mathrm{id} , \quad 1 \leq i \leq l , \\ O(z) \mid E_i &= \text{the rotation through the angle } ((|z|-1)/(r_0 - 1))\, \theta_i , \\ &\qquad l < i \leq k . \end{aligned}$$

It is clear that $O(z)$ depends only on $r = |z|$. We compute the derivative f_0':

$$f_0'(z)\, h = (O'(z)h)z + O(z)h , \quad h \in \mathbf{R}^n , \quad 1 < |z| < r_0 . \tag{3}$$

As we have already observed it is possible to regard O as a function of r only. We do this and denote it also by O, it being clear from the context whether O is regarded as a function of z $(\in \mathbf{R}^n)$ or r $(\in \mathbf{R})$. We have

$$\begin{aligned} O'(z)h &= (O'(r) \circ dr/dz)h \\ &= O'(r)(z^0 , h) , \quad z , h \in \mathbf{R}^n , \quad 1 < |z| < r_0 , \end{aligned} \tag{4}$$

where $z^0 = z/|z|$,

and where the dual space of $\mathbf{R}^n$ is identified with $\mathbf{R}^n$ via the usual inner product $(\, , \,)$ of $\mathbf{R}^n$.

Thus

$$\begin{aligned} f_0'(z)\, h &= ((O'(r) \circ dr/dz)\, h)z + O(z)h \\ &= r\, (O'(r)(z^0 , h))z^0 + O(z)h , \quad z , h \in \mathbf{R}^n, \quad 1 < |z| < r_0 . \end{aligned}$$

The function f_0 defined by (2) has continuous derivatives in the shell $\{ z : 1 \leq |z| \leq r_0 \}$, $f(z) = z$ if $|z| = 1$ and $f(z) = Q(z)$ for $|z| = r_0$.

Thus we see that there is such a function as specified by the lemma for this particular r_0. We must study the behaviour of K as r varies. To do this we choose another $r_1 > 1$ and define a function f_1 using f_0.

Let $r_1 > 1$ be arbitrary and set

$$\alpha = \frac{\log r_0}{\log r_1}.$$

Then $r_1^\alpha = r_0$. Further, we define

$$\begin{aligned} f_1(z) &= f_0(|z|^\alpha z^0)\, |z|^{1-\alpha} = O(|z|^\alpha z^0) z \\ &= O(r^\alpha) z \qquad (\text{for } z \in \mathbf{R}^n, \quad 1 \le |z| \le r_1, \quad z^0 \text{ defined in eq. (4)}). \end{aligned}$$

Then

$$\begin{aligned} f_1'(z)h &= \alpha\, r^{\alpha-1}\, ((O'(r^\alpha) \circ dr/dz)\, h) z + O(r^\alpha) h \\ &= \alpha\, r^\alpha (O'(r^\alpha)(z^0, h)) z^0 + O(r^\alpha) h . \end{aligned}$$

Let $\| \ \|$ denote the usual norm in the space of linear mappings between two linear spaces. Since the matrix function O has continuous derivatives,

$$\sup_{1 \le r \le r_0} \|O'(r)\| = c_1 < \infty$$

and $c_1 \to 0$ as the original orthogonal mapping Q tends to the identity. We have

$$\sup_{1 \le r \le r_1} \|\alpha\, r^\alpha\, (O'(r^\alpha) \circ dr/dz)\| = \alpha\, c_2, \qquad c_2 < \infty, \tag{5}$$

where the same remark applies to c_2 as to c_1.

Since $\alpha = \log r_0 / \log r_1$,

$$f_1'(z) = O(z) \circ (\mathrm{id} + C(r)/\log r_1) \tag{6}$$

where $C(r)$ is a linear mapping $\mathbf{R}^n \to \mathbf{R}^n$ depending on r in such a way that

$$\sup_{1 \le r \le r_1} \|C(r)\| \le c_3 < \infty \tag{7}$$

where c_3 depends only on r_0 and Q and c_3 tends to zero as Q tends to the identity. But if the derivative of f_1 is of the form (6) and $C(r)$ fulfils (7), then the dilatation K of f_1 satisfies

$$K \le 1 + c/\log r_1$$

where $c \to 0$ if the original orthogonal transformation Q tends to the identity. Thus we have proved Lemma 1.

