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# NIELSEN EXTENSIONS OF RIEMANN SURFACES*

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Let $S$ be a Riemann surface of finite type $(p, n, m)$, that is, a sphere with $p$ handles, $n$ punctures and $m$ holes. We assume that $m > 0$ and

$$6p - 6 + 2n + 3m > 0$$

(so that $S$ is not a simply or doubly connected plane domain). Nielsen assigned to $S$ a subdomain $S_0 \subset S$ which is a deformation retract of $S$ and again a Riemann surface of type $(p, n, m)$; we call $S_0$ the Nielsen kernel of $S$. It turns out that every $S$ is the Nielsen kernel of a Riemann surface $S_1$, called the Nielsen extension of $S$. The purpose of this note is to record some simple but useful properties of the Nielsen extension.

We represent $S$ as $U/G$ where $U$ denotes the upper half-plane and $G$ a torsion-free Fuchsian group of the second kind. (The fact that $G$ is determined only up to a conjugation in the real Möbius group will cause no difficulties.) There is an infinite set $\beta$ of open intervals $I$ on the extended real axis $\boldsymbol{R} \cup \{\infty\}$ such that $G$ acts properly discontinuously on $\Omega = U \cup L \cup J$, where $L$ denotes the lower half-plane and $J$ the union of all $I \in \beta$, and on no larger open subset of $\boldsymbol{C} \cup \{\infty\}$. The quotient $S^d = \Omega/G$ is the (Schottky) *double* of $S$, and the conjugation $z \mapsto \bar{z}$ induces a canonical anti-conformal involution $j$ of $S^d$. Note that there is a canonical embedding $S \to S^d$.

For each $I \in \beta$, the stabilizer $G(I)$ of $I$ in $G$ is generated by a hyperbolic element whose axis $A(I)$ is the non-Euclidean line joining the endpoints of $I$. The quotient $I/G(I)$ is homeomorphic to a circle and is identified with an (ideal) boundary curve of $S$. Each of the $m$ ideal boundary curves of $S$ can be so obtained.

We denote by $D(I)$ the non-Euclidean half-plane bounded by $I$ and $A(I)$ and by $N$ the complement in $U$ of the union of the closures of all $D(I)$, $I \in \beta$. $N$ is called the *Nielsen region* of $G$. It is convex in the non-

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Euclidean sense (being the intersection of non-Euclidean half-planes) and open (because, for $I_1$, $I_2 \in \beta$ and $I_1 \neq I_2$, $D(I_1) \cap D(I_2) = \emptyset$). Hence $N$ is a simply connected domain, invariant under $G$. The Riemann surface $S_0 = N/G$ is the *Nielsen kernel* of $S$; it comes equipped with a canonical embedding into $S$.

Noting that for every $I \in \beta$ and every $g \in G$, $g(D(I)) = D(I')$ for some $I' \in \beta$, we conclude that the image of every $D(I)$ under the natural mapping $U \to U/G = S$ is $D(I)/G(I)$. This is a Riemann surface of type $(0, 0, 2)$, that is, homeomorphic to an annulus. Its two (ideal) boundary curves are: the boundary curve $C = I/G(I)$ and $C' = A(I)/G(I)$. $C'$ is the geodesic, in the Poincaré metric of $S$, which is freely homotopic to $C$. We call $D(I)/G(I)$ the *funnel* adjacent to $C'$. The complement of the closure of $S_0$ in $S$ is a disjoint union of such funnels. This remark characterizes the Nielsen kernel $S_0$ of $S$.

L e m m a 1. *Let $C$ be a boundary curve of $S$, $C'$ the geodesic freely homotopic to $C$, $L$ the length of $C'$, and $M$ the module of the funnel adjacent to $C'$. Then*

$$L M = \pi^2 .$$

*Proof.* We may assume, without loss of generality, that $C = I/G(I)$ where $I \in \beta$ is the positive real axis. Then $G(I)$ is generated by a Möbius transformation $g(z) = a z$, for some $a > 1$, $A(I)$ is the positive imaginary axis, $D(I)$ is the first quadrant $Q$, and the length $L$ of $C' = A(I)/G(I)$ is the non-Euclidean distance between $i$ and $a i$, so that

