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WEIERSTRASS DIVISION WITH QUASIANALYTIC BOUNDARY VALUES

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1. Introduction

Let $g\colon \mathbf{R}^+ \to \mathbf{R}^+$ ($\mathbf{R}^+$ is the set of nonnegative real numbers) be a convex increasing function such that $g(0)=0$ and $t^{-1}g(t)\to+\infty$ as $t\to+\infty$. Define a sequence $\{M_n\}_{n\in Z^+}$ ($\mathbf{Z}^+$ is the set of nonnegative integers) by $M_n=\exp g(n)$, $n\in\mathbf{Z}^+$. We assume g grows fast enough to ensure that $M_n\geqq n!$, $n\in\mathbf{Z}^+$.

Let Ω be a domain in $\mathbf{C}^k$ with 0 a point on its boundary bΩ. We denote by $\mathscr{A}_k={}_0\mathscr{A}_k(\{M_n\},\Omega)$ the set of germs at 0 of complex-valued Whitney C^∞ functions f on $\bar\Omega$ (the closure of Ω) which are analytic in Ω and satisfy the following growth conditions on their derivatives: for each $r>0$ sufficiently small that f is represented by a function on $\bar\Omega\cap\Delta_k(r)$ $(\Delta_k(r)=\{z\in\mathbf{C}^k\colon |z_j|<r,\ 1\leqq j\leqq k\})$ there exist constants A and B, which depend in general on both f and r but not on $n\in\mathbf{Z}^+$, such that for all $n\in\mathbf{Z}^+$,

$$\sup_{\substack{\alpha\in(\mathbf{Z}^+)^k,\ |\alpha|\leqq n,\\ z\in\bar\Omega\cap\Delta_k(r)}} |D^\alpha f(z)| \leqq AB^n M_n. \tag{1.1}$$

$(D^\alpha=D^\alpha_z=\partial^{|\alpha|}/\partial z_1^{\alpha_1}\dots\partial z_k^{\alpha_k}$, where $z=(z',z_k)=(z_1,\dots,z_k)$ are coordinates on $\mathbf{C}^k$, $\alpha=(\alpha_1,\dots,\alpha_k)\in(\mathbf{Z}^+)^k$, and $|\alpha|=\alpha_1+\dots+\alpha_k$.) We assume that $\mathscr{A}_k$ is quasianalytic in the sense of Denjoy and Carleman:

$$f\in\mathscr{A}_k \text{ and } D^\alpha f(0)=0 \text{ for all } \alpha\in(\mathbf{Z}^+)^k \text{ imply } f=0\in\mathscr{A}_k. \tag{1.2}$$

Before going on, we remark that by the use of the logarithmic convexity of the sequence $\{M_n\}$, it is not difficult to show that $\mathscr{A}_k$ is a local algebra with maximal ideal $m_k=\{f\in\mathscr{A}_k\colon f(0)=0\}$. The quasianalyticity assumption is independent of the dimension k. If the sequence $\{M_n\}$ satisfies certain additional hypotheses, then $\mathscr{A}_k$ is closed under composition whenever the composition makes sense, and $\mathscr{A}_k$ is also closed under differentiation. For a more complete discussion, see [2].

In this paper we consider a quasianalytic local algebra $\mathscr{A}_k(\{M_n\})$. We show a Weierstrass—Malgrange—Mather type division theorem does not hold in $\mathscr{A}_k(\{M_n\})$

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if $k \geqq 2$, $\mathscr{A}_k(\{n!\})$ is a proper subset of $\mathscr{A}_k(\{M_n\})$, and $b\Omega$ is C^2 smooth and strongly Levi pseudoconvex at 0. If, however, $b\Omega$ is Levi pseudoflat at 0, we prove a generic division theorem holds in $\mathscr{A}_k(\{M_n\})$, $k \geqq 2$. We further show in this case that division is possible in $\mathscr{A}_k(\{M_n\})$ by every regular element of $\mathscr{O}_k$, the local algebra of germs at 0 of analytic functions. (The case in which $b\Omega$ is pseudoconcave at 0 is trivial, since in this case $\mathscr{A}_k(\{M_n\})$ reduces to $\mathscr{O}_k$. See L. Hörmander, [4].)