Lemma 2. *Let S and T be two loxodromic transformations of S^n. Then, if $P(S) = P(T)$, (or $N(S) = N(T)$),*

$$\operatorname{mul} S^l T^m = ((\operatorname{mul} S)^l (\operatorname{mul} T)^m)^{\pm 1},$$

for l, $m \in \mathbf{Z}$ *(the exponent* ± 1 *is so chosen that the resulting number is greater than* 1).

Proof. We can assume that S is of the form

$$S(z) = \lambda_1 O_1(z) \quad \text{for } z \in \mathbf{R}^n \quad (O_1 \in \mathrm{O}(n), \quad 1 < \lambda_1 \in \mathbf{R}),$$

and that T is of the form

$$T(z) = \lambda_2 O_2(z-a) + a \quad \text{for } z \in \mathbf{R}^n \quad (a \in \mathbf{R}^n, \quad O_2 \in \mathrm{O}(n), \quad 1 < \lambda_2 \in \mathbf{R}).$$

Then we have

$$S^l T^m(z) = \lambda_1^l \lambda_2^m O_1^l(O_2^m(z)) - \lambda_1^l \lambda_2^m O_1^l(O_2^m(a)) + \lambda_1^l O_1^l(a)$$

for $z \in \mathbf{R}^n$, and the lemma follows.

L e m m a 3. *Let* G *be a subgroup of* $\mathrm{GM}(n)$ *generated by two loxodromic transformations* S *and* T *of* S^n *without common fixed points. Let* G' *be another subgroup of* $\mathrm{GM}(n)$ *and* $\varphi : G \to G'$ *be a multiplier preserving isomorphism. Then* $\varphi(S)$ *and* $\varphi(T)$ *do not have common fixed points.*

Proof. We assume that $S' = \varphi(S)$ and $T' = \varphi(T)$ have common fixed points and derive a contradiction from this. If this were the case, there would be, by Lemma 2, infinitely many $(l, m) \in \mathbf{Z} \times \mathbf{Z}$ such that

$$\operatorname{mul} S^l T^m \in U$$

where U is a given neighbourhood of $1 \in \mathbf{R}$.

Since S and T do not have common fixed points, G can be supposed to be Schottky-like. (We can replace G by the group generated by S^l and T^l for large enough l.) This means that there are four disjoint closed balls A, B, C, $D \subset S^n$ such that

$$P(S) \in A, \quad N(S) \in B, \quad P(T) \in C, \quad N(T) \in D$$

and that $(S^n \setminus (A \cup B \cup C \cup D)) \cap R(S^n \setminus (A \cup B \cup C \cup D)) = \emptyset$ for $R \in G \setminus \{\mathrm{id}\}$. Then it is easy to see that

$$S^l T^m(A) \subset A, \quad (S^l T^m)^{-1}(D) \subset D, \quad l, m > 0,$$

i.e. $S^l T^m$ is loxodromic with $P(S^l T^m) \in A$, $N(S^l T^m) \in D$. Moreover, there is a real number $k > 1$ depending on A and D (but not on l and m) such that

$$\operatorname{mul} S^l T^m > k$$

is valid for all $l, m > 0$, as is easily seen. In the same manner one can see that $\operatorname{mul} S^l T^m$, $(l, m) \neq 0$, is bounded away from 1. But this contradicts Lemma 2 if $\varphi(T)$ and $\varphi(S)$ have common fixed points and φ preserves multipliers.