$$L = \log a . \tag{1}$$

The function

$$\zeta(z) = e^{-2\pi i \log z/\log a} \tag{2}$$

(where $\log z$ denotes the principal branch) is holomorphic in $Q$ and satisfies $\zeta'(z) \neq 0$, $\zeta \circ g(z) = \zeta(a z) = \zeta(z)$. Also, $\zeta(Q)$ is the annulus $1 < |\zeta| < \varrho$ where

$$\varrho = e^{\pi^2/\log a} . \tag{3}$$

Hence the funnel, $Q/G(I)$, is conformal to this annulus, and the module is

$$M = \log \varrho = \pi^2 / \log a = \pi^2 / L .$$

If $S$ is a Riemann surface of finite type, we call the restriction to $S$ of the Poincaré metric on $S^d$, the *intrinsic metric* on $S$. Note that the boundary curves of $S$ are geodesics of finite length in the intrinsic metric. Indeed, such a curve $C$, considered as a Jordan curve on $S^d$, is freely homotopic to a unique geodesic $C_0$. If $C_0$ were distinct from $C$, $j(C_0)$ would be another geodesic freely homotopic to $C$.

L e m m a 2. *The restriction of the Poincaré metric on $S$ to the Nielsen kernel $S_0$ coincides with the intrinsic metric on $S_0$.*

*Proof.* We consider $S_0$ as embedded in $S_0^d = (S_0)^d$. Let $ds$ denote the restriction to $S_0$ of the Poincaré metric on $S$. The canonical involution $j$ transplants $ds$ to $j(S_0)$; the metric $ds$ is now a conformal metric, with constant Gaussian curvature equal to $(-1)$, defined on $S_0^d$ except on the boundary curves $C'$ of $S_0$, that is on the geodesics on $S$ freely homotopic to the boundary curves $C$ of $S$. Consider a point $P$ on one of such curves $C'$, and a sufficiently small neighborhood $\varDelta$ of $P$ which is divided by an arc $\alpha = \varDelta \cap C'$ into two disjoint sudbomains $\varDelta_1$ and $\varDelta_2 = j(\varDelta_1)$. Since $C'$ is a geodesic on $S$ and since $j$ leaves $C'$ pointwise fixed, there is a homeomorphism $\psi$ of $\varDelta$ onto a neighborhood $\psi(\varDelta)$ of $i \in U$ such that $\psi \mid \varDelta - \alpha$ takes the metric $ds$ into the Poincaré metric $|dz| / y$ on $U$. Now $\psi$ is conformal on $\varDelta - \alpha$, and hence also on $\varDelta$. We conclude that on $S_0^d$ there is a conformal metric $d\hat{s}$, of Gaussian curvature $(-1)$, with $d\hat{s} \mid S_0 \cup j(S_0) = ds$.

To show that $d\hat{s}$ is the Poincaré metric on $S_0^d$ we must verify that it is complete. This so since $S_0^d$ is of type $(2p + m - 1, 2n, 0)$, hence compact except for $2n$ punctures, and near each puncture $d\hat{s}$ coincides with $ds$, that is, with a complete metric.

T h e o r e m 1. *$S$ is the Nielsen kernel of a uniquely determined Riemann surface $S_1$ (called the Nielsen extension of $S$).*

*Proof.* In view of Lemmas 1 and 2 and the relations (1), (2), (3) in the proof of Lemma 2, the Nielsen extension $S_1$ of $S$, if such an extension exists, *must* be constructed as follows. For each of the $m$ boundary curves $C$ of $S$, let $L_C$ denote its length in the intrinsic metric of $S$. Set

$$\varrho_C = e^{\pi^2/L_C}$$

and let $F_C$ denote the annulus $1 < |\zeta| < \varrho_C$. Define a Riemannian metric $ds_C$ on $F_C$ by transplanting to $F_C$ the Poincaré metric $|dz| / y$ in $U$ by means of the mapping

$$\zeta(z) = e^{-2\pi i \log z / L_C}$$

of the first quadrant $Q$ onto $F_C$, and note that the circle $|\zeta| = \varrho_C$ is a geodesic in this metric, of length $L_C$. Now attach $F_C$ to $S$, by means of an isometry between $C$, endowed with the intrinsic metric of $S$, and the circle $|\zeta| = \varrho_C$, endowed with the metric $ds_C$.

