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HAUSDORFF DIMENSION AND EXCEPTIONAL SETS OF LINEAR TRANSFORMATIONS

R. KAUFMAN and P. MATTILA

1. *Introduction.* Let m and n be positive integers with $m \leq n$ and let $O^*(n, m)$ be the set of all orthogonal projections of R^n onto R^m (see [3, 1.7.4]). A linear mapping $p: R^n \to R^m$ belongs to $O^*(n, m)$ if and only if p maps the orthogonal complement of $\ker p$ isometrically onto R^m. Considered as a subset of the space of all linear mappings of R^n into R^m, which is denoted by $\operatorname{Hom}(R^n, R^m)$ and identified with R^{nm}, $O^*(n, m)$ has positive and finite $m(2n - m - 1)/2$-dimensional Hausdorff measure (see [3, 3.2.28(5)]). It was proved first in [5] for $m = 1$, $n = 2$ and then in [8] for general m, n that if E is a Borel (or, more generally, Suslin) set in R^n with $\dim E = s \leq m$, then $\dim p(E) = s$ for $\mathscr{H}^{s+m(2n-m-3)/2}$ almost all $p \in O^*(n, m)$. Here dim means Hausdorff dimension and $\mathscr{H}^t$ is the t-dimensional Hausdorff measure. In [6] an example of a compact plane set was given such that the corresponding exceptional set of projections has positive Hausdorff dimension. In this paper we shall show that the number $s + m(2n - m - 3)/2$ is the best possible upper bound for the Hausdorff dimension of the exceptional set. We shall also indicate how the result mentioned above can be generalized to a larger class of sets of linear transformations.

2. *Preliminaries.* Besides what is presented in the introduction, we shall use the following notation: The orthogonal group of R^n is denoted by $O(n)$. The Grassmann manifold of all m-dimensional linear subspaces of R^n is $G(n, m)$. For $V \in G(n, m)$, P_V is the orthogonal projection of R^n onto V and $V^\perp$ is the orthogonal complement of V. If $f \in \operatorname{Hom}(R^n, R^m)$, then

$$\|f\| = \max\{ |f(x)| : x \in S^{n-1} \},$$
$$l(f) = \min\{ |f(x)| : x \in S^{n-1} \cap (\ker f)^\perp \}$$

and $r(f)$ is the rank of f.

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There exists a positive number c depending only on m with the following properties:

2.1. If V, $W \in G(n, m)$ and $|w - P_V(w)| \leq \delta |w|$ for $w \in W$, then $\|P_V - P_W\| \leq c\,\delta$.

2.2. If f, $g \in \operatorname{Hom}(R^n, R^m)$ and $r(f) = r(g)$, then

$$l(f)\,|x - P_{\ker f}(x)| \leq |f(x)| \text{ for } x \in R^n,$$

$$l(f)\,\|P_{\ker f} - P_{\ker g}\| \leq c\,\|f - g\|.$$

One can prove 2.1 by choosing an orthonormal base $\{w_1, \ldots, w_m\}$ for W and by constructing an orthonormal base $\{v_1, \ldots, v_m\}$ for V such that $|v_i - w_i| \leq c'\,\delta$ where c' depends only on m. The first inequality in 2.2 is an immediate consequence of the definition of $l(f)$, and the second follows from the first and 2.1.

The metric d and Hausdorff measures on $G(n, m)$ are defined via the identification used in [3, 3.2.28]. The inequalities [9, Chapter I, 12 (17), 15 (7)] and 2.1 imply that there is a positive number b such that

$$b^{-1} d(V, W) \leq \|P_V - P_W\| \leq b\,d(V, W)$$

for V, $W \in G(n, m)$. Therefore, for the purposes of this paper, we could as well identify $V \in G(n, m)$ with P_V.

3. Proposition. *Let k and n be positive integers and let M be a Borel set in* $\operatorname{Hom}(R^n, R^k)$. *For $x \in S^{n-1}$ denote*

$$M_x = \{f \in M : f(x) = 0\}.$$

Suppose that there exist real numbers $c > 0$, $C > 0$ and $\nu \geq 0$ and that to each $g \in O(n)$ corresponds a mapping $A_g : M \to M$ such that the following conditions are satisfied for all $f \in M$ and $g \in O(n)$:

(1) $\ker f \circ g \subset \ker A_g f$,

(2) $\|f - A_g f\| \leq C\,\|f\|\,\|\mathbf{1}_{R^n} - g\|$, *where $\mathbf{1}_{R^n}$ is the identity mapping of R^n,*

(3) $\mathscr{H}^\nu(M_x) \leq C$ *for all* $x \in S^{n-1}$,

(4) $\mathscr{H}^\nu\{h' \in M_x : \|h' - h\| \leq r\} \geq c\,r^\nu$ *for all* $x \in S^{n-1}$, $h \in M_x$ *and* $0 < r < 1$.