L e m m a 4. *Let* G *and* G' *be two subgroups of* $\mathrm{GM}(n)$ *such that* G *is generated by loxodromic* T *and* S *without common fixed points. If* $\varphi : G \to G'$ *is multiplier preserving, then there is a Möbius transformation* $R \in \mathrm{GM}(n)$ *such that*

$$R(P(T)) = P(\varphi(T)), \qquad R(N(T)) = N(\varphi(T)),$$
$$R(P(S)) = P(\varphi(S)), \qquad R(N(S)) = N(\varphi(S)).$$

Proof. We assume $n \geq 3$. (If $G \subset \mathrm{GM}(n)$, then $G \subset \mathrm{GM}(m)$ for $m > n$.) Let $T' = \varphi(T)$, $S' = \varphi(S)$. We may suppose that S and T are conformal; otherwise we replace S and T by S^2 and T^2. (We do this substitution also if S' or T' is anticonformal.) Further, we can suppose that T and T' fix 0 and ∞ and that S and S' fix points in $\mathbf{R}^2$. If R is a loxodromic Möbius transformation we define R_h to be the unique hyperbolic transformation with the same attracting and repelling fixed points as R and for which

$$\mathrm{mul}\, R_h = \mathrm{mul}\, R.$$

Let $G^m = \langle T^m, S^m\rangle$ be the group generated by T^m and S^m, and denote $G'^m = \langle T'^m, S'^m\rangle$, $G_h^m = \langle T_h^m, S_h^m\rangle$, $G_h'^m = \langle T_h'^m, S_h'^m\rangle$, and let $\varphi^m : G^m \to G_h^m$ and $\varphi'^m : G'^m \to G_h'^m$ be the mappings defined by

$$T^m \mapsto T_h^m, \quad S^m \mapsto S_h^m \ \text{ and } \ T'^m \mapsto T_h'^m, \quad S'^m \mapsto S_h'^m.$$

We wish to obtain an estimate for the dilatation K_m of φ^m and K_m' of φ'^m. We show that, beginning from some m,

$$(8) \qquad \begin{aligned} K_m &\leq 1 + c(\mathrm{rot}\, T^m, \mathrm{rot}\, S^m)/m = 1 + c_m/m, \\ K_m' &\leq 1 + c'(\mathrm{rot}\, T'^m, \mathrm{rot}\, S'^m)/m = 1 + c_m'/m, \end{aligned}$$

where $c(O, P)$, $O, P \in \mathrm{SO}(n)$, are bounded and $c(O, P) \to 0$ as $O, P \to \mathrm{id}$ in $\mathrm{SO}(n)$. (Since $\mathrm{rot}\, T$ and $\mathrm{rot}\, S$ are determined only up to the conjugacy class in $\mathrm{SO}(n)$, $c(O, P)$ depends only on the conjugacy class of O and P.) A similar remark applies to $c'(O, P)$.

To prove (8) we note that, for large m, G_m is Schottky-like, i.e. there are closed disjoint n-balls of S^n, denoted by A_m, B_m, C_m, D_m, such that

$$P(T) \in A_m, \qquad N(T) \in B_m, \qquad P(S) \in C_m, \qquad N(S) \in D_m,$$

$F_m = \mathrm{cl}\,(S^n \setminus (A_m \cup B_m \cup C_m \cup D_m))$ being a fundamental domain for G^m. It is clear, at least for large m, that the balls can be so chosen that F_m is also a fundamental domain for G_h^m. Schottky-groups are free, and so G^m and G_h^m are free, hence φ^m is an isomorphism. Since T is normalized to fix 0 and ∞, we can assume that B_m is the unit ball $S^{n-1} \subset \mathbf{R}^n$ and that $A_m = (\mathrm{mul}\, T)^m B_m$.