In order to show that the Riemann surface $S_1$ so constructed is indeed the Nielsen extension of $S$, it sufficies to show that the restriction of the Poincaré metric on $S_1$ to $S$ is the intrinsic metric on $S$ and the restriction to an $F_C$ is the metric $ds_C$. The required argument is so similar to the proof of Lemma 2 that it may be omitted.

L e m m a 3. *Represent the double* $S^d$ *of* $S$ *as* $U/\Gamma$ *where* $\Gamma$ *is a torsion-free Fuchsian group. Let* $\Delta$ *be a component of the preimage of* $S \subset S^d$ *under the natural mapping* $U \to U/\Gamma = S^d$, *and let* $\Gamma_1$ *be the stabilizer of* $\Delta$ *in* $\Gamma$. *Then* $U/\Gamma_1 = S_1$ *is the Nielsen extension of* $S$.

*Proof.* Let $f$ be the restriction of the natural mapping $U \to U/\Gamma = S^d$ to $\Delta$. Then $f$ is holomorphic and locally one-to-one and $f(\Delta) = S$. Also $f(z_1) = f(z_2)$ if and only if there is a $\gamma \in \Gamma$ with $z_2 = \gamma(z_1)$; such a $\gamma$ must belong to $\Gamma_1$, since every element of $\gamma$ permutes the components of the inverse image of $S$. We conclude that $f: \Delta \to S$ is a universal covering with covering group $\Gamma_1$. Furthermore, $f$ transplants the metric $|dz| / y$ on $U$ into the intrinsic metric of $S$.

Now represent the Nielsen extension $S_1$ of $S$ as $U/G$, $G$ a torsion-free Fuchsian group, and let $N$ be the Nielsen region of $G$. One sees as before that the restriction $h$ of the natural mapping $U \to U/G$ to $N$ is a universal covering $N \to S$, with covering group $G$, which transplants the metric $|dz| / y$ into the restriction of the Poincaré metric of $S_1$ to $S$, that is, by Lemma 2, into the intrinsic metric on $S$.

It follows that there exists a homeomorphism $\varphi$ of $\Delta$ onto $N$ such that $f = h \circ \varphi$ and $\varphi \Gamma_1 \varphi^{-1} = G$. This $\varphi$ is an isometry in the metric $|dz| / y$, hence the restriction to $\Delta$ of a real Möbius transformation. Since $G$ is determined but for a conjugation in the real Möbius group, we may assume that $\varphi = \mathrm{id}$. Then $\Delta = N$, $\Gamma_1 = G$, and the assertions of the lemma follow.

T h e o r e m 2. *A quasiconformal homeomorphism of* $S$ *onto another Riemann surface can be extended, canonically, and without increasing the dilatation, to a quasiconformal homeomorphism of the Nielsen extension of* $S$ *onto that of* $\varphi(S)$.

*Proof.* Since $\varphi$ is quasiconformal, it admits a homeomorphic extension to the boundary curves of $S$ and hence can be canonically extended to a homeomorphism of $S^d$ onto $\varphi(S)^d$ which respects the canonical involutions. We denote the extended homeomorphism again by $\varphi$. Its dilatation is the same as before the extension.

Represent $S^d$ and $\varphi(S)^d$ as $U/\Gamma$ and $U/\Gamma'$, respectively, $\Gamma$ and $\Gamma'$ being torsion-free Fuchsian groups. Then $\varphi: S^d \to \varphi(S)^d$ can be lifted, using the natural mappings $U \to U/\Gamma$ and $U \to U/\Gamma'$, to a homeomorphism $w$ of $U$ onto $U$, which has the same dilatation as $\varphi$ and satisfies $w \Gamma w^{-1} = \Gamma'$. Choose a component $\Delta$ of the preimage of $S$ under $U \to U/\Gamma$, and let $\Gamma_1$ be the stabilizer of $\Delta$ in $\Gamma$. Then $w(\Delta)$ is a component of the preimage of $\varphi(S)$ under the mapping $U \to U/\Gamma'$ and $\Gamma_1' = w \Gamma_1 w^{-1}$ is the stabilizer of $w(\Delta)$ in $\Gamma_1$. Clearly, $w$ induces the mapping $\varphi$ of $S = \Delta/\Gamma_1$ onto $\varphi(S) = w(\Delta)/\Gamma_1'$. But $w$ also induces a homeomorphism, with the same dilatation, of $U/\Gamma_1$ onto $U/\Gamma_1'$. In view of Lemma 3 this is a homeomorphism $\psi$, extending $\varphi$, of the Nielsen extension of $S$ onto that of $\varphi(S)$.