2. Preliminaries

The following proposition is any easy consequence of the closed graph theorem:

Proposition 2.1. *Let E be a Banach space and $F=\bigcup_{n=1}^{+\infty} F_n$ be an inductive limit of Banach spaces. If $T\colon E\to F$ is a continuous linear map, then there exists a positive integer n_0 such that $T(E)\subseteq F_{n_0}$.* □

We will apply this proposition to estimate the derivatives of the quotient and remainder when we divide by a fixed regular element $f\in\mathscr{A}_k(\{M_n\})$. The result will be that the growth of the derivatives of the element we are dividing by f determines the growth of the derivatives of the quotient and remainder.

For positive integers ν and N, let

$$A_{k,\nu,N}=\Big\{f: f \text{ is a Whitney } C^\infty \text{ function on } \bar\Omega\cap\Delta_k(1/\nu)$$

$$\text{and } \sup_{n\in Z^+}\ \sup_{\substack{\alpha\in(Z^+)^k,\,|\alpha|\leqq n,\\ z\in\bar\Omega\cap\Delta_k(1/\nu)}} |D^\alpha f(z)|/N^n M_n<+\infty\Big\}.$$

Note that for all positive integers ν and N, $A_{k,\nu,N}$ is a Banach space, and the inductive limit $\bigcup_{\nu,N=1}^{+\infty} A_{k,\nu,N}$ may be identified with $\mathscr{A}_k(\{M_n\})$.

Fix $f\in\mathscr{A}_k(\{M_n\})$, which is regular in z_k of order d. (This means $f(0)=\partial f(0)/\partial z_k=$ $=\dots=\partial^{d-1}f(0)/\partial z_k^{d-1}=0$, while $\partial^d f(0)/\partial z_k^d\neq 0$.) Let ν_0 be the smallest positive integer such that f is represented by a function on $\Delta_k(1/\nu)$ for all $\nu\geqq\nu_0$. We define a map

$$(q, r_1, \dots, r_d)\to g=fq+\sum_{j=1}^{d} r_j z_k^{d-j},$$

$$\bigcup_{\substack{\nu=\nu_0,\\ N=1}}^{+\infty}\Big(A_{k,\nu,N}\oplus\Big(\bigoplus_1^d A_{k-1,\nu,N}\Big)\Big)\to\bigcup_{\substack{\nu=\nu_0,\\ N=1}}^{+\infty} A_{k,\nu,N}.$$

This map is continuous, for its restriction to each direct summand $A_{k,\nu,N}\oplus(\oplus_1^d A_{k-1,\nu,N})$ is continuous. The assumption that the $\mathscr{A}_k(\{M_n\})$ are quasianalytic implies the map is injective. The map is surjective if and only if division by f is possible within $\mathscr{A}_k(\{M_n\})$. If the map is surjective, the closed graph theorem implies its inverse is continuous. It is to this inverse map that we apply Proposition 2.1.

Given positive integers $\nu \geqq \nu_0$ and N, Proposition 2.1 implies that there exist positive integers $\nu' \geqq \nu_0$ and N', as well as a constant A, such that for each $g \in A_{k,\nu,N}$, there exist $q \in A_{k,\nu',N'}$, and $r_j \in A_{k-1,\nu',N'}$, $1 \leqq j \leqq d$, which satisfy

$$g = fq + \sum_{j=1}^{d} r_j z_k^{d-j}, \tag{2.1}$$

and which also satisfy the estimates

$$\sup_{n\in\mathbf{Z}^+} \sup_{\substack{\alpha\in(\mathbf{Z}^+)^{k-1},\,|\alpha|\leqq n,\\ z'\in\Omega\cap\Delta_{k-1}(1/\nu')}} |D^\alpha r_j(z')|/(N')^n M_n \leqq A \sup_{n\in\mathbf{Z}^+} \sup_{\substack{\beta\in(\mathbf{Z}^+)^{k},\,|\beta|\leqq n,\\ z\in\Omega\cap\Delta_{k}(1/\nu)}} |D^\beta g(z)|/N^n M_n, \tag{2.2}$$