Now we can use Lemma 1 to find a K-quasiconformal self-mapping f of the set $\{ x \in \mathbf{R}^n : 1 \leq |x| \leq (\mathrm{mul}\, T)^{k+m} \}$, where m, $k \in \mathbf{Z}$, m is large and fixed and $k > 0$ varies, such that the following conditions are fulfilled:

(i) $f \mid \{ x \in \mathbf{R}^n : 1 \leq |x| \leq (\mathrm{mul}\, T)^m \} = \mathrm{id}$,

(ii) $f \mid (\mathrm{mul}\, T)^{k+m} S^{n-1} = \mathrm{rot}\, T^{k+m} | \, (\mathrm{mul}\, T)^{k+m} S^{n-1}$,

(iii) $K \leq 1 + c/(k \log (\mathrm{mul}\, T))$,

where c depends on $\mathrm{rot}\, T^{k+m}$ in such a way that it tends to 0 as $\mathrm{rot}\, T^{k+m}$ tends to the identity in $\mathrm{SO}(n)$. We fix m so that $C_l \cup D_l \subset \{ z \in \mathbf{R}^n : 1 \leq |z| \leq \mathrm{mul}\, T^m \}$ for every $l > m$.

Define

$$F^k = \langle T_h^k , S^k \rangle , \qquad k > 0 ,$$

and let $\psi^k : G^k \to F^k$ be defined by

$$\psi^k(T^k) = T_h^k , \qquad \psi^k(S^k) = S^k .$$

Then ψ^k are isomorphisms for large k. Let $k > 0$. Now we define a homeomorphism $f' : S^n \to S^n$,

$$f'(x) = (\psi^{m+k})^{-1}(R)(f(R(x))) ,$$

if $R \in F^{m+k}$ and $R(x) \in \mathrm{cl}\,(S^n \setminus (A_{m+k} \cup B_{m+k} \cup C_{m+k} \cup D_{m+k}))$. Since a $(n-1)$-sphere is a removable singularity for quasiconformal mappings in n-dimensional space, f' is K-quasiconformal in the set where it is now defined, i.e. in the regular set of F^{m+k}. We can extend it also to the limit set of F^{m+k}. Notice that if x is a point of the limit set of F^{m+k}, then there is a sequence of elements $T_i \in F^{m+k}$, $i > 0$, and $n-1$ spheres $E_i \in \{ \mathrm{bd}\, A_{m+k} , \ \mathrm{bd}\, B_{m+k} , \ \mathrm{bd}\, C_{m+k} , \ \mathrm{bd}\, D_{m+k} \}$, $i > 0$, such that $\lim_{i \to \infty} \mathrm{diam}\, T_i(E_i) = 0$ (in the spherical metric of S^n) and that $\{ T_i(E_i) \}_{i > j}$ and x are in the same component of $S^n \setminus T_j(E_j)$ for all j. Now it is easy to see that $f'(T_i(E_i))_{i > 0}$ converges to a point $y \in S^n$. We set $f'(x) = y$. Extended this way, f' is a homeomorphism that induces $(\psi^{m+k})^{-1}$. We know that it is K-quasiconformal outside the limit set of F^{m+k}.

Let $\psi'^k : F^k \to G_h^k$ be defined by

$$\psi'^k(T_h^k) = T_h^k , \qquad \psi'^k(S^k) = S_h^k \qquad \text{for } k \geq m .$$

As above, we can show that there is a homeomorphism $f'' : S^n \to S^n$ such that f''^{-1} induces ψ'^{m+k} for $k > 0$, and that f'' is K'-quasiconformal outside the limit set of G_h^{m+k} with

$$K' < 1 + c'/(k \log (\mathrm{mul}\, S)) ,$$

where $c' \to 0$ as $\operatorname{rot} S^{m+k} \to \mathrm{id}$. Thus $f' \circ f''$ is $K K'$-quasiconformal outside the limit set of G_h^{m+k}. But G_h^{m+k} fixes S^2, and consequently its limit set is contained in S^2. But S^2 is a removable singularity in S^n. Thus $f' \circ f''$ is $K K'$-quasiconformal in the whole S^n. Since φ^{m+k} is induced by $(f' \circ f'')^{-1}$, the first of the equalities in (8) is seen to be valid. The other equality in (8) is proved in the same manner.

