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A STRICT INCLUSION RELATED TO BIHARMONIC GREEN'S FUNCTIONS OF CLAMPED AND SIMPLY SUPPORTED BODIES

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Let O_β^N and O_γ^N be the classes of Riemannian N-manifolds, $N \geqq 2$, which do not carry biharmonic Green's functions β of clamped bodies, characterized by "boundary data" $\beta = \partial\beta/\partial n = 0$, or biharmonic Green's functions γ of simply supported bodies, characterized by boundary data $\gamma = \Delta\gamma = 0$, respectively. It is known that $O_\beta^N \subset O_\gamma^N$ (Ralston—Sario [4]), but whether or not this inclusion is strict has been an open problem. The main purpose of the present study is to show that the inclusion *is* strict:

$$O_\beta^N < O_\gamma^N.$$

Let O_G^N be the class of parabolic Riemannian manifolds, i.e., those not carrying harmonic Green's functions. For any null class O^N, denote by $\tilde{O}^N$ its complement. It is known that $O_G^N < O_\gamma^N$ (Sario [5]), but the relation of O_G^N to O_β^N has been unknown, except for the special case $N=2$, in which the invariance of harmonicity under conformal metrics allowed us to construct 2-manifolds which belong to $O_G^2 \cap \tilde{O}_\beta^2$, (Nakai—Sario [3]). We shall now show that, for any $N>2$ as well, there exist N-manifolds which are parabolic but nevertheless carry β:

$$O_G^N \cap \tilde{O}_\beta^N \neq \emptyset.$$

This relation is sharper than $O_\beta^N < O_\gamma^N$.

A perhaps somewhat unexpected consequence of our reasoning will be that, for $N=2, 3$, every compact Riemannian manifold punctured at a point carries β.

We start by giving, in No. 1, a new short proof of the Ralston—Sario relation $O_\beta^N \subset O_\gamma^N$. In No. 2, we introduce a useful sufficient condition for the existence of β on parabolic manifolds: an Evans kernel is square integrable off its pole. We use this test to show, in No. 3, that the parabolic Riemannian ball constructed in Nakai—Sario [2] actually carries β.

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L. Chung has reported to the authors that he has constructed another counterexample to show the strictness of $O_\beta^N < O_\gamma^N$. His manifold is the N-space with a nonconformal metric.

1. New proof of $O_\beta^N \subset O_\gamma^N$. To show that $O_\beta^N \subset O_\gamma^N$, take a manifold R in $\tilde{O}_\gamma^N$, $N \geqq 2$. Choose a regular subregion Ω of R and denote by $g_\Omega(x, y)$ the harmonic Green's function in $\bar{\Omega}$. Let $\beta_\Omega(x, y)$ be the biharmonic Green's function of the clamped body on $\bar{\Omega}$, characterized by $\Delta^2 \beta_\Omega = \Delta\Delta\beta_\Omega = u$ on $\Omega - \{y\}$, the biharmonic fundamental singularity at y, and the conditions $\beta_\Omega = \partial\beta_\Omega/\partial n = 0$ on $\partial\Omega$. Here Δ is the Laplace—Beltrami operator $d\delta + \delta d$. Write

$$s_\Omega(x, y) = \Delta_x \beta_\Omega(x, y)$$

and set $s_\Omega(\cdot, y) = g_\Omega(\cdot, y) = 0$ on $R - \bar{\Omega}$. Then

$$\beta_\Omega(x, y) = (g_\Omega(x, \cdot), s_\Omega(\cdot, y)).$$

Let $h \in H(\Omega) \cap C^1(\bar{\Omega})$, where H stands for the class of harmonic functions. In view of

$$\int_{\partial\Omega} h(x) *_x d\beta_\Omega(x, y) = 0$$

and

$$(dh, d\beta_\Omega)_\Omega = \int_{\partial\Omega} \beta_\Omega * dh = 0,$$

we have

$$(h(\cdot), s_\Omega(\cdot, y))_\Omega = 0$$

for all $h \in H(\Omega) \cap C^1(\bar{\Omega})$.

Fix $x, y \in R$ and take regular subregions Ω_0, Ω_1 with $\bar{\Omega}_0 \subset \Omega_1$ and $x, y \in \Omega_0$. By Harnack's inequality, the existence of the biharmonic Green's function of a simply supported body on R,

$$\gamma_R(x, y) = (g_R(\cdot, x), g_R(\cdot, y)),$$

is equivalent to

$$\|g_R(\cdot, x)\|_{R-\Omega_1} < \infty$$

for every $x \in \Omega_0$. Since $\beta_\Omega(x, y) = (g_\Omega(\cdot, x), s_\Omega(\cdot, y))_\Omega$, we obtain

$$\beta_{\Omega'}(x, y) - \beta_\Omega(x, y) = (g_R(\cdot, x) - s_{\Omega_1}(\cdot, x), s_{\Omega'}(\cdot, y) - s_\Omega(\cdot, y))_{\Omega'}$$