$1 \leqq j \leqq d$, and

$$\sup_{n\in\mathbf{Z}^+} \sup_{\substack{\beta\in(\mathbf{Z}^+)^{k},\,|\beta|\leqq n,\\ z\in\Omega\cap\Delta_{k}(1/\nu')}} |D^\beta q(z)|/(N')^n M_n \leqq A \sup_{n\in\mathbf{Z}^+} \sup_{\substack{\beta\in(\mathbf{Z}^+)^{k},\,|\beta|\leqq n,\\ z\in\Omega\cap\Delta_{k}(1/\nu)}} |D^\beta g(z)|/N^n M_n. \tag{2.3}$$

We summarize these results as a lemma.

Lemma 2.2. *Let $f \in \mathscr{A}_k(\{M_n\})$ be regular in z_k of order d, and suppose that division by f is possible within $\mathscr{A}_k(\{M_n\})$. Let ν_0 be the smallest positive integer such that f is represented by a function on $\Delta_k(1/\nu)$ for all $\nu \geqq \nu_0$. Given positive integers $\nu \geqq \nu_0$ and N, there exist positive integers $\nu' \geqq \nu_0$ and N', as well as a constant A, such that for each $g \in A_{k,\nu,N}$, there exist $q \in A_{k,\nu',N'}$ and $r_j \in A_{k-1,\nu',N'}$, $1 \leqq j \leqq d$, which satisfy equation* (2.1) *and estimates* (2.2) *and* (2.3). □

We will need one technical lemma, which we now state. The proof may be found in [2].

Lemma 2.3. *Let $\lambda(a) = \sup_{n\in Z^+} |a|^n/M_n$ for $a \in \mathbf{C}$, and suppose there exist $\varepsilon > 0$, $A > 0$, and $C > 0$ such that*

$$\exp(\varepsilon a) \leqq C\lambda(a), \quad a \in \mathbf{R}, \quad a > A. \tag{2.4}$$

Then there exist $\alpha > 0$ and $\beta > 0$ such that

$$M_n \leqq \alpha \beta^n n!, \quad n \in \mathbf{Z}^+. \quad \square \tag{2.5}$$

3. The case of a strongly pseudoconvex boundary

Let $\Omega\subseteqq\mathbf{C}^k$ be a domain with C^2 smooth boundary $b\Omega$, and assume Ω is strongly pseudoconvex at $0\in b\Omega$. Then there is an open neighborhood U of 0 in $\mathbf{C}^k$ and a C^2 smooth function $\varphi\colon U\to\mathbf{R}$ with the following properties:

$$\Omega\cap U=\{z\in U:\varphi(z)<0\}$$
$$\varphi(0)=0,$$
$$\operatorname{grad}\varphi(0)\neq 0,\quad\text{and}$$
$$\text{the Levi form}\quad (\partial^2\varphi(0)/\partial z_i\,\partial\bar z_j)_{1\leqq i,j\leqq k}\quad\text{is strictly positive definite.}$$

After analytic change of coordinates in $\mathbf{C}^k$, we may assume φ has the form

$$\varphi(z)=\operatorname{Im} z_k+\sum_{j=1}^{k}c_j|z_j|^2+O(|z|^3), \tag{3.1}$$

where $c_j>0$ is constant, $1\leqq j\leqq k$. (A proof of this fact may be found in Hörmander, [4].)