Since T_h, S_h, T'_h and S'_h are Möbius transformations of $\mathbf{R}^2$, they can be represented as matrices of $\mathrm{SL}(2, \mathbf{C})$. In view of the above normalization we have

$$T_h = T'_h = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \qquad \lambda^2 = \operatorname{mul} T = \operatorname{mul} T',$$

$$S_h = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad a d - b c = 1, \; a + d = \zeta + \zeta^{-1}, \; \zeta = (\operatorname{mul} S)^{1/2},$$

$$S'_h = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}, \qquad a' d' - b' c' = 1, \; a' + d' = \zeta + \zeta^{-1}.$$

In diagonalized form we have

$$(*) \quad S_h = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix} \begin{pmatrix} h & -f \\ -g & e \end{pmatrix} = \begin{pmatrix} e h \zeta - f g \zeta^{-1} & \cdot \\ \cdot & -f g \zeta + e h \zeta^{-1} \end{pmatrix},$$

$e h - f g = 1$. Similarly, S'_h can be diagonalized by a matrix $(e' f' \mid g' h')$ with the same diagonal matrix. We have

$$T_h^m S_h^m = \begin{pmatrix} \lambda^m \zeta^m e h - \lambda^m \zeta^{-m} f g & \cdot \\ \cdot & -\lambda^{-m} \zeta^m f g + \lambda^{-m} \zeta^{-m} e h \end{pmatrix}.$$

It follows from (8) that the mapping $\varphi_h^m : G_h^m \to G_h'^m$, $T_h^m \mapsto T_h'^m$, $S_h^m \mapsto S_h'^m$ has bounded dilatation for large m with dilatation less than or equal to

$$(9) \qquad 1 + c''_m / m$$

where $c''_m \leq M < \infty$ beginning from some m and c''_m is near 0 if $\operatorname{rot} T^m$ is near id and $\operatorname{rot} S^m$ is near id. On the other hand, the multiplier of an element of $\mathrm{SL}(2, \mathbf{C})$ is

$$\operatorname{mul} \begin{pmatrix} i & j \\ k & l \end{pmatrix} = (|i + l| + o(|i + l|^{-1}))^2$$

where $o(x)$ tends to 0 as x tends to 0. Thus

$$\tfrac{1}{2} \log \operatorname{mul} T_h^m S_h^m = m \log \lambda + m \log \zeta + \log |e h| + o(\lambda^{-m} + \zeta^{-m})$$

$$\tfrac{1}{2} \log \operatorname{mul} T_h'^m S_h'^m = m \log \lambda + m \log \zeta + \log |e' h'| + o(\lambda^{-m} + \zeta^{-m}).$$

In view of (9) we have

$$m \log \lambda + m \log \zeta + \log |e\,h| + o(\lambda^{-m} + \zeta^{-m})$$
$$\leq (1 + c''_m/m)(m \log \lambda + m \log \zeta + \log |e'\,h'| + o(\lambda^{-m} + \zeta^{-m}))$$

or

$$\log |e\,h| \leq \log |e'\,h'| + (c''_m/m) \log |e'\,h'|$$
$$+ c''_m (\log \lambda + \log \zeta) + o(\lambda^{-m} + \zeta^{-m}) .$$

Since the group $\mathrm{O}(n)$ is compact, there are arbitrarily large values of m such that $\operatorname{rot} T^m$ and $\operatorname{rot} S^m$ are arbitrarily near $\mathrm{id} \in \mathrm{O}(n)$ and thus, given m_0, there is always $m > m_0$ such that $c''_m < 1/m_0$. Therefore we must have