If E is a Suslin set in R^n and $\dim E = s$, *then* $\dim f(E) = s$ *for $\mathscr{H}^{s+\nu}$ almost all $f \in M$.*

Since $\dim f(E) = s$, if $\ker f = \emptyset$, we may assume that $\ker f \neq \emptyset$ for $f \in M$. Further, since

$$M \setminus \{0\} = \bigcup_{j=1}^{\infty} N_j, \text{ where } N_j = \{f \in M : l(f) > 1/j,\ \|f\| < j\},$$

it is sufficient to prove that $\dim f(E) = s$ for $\mathscr{H}^{s+\nu}$ almost all $f \in N_j$ for all j. The proof of this fact is almost identical with that of Theorem 5.6(b) in [8]. Indeed, after fixing j we only have to verify the following analogue of the formula (1) occurring in the proof of Lemma 5.5 in [8]:

There exists a positive number c_1 such that if $f \in N_j$, $x \in S^{n-1}$, $\delta > 0$ and $|f(x)| \leq \delta$, then $\|f - h\| \leq c_1 \delta$ for some $h \in M_x$.

To see this, take $z \in S^{n-1} \cap \ker f$ such that $|P_{\ker f}(x)|\, z = P_{\ker f}(x)$. Then $|x - z| \leqq 2 j \delta$ by 2.2. Choose $g \in O(n)$ such that $g(x) = z$ and $\|\mathbf{1}_{R^n} - g\| = |x - z|$. Then $f(g(x)) = 0$, $A_g f \in M_x$ by (1), and $\|f - A_g f\| \leq c_1 \delta$ with $c_1 = 2 j^2 C$ by (2).

4. *Remarks.* One can often apply this proposition in the following situation: To each pair of positive integers m, n, $m \leq n$, corresponds a smooth submanifold $M(n, m)$ of some space Hom (R^n, R^k) such that $M(n, m)_x$ is isometric with $M(n-1, m)$ for $m < n$ and $x \in S^{n-1}$. Then the conditions (3) and (4) are satisfied with $\nu = \dim M(n-1, m)$ if $\mathscr{H}^\nu(M(n-1, m)) < \infty$ and $M(n-1, m)$ is "sufficiently homogeneous". Such is the case if $M(n, m)$ is $O^*(n, m)$ or $\{ P_V : V \in G(n, m) \}$. In the first case one can define A_g by $A_g p = p \circ g$ and in the second by $A_g P_V = P_{g^{-1}(V)}$. Then all the assumptions are in force. If $M(n, m) =$ Hom (R^n, R^m), then (3) is false. However the proposition is true even in this case, because all the assumptions hold with $A_g f = f \circ g$ for $M = \{ f \in \mathrm{Hom}\,(R^n, R^m) : \|f\| < R \}$ whenever $0 < R < \infty$.

The dimensions of the manifolds $O^*(n, m)$, $G(n, m)$ and Hom (R^n, R^m) are $m(2n - m - 1)/2$, $m(n-m)$ and mn (see [3, 3.2.28(5)]), so that the upper bounds for the dimensions of the exceptional sets are $s + m(2n - m - 3)/2$, $s + m(n - m - 1)$ and $s + m(n-1)$, respectively. We shall next show that these upper bounds may be attained. More precisely, we shall prove

5. Theorem. *If m and n are integers and s is a real number with $0 < s \leq m < n$, then there exists a compact set $E \subset R^n$ such that* $\dim E = s$ *and*

$$\dim \{ p \in O^*(n, m) : \dim p(E) < s \} = s + m(2n - m - 3)/2,$$

$$\dim \{ V \in G(n, m) : \dim P_V(E) < s \} = s + m(n - m - 1),$$

$$\dim \{ f \in \mathrm{Hom}\,(R^n, R^m) : r(f) = m, \ \dim f(E) < s \} = s + m(n-1).$$

The following lemma is an immediate consequence of [3, 2.10.25]:

6. Lemma. *If X and Y are metric spaces, Y is σ-compact, $F : X \to Y$ is Lipschitzian, $\emptyset \neq B \subset Y$, $0 \leq \nu < \infty$ and $\mathscr{H}^\nu(F^{-1}\{y\}) > 0$ for $y \in B$, then* $\dim F^{-1}(B) \geq \dim B + \nu$.