Let $\mathscr{A}_k(\{n!\})$ be a proper subset of $\mathscr{A}_k(\{M_n\})$, $k\geqq 2$, a quasianalytic local algebra as defined in the Introduction. Set $z'=(z_1,\dots,z_{k-1})$ and $f(z)=f(z',z_k)=$ $=z_k^2+z_1$. Then f is an analytic Weierstrass polynomial of degree two in z_k. For $a\in\mathbf{C}$, set $g(z,a)=e^{iaz_k}$. Note that for each $a\in\mathbf{C}$, $g\in\mathcal{O}_k\subseteqq\mathscr{A}_k(\{M_n\})$. Suppose it were possible to write for each $a\in\mathbf{C}$

$$g=fq+r_1z_k+r_2, \tag{3.2}$$

where $q=q(z,a)\in\mathscr{A}_k(\{M_n\})$ and $r_1=r_1(z',a)$, $r_2=r_2(z',a)\in\mathscr{A}_{k-1}(\{M_n\})$. Since the roots of $f(z',z_k)=0$ are $z_k=\pm i\sqrt{z_1}$, it would follow from equation (3.2) that

$$r_1(z',a)=i\left(e^{a\sqrt{z_1}}-e^{-a\sqrt{z_1}}\right)/2\sqrt{z_1}. \tag{3.3}$$

Now consider only values of $a\in\mathbf{R}$ with $a<0$. If $z\in\bar\Omega$ and $|z|$ is sufficiently small, it follows from equation (3.1) that $\operatorname{Im} z_k\leqq 0$. Thus, if $\alpha\in(\mathbf{Z}^+)^k$, $|\alpha|\leqq n\in\mathbf{Z}^+$, $z\in\bar\Omega$, and $|z|$ is sufficiently small, then we get

$$\begin{aligned}|D_z^\alpha g(z,a)| &\leqq |a|^{|\alpha|}e^{-a\operatorname{Im} z_k}\\ &\leqq |a|^{|\alpha|}\\ &\leqq (|a|^{|\alpha|}/M_{|\alpha|})M_{|\alpha|}\\ &\leqq \lambda(a)M_n.\end{aligned}$$

Thus, for each $a<0$, $g=g(z,a)\in A_{k,1,1}$. If we apply Lemma 2.2, it follows that there exist $\varepsilon_1>0$ and $A_1>0$, both independent of a, such that

$$\sup_{z'\in\bar\Omega\cap\Delta_{k-1}(\varepsilon_1)}|r_1(z',a)|\leqq A_1\lambda(a),\quad a<0. \tag{3.4}$$

Let $\varepsilon>0$, $z'=(\varepsilon, 0, \dots, 0)$, and $z_k=-i\varepsilon$. If ε is chosen sufficiently small, then $z'\in\Delta_{k-1}(\varepsilon_1)$ and $\varphi(z)=-\varepsilon+c_1\varepsilon^2+c_k\varepsilon^2+O(\varepsilon^3)<0$, so that $z\in\bar{\Omega}$. Thus estimate (3.4) yields

$$|e^{a\sqrt{\varepsilon}}-e^{-a\sqrt{\varepsilon}}|/2\sqrt{\varepsilon} \leqq A_1\lambda(a), \quad a<0. \tag{3.5}$$

Since $(e^{a\sqrt{\varepsilon}}-e^{-a\sqrt{\varepsilon}})/2\sqrt{\varepsilon}$ is asymptotic to $e^{-a\sqrt{\varepsilon}}/2\sqrt{\varepsilon}$, inequality (3.5) implies there exist constants $A>0$ and $K>0$, both independent of a, such that

$$e^{a\sqrt{\varepsilon}} \leqq K\lambda(a), \quad a>A. \tag{3.6}$$

Inequality (3.6) together with Lemma 2.3 now imply the existence of constants $\alpha>0$ and $\beta>0$ such that

$$M_n \leqq \alpha\beta^n n!, \quad n\in\mathbf{Z}^+.$$

This implies $\mathscr{A}_k(\{M_n\})=\mathscr{A}_k(\{n!\})$, contrary to assumption. We conclude the Weierstrass division theorem does not generalize to $\mathscr{A}_k(\{M_n\})$ when $\mathscr{A}_k(\{M_n\})\supsetneqq\mathscr{A}_k(\{n!\})$ and $k\geqq 2$. Indeed, we have shown that it isn't always possible to divide in $\mathscr{A}_k(\{M_n\})$ by Weierstrass polynomials from $\mathcal{O}_{k-1}[z_k]$.