$$\log |e\,h| \leq \log |e'\,h'| .$$

Since the reversed inequality is also valid,

$$|e\,h| = |e'\,h'| . \tag{10}$$

If we replace $T_h^m S_h^m$ by $T_h^m S_h^{-m}$, a similar argument shows that

$$|f\,g| = |f'\,g'| . \tag{11}$$

Since the matrices of $\mathrm{SL}(2, \mathbf{C})$ have determinant equal to 1, $e\,h - f\,g = 1$ and $e'\,h' - f'\,g' = 1$. If we combine this with (10) and (11), we see that the triangle with vertices 0, 1 and $e\,h$ is equilateral with the triangle with vertices 0, 1, $e'\,h'$. Since they have the common side $[0, 1]$, we must have either

$$e\,h = e'\,h' \quad \text{and} \quad f\,g = f'\,g' \tag{12}$$

or, alternatively,

$$e\,h = \overline{e'\,h'} \quad \text{and} \quad f\,g = \overline{f'\,g'} , \tag{13}$$

the bar $^{-}$ denoting the complex conjugation.

Suppose we have the case (12). Equation (*) for S_h and a similar (omitted) equation for S'_h show that

$$a = a' , \qquad d = d' ,$$

i.e., S_h and S'_h have equal diagonal elements. If we conjugate S'_h by a diagonal $R \in \mathrm{SL}(2, \mathbf{C})$ we have

$$R\,S'_h\,R^{-1} = \begin{pmatrix} \varkappa & 0 \\ 0 & \varkappa^{-1} \end{pmatrix} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \begin{pmatrix} \varkappa^{-1} & 0 \\ 0 & \varkappa \end{pmatrix} = \begin{pmatrix} a' & \varkappa^2 b' \\ \varkappa^{-2} c' & d' \end{pmatrix} .$$

There is always $\varkappa \in \mathbf{C}$ such that $\varkappa^2 b = b'$ and, consequently, as the determinant of the matrix is 1, $\varkappa^{-2} c' = c$. But then $S_h = R\,S'_h\,R^{-1}$

and in this case certainly $P(S) = P(S_h) = R(P(S_h')) = R(P(S'))$, and $N(S) = N(S_h) = R(N(S_h')) = R(N(S'))$. Since T and T' were normalized to fix 0 and ∞ and R can be extended to a Möbius transformation of S^n, the lemma follows in case (12).

In case (13) we denote by R an element of $\mathrm{GM}(n)$ whose restriction to S^2 is the complex conjugation $z \mapsto \bar{z}$. Then conjugation by R reduces this case to the above case.

Theorem 2. *Let G and G' be two groups of Möbius transformations of S^n such that G has at least two loxodromic transformations without common fixed points. Let $\varphi : G \to G'$ be a multiplier preserving isomorphism. Further, suppose that*

$$\mathrm{Fix}\, G = \{ P(T) : T \in G \text{ loxodromic} \}$$

is not contained in any k-sphere S^k or its image $T(S^k)$ for $k < n$, $T \in \mathrm{GM}(n)$. Then φ is a conjugation in the group of all (conformal and anticonformal) Möbius transformations of S^n.

Proof. By assumption, there are two loxodromic transformations T, $S \in G$ without common fixed points. We assume that $N(T) = 0$, $P(T) = \infty$ and prove Theorem 2 step by step.

A. *If Q, $R \in G$ are loxodromic and*

$$\{ P(Q), N(Q) \} \cap \{ P(R), N(R) \} = \emptyset ,$$

then $\lim_{n\to\infty} P(Q^n R^{-n}) = P(Q)$ and $\lim_{n\to\infty} N(Q^n R^{-n}) = P(R)$.

The proof of A is based on the fact that for large n the group generated by Q^n and R^n is Schottky-like. It is exactly similar to the proof of the case where Q, $R \in \mathrm{SL}(2, \mathbf{R})$, presented in Tukia [6], p. 9.

B. *If $R \in G$ is loxodromic, then there is $Q \in G$ such that R and Q do not have common fixed points.*

For in the set $\{ P(T), N(T), P(S), N(S) \}$ there are at least two points that do not belong to $\{ P(R), N(R) \}$. Then the result follows by A.