for $\Omega \subset \Omega'$ with $\Omega_1 \subset \Omega$ and for any $x \in \Omega_0$. The quantity

$$K = \sup_{x \in \Omega_0} \|g_R(\cdot, x) - s_{\Omega_1}(\cdot, x)\|_R$$

is finite by virtue of the continuity of $g_R(z, x) - s_{\Omega_1}(z, x)$ on $\Omega_1 \times \Omega_1$. The Schwarz inequality yields

$$|\beta_{\Omega'}(x, y) - \beta_\Omega(x, y)|^2 \leqq K^2 \|s_{\Omega'}(\cdot, y) - s_\Omega(\cdot, y)\|_{\Omega'}^2$$

for $\Omega' \supset \Omega \supset \Omega_1$ and $x \in \Omega_0$. Here,

$$\|s_{\Omega'}(\cdot, y) - s_\Omega(\cdot, y)\|_{\Omega'}^2 = \|s_{\Omega'}(\cdot, y) - s_{\Omega_1}(\cdot, y)\|_{\Omega'}^2 - \|s_\Omega(\cdot, y) - s_{\Omega_1}(\cdot, y)\|_\Omega^2.$$

Since

$$(g_R(\cdot, y)-s_\Omega(\cdot, y), s_\Omega(\cdot, y)-s_{\Omega_1}(\cdot, y))_\Omega = 0,$$

we obtain

$$(g_R(\cdot, y)-s_{\Omega_1}(\cdot, y), s_\Omega(\cdot, y)-s_{\Omega_1}(\cdot, y))_\Omega = \|s_\Omega(\cdot, y)-s_{\Omega_1}(\cdot, y)\|_\Omega^2.$$

The Schwarz inequality gives

$$\|s_\Omega(\cdot, y)-s_{\Omega_1}(\cdot, y)\|_\Omega \leqq \|g_R(\cdot, y)-s_{\Omega_1}(\cdot, y)\|_R \leqq K$$

for every Ω. Therefore,

$$\lim_{\Omega'\supset\Omega\nearrow R} \|s_{\Omega'}(\cdot, y)-s_\Omega(\cdot, y)\|_{\Omega'}^2 = 0$$

and

$$\lim_{\Omega'\supset\Omega\nearrow R} |\beta_{\Omega'}(x, y)-\beta_\Omega(x, y)| = 0,$$

uniformly for $x\in\Omega_0$. Thus,

$$\beta_R(x, y) = \lim_{\Omega\nearrow R} \beta_\Omega(x, y)$$

exists on R for any fixed y, and the convergence is uniform for x in any compact subset of R.

The proof of $O_\beta^N \subset O_\gamma^N$ is complete.

2. A criterion for the existence of β. Suppose $R\in O_G^N$, and let $e(x, y)$ be an *Evans kernel* in the sense of Nakai [1]. For the definition and properties of $e(x, y)$ to be used below, we refer to Sario—Nakai [6, pp. 353—361]; the discussion there is for Riemann surfaces, but it applies verbatim to Riemannian manifolds. Let B_y be a geodesic ball $|x-y|<\varepsilon$ about y.

Theorem 1. *If an Evans kernel e on $R\in O_G^N$ satisfies*

$$\|e(\cdot, y)\|_{R-B_y} < \infty$$

for every y, then $R\in\tilde{O}_\beta^N$.

Proof. Using $h(\cdot)=e(\cdot, y)-s_\Omega(\cdot, y)$, we have, by the convention $s_{\Omega'}(\cdot, y)=0$ on $R-\bar{\Omega}'$,

$$(e(\cdot, y)-s_\Omega(\cdot, y), s_{\Omega'}(\cdot, y))_\Omega = 0$$

for $\Omega'=\Omega\supset\bar{B}_y$ and $\Omega'=B_y$. We set $f(\cdot)=e(\cdot, y)-s_{B_y}(\cdot, y)$ and $t_\Omega(\cdot)= =s_\Omega(\cdot, y)-s_{B_y}(\cdot, y)$ and obtain

$$(f(\cdot)-t_\Omega(\cdot), t_\Omega(\cdot))_\Omega = 0.$$

By the Schwarz inequality,

$$\|t_\Omega(\cdot)\|_\Omega^2 = (f(\cdot), t_\Omega(\cdot))_\Omega \leqq \|f(\cdot)\|_\Omega \cdot \|t_\Omega(\cdot)\|_\Omega.$$