It is easy to verify that $\psi$ does not depend on the choices (of $\Gamma$, $\Gamma''$, $w$ and $\Delta$) made during the above construction.

Lemma 4. *Let $C$ be a boundary curve of $S$, $C_1$ the boundary curve of the Nielsen extension $S_1$ of $S$ freely homotopic to $C$. Let $L$ be the length of $C$ in the Poincaré metric of $S_1$, $L_1$ the length of $C_1$ in the Poincaré metric of the Nielsen extension $S_2$ of $S_1$. Then*

$$L_1 < L .$$

*Proof.* Let $ds_1$ denote the Poincaré metric on $S_1$, $ds_2$ the Poincaré metric on $S_2$. Since $S_1 \subset S_2$, and $S_1 \neq S_2$, we have $ds_2 < ds_1$. On the other hand, $C_1$ is the unique geodesic, in its homotopy class, with respect to the metric $ds_2$. Hence

$$L_1 = \int_{C_1} ds_2 < \int_C ds_2 < \int_C ds_1 = L .$$

Let $S_1$ be the Nielsen extension of $S$ and let $S_{k+1}$ be the Nielsen extension of $S_k$, $k = 1, 2, \ldots$. In view of the canonical embeddings $S_k \to S_{k+1}$ we can define the Riemann surface $S_\infty = S_1 \cup S_2 \cup \ldots$ which we call the *infinite Nielsen extension* of $S$. It is, of course, homeomorphic to $S$, but, as the next theorem shows, if $S$ is of type $(p, n, m)$, then $S_\infty$ is of type $(p, n+m, 0)$. Thus the construction of $S_\infty$ gives us a canonical way of "filling in the holes" in $S$.

Theorem 3. *The infinite Nielsen extension $S_\infty$ of $S$ has no ideal boundary curves.*

*Proof.* Let $C_0$ be a boundary curve of $S$. Then $C_0$ divides $S_\infty$ into two components, one of which, call it $X$, is conformally equivalent to a doubly connected domain. We must show that the module $M_\infty$ of $X$ is infinite.

Let $C_1$ be the boundary curve of $S_1$ (the Nielsen extension of $S$) freely homotopic to $C_0$, and let $C_{k+1}$ be the boundary curve of $S_{k+1}$ (the Nielsen extension of $S_k$) freely homotopic to $C_k$, $k = 1, 2, \ldots$. Also let $F_k$ be the funnel bounded by $C_k$ and $C_{k+1}$ and $M_k$ the module of $F_k$. Finally let $L_k$ be the length of $C_k$ in the Poincaré metric of $S_{k+1}$. Then $X = F_0 \cup F_1 \cup \ldots$, $F_k$ and $F_{k+1}$ have no inner points in common, and each $F_k$ separates the boundary continua of $X$. By Grötzsch's inequality,

$$M_\infty \geq M_0 + M_1 + \ldots$$

that is, in view of Lemma 1,

$$M_\infty \geq \pi^2 (L_0^{-1} + L_1^{-1} + \ldots) .$$

Since, by Lemma 4, $L_0 > L_1 > \ldots$, we have that $M = +\infty$, as asserted.

Remark. Let $S_0$ be the Nielsen kernel of $S$, $S_{k-1}$ that of $S_k$, for $k = 0, -1, -2, \ldots$. It would be interesting to investigate the set $S_0 \cap S_{-1} \cap S_{-2} \cap \ldots$.

We conclude with an immediate corollary of Theorem 2.

Theorem 4. *A quasiconformal homeomorphism of $S$ onto another Riemann surface can be extended, canonically and without increasing the dilatation, to a quasiconformal homeomorphism of the infinite Nielsen extension of $S$ onto that of $\varphi(S)$.*

Applications of Theorems 2, 3, 4 will be found in a forthcoming paper on a theorem by Thurston.

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