C. *Suppose R, $Q \in G$ are loxodromic, $P(R) \neq P(Q)$. Then there is a sequence of loxodromic $T_n \in G$ such that, if $T_n' = \varphi(T_n)$,*

$$\lim_{n\to\infty} P(T_n) = P(R), \qquad \lim_{n\to\infty} P(T_n') = P(R') = P(\varphi(R)),$$
$$\lim_{n\to\infty} N(T_n) = P(Q), \qquad \lim_{n\to\infty} N(T_n') = P(Q') = P(\varphi(Q)).$$

If the set $\{ P(R), N(R), P(Q), N(Q) \}$ contains four elements, C follows by A and by Lemma 3. If it contains three or two elements, we can find by A and B a series of loxodromic elements $S_n \in G$ such that $\lim_{n\to\infty} P(S_n) = P(Q)$, $\lim_{n\to\infty} N(S_n) \neq P(R)$ and $\neq N(R) = N(Q)$. Now we can form a double series T_{nm} with a subseries $T_{n_k m_k}$ fulfilling our conditions.

Note that C is of course valid also if $P(R) = P(Q)$, but we do not need this.

We can define a mapping f_φ : Fix $G \to$ Fix G' as follows. Let $R \in G$ be loxodromic. Then we set

$$f_\varphi(P(R)) = P(\varphi(R)) .$$

We must show that this does not depend on the choice of R. Let $P(Q) = P(R)$, $Q \in G$ loxodromic. Then by Lemma 3, $Q' = \varphi(Q)$ and $R' = \varphi(R)$ have common fixed points. To see that $P(R') = P(Q')$, we use the equation of Lemma 2. For then

$$\text{(14)} \qquad \operatorname{mul} R^l Q^m = ((\operatorname{mul} R)^l (\operatorname{mul} Q)^m)^{\pm 1} , \qquad l , m \in \mathbf{Z} .$$

If $P(R) = N(Q)$, we must replace m by $-m$ in the right side of (14). Since φ is multiplier preserving, we must have $P(R') = P(Q')$.

We denote $T' = \varphi(T)$ and $S' = \varphi(S)$, where T and S are the elements of G defined in the beginning of the proof. We suppose that T' fixes 0 and ∞ with $P(T') = \infty$. Moreover we suppose that $P(S) = P(S')$. This can always be achieved by conjugation which leaves 0 and ∞ fixed.

Now we claim:

(i) $|P(R)| = |P(R')|$, for $R \in G$ loxodromic, $R' = \varphi(R)$, $|P(R)| < \infty$,

(ii) $(P(R), P(Q)) = (P(R'), P(Q'))$, for R, $Q \in G$ loxodromic

$$R' = \varphi(R) , \quad Q' = \varphi(Q') , \quad (|P(R)| , \ |P(Q)| < \infty) .$$

We prove (i) first. Assume $P(R) \neq P(S)$. By C there is a sequence T_n of loxodromic elements of G such that

$$\lim_{n\to\infty} P(T_n) = P(R) , \quad \lim_{n\to\infty} P(T_n') = P(R') , \quad (T_n' = \varphi(T_n)) ,$$
$$\lim_{n\to\infty} N(T_n) = P(S) , \quad \lim_{n\to\infty} N(T_n') = P(S') .$$

By Lemma 4 there are orthogonal linear mappings $O_m \in \mathrm{O}(n)$ and real numbers $\lambda_m > 0$, $m > 0$, such that

$$\lambda_m O_m(P(T_m)) = P(T_m') \text{ and } \lambda_m O_m(N(T_m)) = N(T_m') .$$

The family $\{O_m\}$, $m > 0$, is a normal family and therefore there is a subsequence O_{m_1}, $O_{m_2}, \ldots$ for which the limit $\lim_{i\to\infty} O_{m_i}$ is an orthogonal linear mapping. Since $|P(S)| = |P(S')|$, we must also have $\lim_{i\to\infty} \lambda_i = 1$. Therefore $|P(R')| = \lim_{n\to\infty} |P(T_n')| = \lim_{i\to\infty} \lambda_{m_i} |O_{m_i}(P(T_{m_i}))| = |P(R)|$.