In view of the assumption of the theorem, and the joint continuity of $e(x, y)$ on $R\times R$,

$$\|s_\Omega(\cdot, y)-s_{B_y}(\cdot, y)\|_\Omega^2 \leqq \|e(\cdot, y)-s_{B_y}(\cdot, y)\|_R^2 = K(y) < K(R_0) < \infty$$

for every Ω and for all y in an arbitrarily chosen compact subset R_0 of R. We recall that

$$\beta_\Omega(x, y) = (s_\Omega(\cdot, x), s_\Omega(\cdot, y))_\Omega$$
$$\beta_{\Omega_0}(x, y) = (s_\Omega(\cdot, x), s_{\Omega_0}(\cdot, y))_\Omega,$$

where we again use the convention $s_{\Omega'}(\cdot, x)=0$ on $R-\bar{\Omega}'$ for every $\Omega'=\Omega$, $\Omega_0 \subset \Omega$. It follows that

$$\beta_\Omega(x, y)-\beta_{\Omega_0}(x, y) = (s_\Omega(\cdot, x), s_\Omega(\cdot, y)-s_{\Omega_0}(\cdot, y))_\Omega$$
$$= (s_\Omega(\cdot, x)-s_{\Omega_0}(\cdot, x), s_\Omega(\cdot, y)-s_{\Omega_0}(\cdot, y))_\Omega.$$

By the Schwarz inequality,

$$|\beta_\Omega(x, y)-\beta_{\Omega_0}(x, y)|^2 \leqq \|s_\Omega(\cdot, x)-s_{\Omega_0}(\cdot, x)\|_\Omega^2 \cdot \|s_\Omega(\cdot, y)-s_{\Omega_0}(\cdot, y)\|_\Omega^2$$
$$= I_1(x)^2 \cdot I_2(y)^2,$$

where

$$I_1(x) = \|s_\Omega(\cdot, x)-s_{\Omega_0}(\cdot, x)\|_\Omega$$
$$\leqq \|s_\Omega(\cdot, x)-s_{B_x}(\cdot, x)\|_\Omega+\|s_{\Omega_0}(\cdot, x)-s_{B_x}(\cdot, x)\|_\Omega$$
$$\leqq 2K(R_0)^{1/2} < \infty$$

for all $x \in R_0$. Since

$$\|s_\Omega(\cdot, y)-s_{B_y}(\cdot, y)\| \leqq K(y)^{1/2} < \infty$$

for all Ω,

$$I_2(y) = \|(s_\Omega(\cdot, y)-s_{B_y}(\cdot, y))-(s_{\Omega_0}(\cdot, y)-s_{B_y}(\cdot, y))\|_\Omega \to 0$$

as $\Omega \supset \Omega_0 \nearrow R$. We conclude that

$$\beta(x, y) = \lim_{\Omega \to R} \beta_\Omega(x, y) = \lim_{\Omega \to R} \beta_\Omega(y, x)$$

exists and the convergence is uniform on every compact subset of R for any fixed $y \in R$. The proof of Theorem 1 is complete.

We note in passing the following immediate consequence of Theorem 1:

Corollary. *For $N=2, 3$, every compact Riemannian manifold punctured at a point carries β.*

3. Strictness of the inclusion. We are ready to establish our main result:

Theorem 2. *For $N \geqq 2$,*

$$O_\beta^N < O_\gamma^N.$$

More precisely,

$$O_G^N \cap \tilde{O}_\beta^N \neq \emptyset.$$

Proof. For $N=2$, the proof was given in Nakai—Sario [3], where a necessary and sufficient condition was established for the complex plane with a conformal radial metric to carry β. For $N>2$, consider the N-ball

$$R = \{r < 1, ds\}$$

with the metric $ds=\lambda(x)^{1/2}|dx|$, where $r=|x|$, $x=(x^1, \dots, x^N)$, $\lambda \in C^\infty(R)$, $\lambda>0$, and on $\{1/2<r<1\}$,

$$\lambda(x) = |x|^{(2-2N)/(N-2)}(1-|x|)^{4/(N-2)}.$$

Since the function $h(x)=1/(1-|x|)$ satisfies on $\{1/2<|x|<1\}$ the harmonic equation

$$\Delta h(r) = -g^{-1/2}(g^{1/2}g^{rr}h'(r))' = 0,$$

R is parabolic. Therefore, there exists an Evans kernel $e(x, y)$ on R such that

$$e(x, y) \sim \frac{1}{1-|x|} \quad \text{as} \quad |x| \to 1.$$

By virtue of

$$g(x)^{1/2} = \lambda(x)^{N/2} \sim [(1-|x|)^{4/(N-2)}]^{N/2} = (1-|x|)^{2N/(N-2)}$$

we obtain for $\varrho \in (|y|, 1)$,

$$\|e(\cdot, y)\|^2_{|x|>\varrho} \sim \int_\varrho^1 (1-r)^{-2}(1-r)^{2N/(N-2)} r^{N-1}\, dr$$

$$\sim \int_\varrho^1 (1-r)^{-2+2N/(N-2)}\, dr$$

$$= \int_\varrho^1 (1-r)^{4/(N-2)}\, dr < \infty.$$

By Theorem 1, β exists on R, hence $O_G^N \cap \tilde{O}_\beta^N \neq \emptyset$.

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