Proof. We may assume that Y is compact. If $t \geq 0$, it follows from [3, 2.10.25] that for a positive number c

$$\int_B^* \mathscr{H}^\nu(F^{-1}\{y\})\, d\mathscr{H}^t y \;\leq\; c\mathscr{H}^{t+\nu}(F^{-1}(B)) \,.$$

Hence $\mathscr{H}^{t+\nu}(F^{-1}(B)) = 0$ implies $\mathscr{H}^t(B) = 0$, which means that $\dim F^{-1}(B) \geq \dim B + \nu$.

7. *Proof of Theorem* 5. We begin with the case of $G(n, m)$ and assume first that $m = n-1$. Let (N_k) be a strictly and rapidly increasing sequence of positive integers, e.g. $N_{k+1} > N_k^k$, and let E be the set of all $(x_1, \ldots, x_n) \in R^n$ such that

$$-1 \leq x_j \leq 1\,, \qquad \|N_k\, x_j\| \leq N_k^{1-n/s} \qquad \text{for } 1 \leq j \leq n\,,\ k \geq 1.$$

(For a real number t, $\|t\|$ is the distance from t to the nearest integer.) Then $\dim E = s$. This follows from [2, Theorem 10]. The method goes back to Jarnik [4], and a similar problem occurs in [7]. (For further information on number-theoretic methods, see [1] and its bibliography.) For $y = (1, y_2, \ldots, y_n)$ we denote by T_y the orthogonal projection from R^n onto the orthogonal complement of y. We shall show that

$$\dim \{\, y :\ \dim T_y(E) < s \,\} \geq s\,,$$

which is equivalent with

$$\dim \{\, V \in G(n\,, n-1) :\ \dim P_V(E) < s \,\} \geq s\,.$$

Fix $\delta > 0$ and let A_δ be the set of all $y = (1, y_2, \ldots, y_n)$ such that for arbitrarily large k there is an integer H for which $k < H < N_k^{1-\delta}$ and $\|H\, y_j\| < H\, N_k^{-n/s}$ for $2 \leq j \leq n$. Using [4, Satz 4] or [7], one finds that $\dim A_\delta \geq s\,(1-\delta)$. Since $s\,(1-\delta) \to s$ as $\delta \to 0$, it is sufficient to show that $\dim T_y(E) < s$ for $y \in A_\delta$.

Let $y \in A_\delta$ and let k and H be as above. Choose integers $b_2, \ldots, b_n$ such that $|H\, y_j - b_j| < H\, N_k^{-n/s}$ for $2 \leq j \leq n$ and denote $y^* = (1, b_2\, H^{-1}, \ldots, b_n\, H^{-1})$. Then $|y - y^*| < n\, N_k^{-n/s}$, whence $\|T_y - T_{y^*}\| < O(1)\, N_k^{-n/s}$. ($O(1)$ stands for a constant independent of k.) Let $v = (v_1, \ldots, v_n)$ with each v_j an integer and $|v_j| \leq N_k$. We write $v_1 = q_1\, H + r_1$, where q_1 and r_1 are the uniquely determined integers such that $0 \leq r_1 < H$. Then

$$v \;=\; (q_1\, H\,, q_1\, b_2\,, \ldots\,, q_1\, b_n) + (r_1\,, v_2 - q_1\, b_2\,, \ldots\,, v_n - q_1\, b_n)\,.$$

The first member on the right is parallel to y^*, while the second takes at most $H\,(3\, N_k)^{n-1} = O(1)\, H\, N_k^{n-1}$ values, when v varies. Hence also

$T_{y*}(N_k^{-1} v)$ assumes at most $O(1)\,H\,N_k^{n-1}$ values, and, by the definition of E, $T_{y*}(E)$ can be covered by that many balls of radius less than $O(1)\,N_k^{-n/s}$. Inasmuch as $\|T_y - T_{y*}\| < O(1)\,N_k^{-n/s}$, the same is true for $T_y(E)$. For $(n-\delta)\,s/n < \alpha < s$, $H\,N_k^{n-1}\,(N_k^{-n/s})^\alpha < N_k^{n-\delta-n\alpha/s}$ tends to zero as $k \to \infty$; hence $\mathscr{H}^\alpha(T_y(E)) = 0$. Therefore $\dim T_y(E) < s$, as required.