The proof of (ii) is the same; we only replace S by Q.

But if a mapping satisfies (i) and (ii) then it must be the restriction of an orthogonal mapping of $\mathbf{R}^n$, which, moreover, is unique, since Fix G

spans $\mathbf{R}^n$. For simplicity, we can now assume that this orthogonal transformation is the identity and thus $\operatorname{Fix} G = \operatorname{Fix} G'$ and the mapping $P(R) \mapsto P(\varphi(R))$ is the identity, $R \in G$ loxodromic.

The proof of the theorem can now be concluded. We show that $T = T'$, the normalized loxodromic transformation of G. We have

$$T(z) = \lambda\, O(z), \quad z \in \mathbf{R}^n, \quad O \in \mathrm{O}(n), \quad \lambda = \operatorname{mul} T = \operatorname{mul} T',$$
$$T'(z) = \lambda\, O'(z), \quad z \in \mathbf{R}^n, \quad O' \in \mathrm{O}(n).$$

Thus $T = T'$ if $O = O'$. But since $O \mid \operatorname{Fix} G = O' \mid \operatorname{Fix} G$ and since $\operatorname{Fix} G$ spans $\mathbf{R}^n$, $O = O'$. It is clear that $R = \varphi(R)$ also for other loxodromic $R \in G$. Finally, if $R \in G$ is arbitrary, there is a loxodromic $Q \in G$ such that RQ is loxodromic. This shows that $\varphi(R) = R$. This concludes the proof.

Some related theorems. Sorvali [5] has proved our theorem for $\mathrm{SL}(2, \mathbf{R})/\{1, -1\}$. His theorem is stated only for discrete groups but the proof does not make use of the discreteness of the groups. Selberg [4] has also proved similar results for deformations of groups of $\mathrm{SL}(n, \mathbf{R})$, stated in terms of traces of matrices of $\mathrm{SL}(n, \mathbf{R})$. The group of Möbius transformations of S^n can be identified with $\mathrm{O}(1, n+1) / \{1, -1\}$ and in view of this we might ask what the relation is between the multiplier and the trace of an element of $\mathrm{O}(1, n+1)$ which is a subgroup of $\mathrm{SL}(n+2, \mathbf{R})$. For $\mathrm{SL}(2, \mathbf{R})$ we have $|\operatorname{tr} T| = (\operatorname{mul} T)^{1/2} + (\operatorname{mul} T)^{-1/2}$.

If $T \in \mathrm{O}(1, n+1)$ is loxodromic when regarded as a transformation of S^n, it can be conjugated to the form

$$T = \begin{pmatrix} A & 0 \\ 0 & O \end{pmatrix}$$

where A is a real 2×2-matrix

$$A = \tfrac{1}{2} \begin{pmatrix} \lambda + \lambda^{-1} & \lambda - \lambda^{-1} \\ \lambda - \lambda^{-1} & \lambda + \lambda^{-1} \end{pmatrix}$$

and $O \in \mathrm{O}(n)$ and $\lambda = \operatorname{mul} T$ (Mostow [3]). It follows:

$$\operatorname{mul} T = \lim_{n\to\infty} |\operatorname{tr} T^n|^{1/n}. \tag{15}$$

Thus Theorem 2 is also valid if we replace the words "multiplier preserving" by "trace-preserving". Finally, we remark that if the Möbius group of S^2 is identified with $\mathrm{SL}(2, \mathbf{C})/\{1, -1\}$, then the right side of (15) gives the square root of mul T. Thus in this case also the property that an isomorphism between Möbius groups preserves multipliers can be replaced by the requirement that it preserves the absolute values of traces.

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