We next assume that $m < n-1$ and proceed by induction on n. So assume that there exists a compact set $E \subset R^{n-1}$ with $\dim E = s$ such that letting

$$A = \{ V \in G(n-1\,,m) : \dim P_V(E) < s \},$$

we have $\dim A = s + m\,(n - m - 2)$. We identify R^{n-1} with $\{ (x_1, \ldots, x_n) \in R^n : x_n = 0 \}$. Let G be the set of all $V \in G(n\,,m)$ such that $(x_1, \ldots, x_n) \in S^{n-1} \cap V$ implies $|x_n| < 1/2$ and define $P : R^n \to R^{n-1}$ by $P(x_1, \ldots, x_n) = (x_1, \ldots, x_{n-1})$ and $F : G \to G(n-1\,,m)$ by $F(V) = P(V)$. Then F is Lipschitzian (this can be seen with the help of 2.1) and $F^{-1}\{V\}$ is isometric with an open subset of $G(m+1\,,m)$ for $V \in G(n-1\,,m)$. Thus $\mathscr{H}^m(F^{-1}\{V\}) > 0$ for $V \in G(n-1\,,m)$, and Lemma 6 implies that $\dim F^{-1}(A) \geq \dim A + m = s + m\,(n - m - 1)$. Since $\dim P_V(E) < s$ for $V \in F^{-1}(A)$, we have shown that the set

$$A_1 = \{ V \in G(n\,,m) : \dim P_V(E) < s \}$$

is of dimension $s + m\,(n - m - 1)$.

To treat the case of $O^*(n\,,m)$, assume that E and A_1 are as above and denote

$$A_2 = \{ p \in O^*(n\,,m) : \dim p(E) < s \}.$$

Define $F : O^*(n\,,m) \to G(n\,,m)$ letting $F(p)$ be the orthogonal complement of $\ker p$. Then F is Lipschitzian by 2.2 and $F^{-1}\{V\}$ is isometric with $O(m)$, whence $\mathscr{H}^{m(m-1)/2}(F^{-1}\{V\}) > 0$, for $V \in G(n\,,m)$. Moreover $A_2 \supset F^{-1}(A_1)$. Using Lemma 6 we infer $\dim A_2 \geq \dim F^{-1}(A_1) \geq \dim A_1 + m\,(m-1)/2 = s + m\,(2\,n - m - 3)/2$. Thus $\dim A_2 = s + m\,(2\,n - m - 3)/\,2$.

Finally we consider $\mathrm{Hom}\,(R^n\,, R^m)$ and set

$$A_3 = \{ f \in \mathrm{Hom}\,(R^n\,, R^m) : r(f) = m\,,\ \dim f(E) < s \}.$$

Let B be the set of all $f \in \mathrm{Hom}\,(R^n\,, R^m)$ such that $l(f) > 1$ and $f = L \circ P_V$ for some $V \in G(n\,,m)$ and some non-singular linear mapping $L :\ V \to R^m$. Then V is uniquely determined by f; in fact $V = (\ker f)^\perp$. We define $F :\ B \to G(n\,,m)$ setting $F(f) = V$ for $f = L \circ P_V$. Then f is Lipschitzian by 2.2 and $F^{-1}\{V\}$ is isometric with an open subset of

Hom (R^m, R^m) for $V \in G(n, m)$, whence $\mathscr{H}^{m^2}(F^{-1}\{V\}) > 0$. Clearly $A_3 \supset F^{-1}(A_1)$. Therefore $\dim A_3 \geq \dim F^{-1}(A_1) \geq \dim A_1 + m^2 = s + m(n-1)$ and $\dim A_3 = s + m(n-1)$.

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University of Illinois
Department of Mathematics
Urbana, Illinois 61801
USA

University of Helsinki
Department of Mathematics
SF-00100 Helsinki 10
Finland

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