4. The case of a pseudoflat boundary

Let $\Omega\subseteqq\mathbf{C}^k$, $k\geqq 2$, be a product domain with 0 a member of the pseudoflat part of $\mathrm{b}\Omega$. Thus, let $U_1\subseteqq\mathbf{C}$ be any plane domain with $0\in \mathrm{b}U_1$, let $U_j=\{z\in\mathbf{C}\colon |z|<1\}$ be the open unit disc in the plane for $2\leqq j\leqq k$, and let $\Omega=U_1\times U_2\times\dots\times U_k$. Let $\mathscr{A}_k(\{M_n\})$ be a quasianalytic local algebra. We show in this section that a generic division theorem holds in $\mathscr{A}_k(\{M_n\})$. We also show that division is possible in $\mathscr{A}_k(\{M_n\})$ by every regular element of $\mathcal{O}_k$.

By a generic monic polynomial in z_k of degree d we mean an element in $\mathbf{C}[z_k]$ of the form $P_d(z_k, \lambda)=z_k^d+\sum_{j=1}^d \lambda_j z_k^{d-j}$, where $\lambda=(\lambda_1, \dots, \lambda_d)\in\mathbf{C}^d$.

Theorem 4.1. (Generic division theorem for $\mathscr{A}_k(\{M_n\})$.) *Let $P_d=P_d(z_k, \lambda)$ be a generic polynomial in z_k of degree d. For each $g\in\mathscr{A}_k(\{M_n\})$, there exists $\varepsilon>0$ such that if $\lambda\in\Delta_d(\varepsilon)$, then there exist unique elements $q=q(z, \lambda)\in\mathscr{A}_k(\{M_n\})$ and $r_j=r_j(z', \lambda)\in\mathscr{A}_{k-1}(\{M_n\})$, $1\leqq j\leqq d$, such that*

$$g = P_d q+\sum_{j=1}^{d} r_j z_k^{d-j}. \tag{4.1}$$

Furthermore, all the germs in equation (4.1) *are defined for* $(z, \lambda)\in(\bar{\Omega}\cap\Delta_k(\varepsilon))\times\Delta_d(\varepsilon)$ *and are analytic in* (z, λ) *on* $(\Omega\cap\Delta_k(\varepsilon))\times\Delta_d(\varepsilon)$.

Proof. Choose $0<r<1$ so that the germ g is defined on $\bar{\Omega}\cap\Delta_k(r)$. Let $0<\delta<r$. By Cauchy's integral formula, if $z\in\bar{\Omega}\cap\Delta_k(\delta/2)$, then

$$g(z) = \frac{1}{2\pi i}\int_{|\zeta|=\delta}\frac{g(z', \zeta)}{\zeta-z_k}\,d\zeta. \tag{4.2}$$

Observe that

$$P_j(\zeta, \lambda) = \zeta^j + \sum_{i=1}^{j} \lambda_i \zeta^{j-i}$$
$$= \zeta\left(\zeta^{j-1} + \sum_{i=1}^{j-1} \lambda_i \zeta^{j-1-i}\right) + \lambda_j$$
$$= \zeta P_{j-1}(\zeta, \lambda) + \lambda_j,$$

and so

$$-\lambda_j = \zeta P_{j-1}(\zeta, \lambda) - P_j(\zeta, \lambda).$$

Thus

$$P_d(\zeta, \lambda) - P_d(z_k, \lambda)$$
$$= \zeta P_{d-1}(\zeta, \lambda) + \lambda_d - \sum_{j=1}^{d} \lambda_j z_k^{d-j} - z_k^d$$
$$= \zeta P_{d-1}(\zeta, \lambda) + \sum_{j=1}^{d-1} (-\lambda_j) z_k^{d-j} - z_k^d$$
$$= \zeta P_{d-1}(\zeta, \lambda) + \sum_{j=1}^{d-1} [\zeta P_{j-1}(\zeta, \lambda) - P_j(\zeta, \lambda)] z_k^{d-j} - z_k^d$$
$$= \sum_{j=1}^{d} \zeta P_{j-1}(\zeta, \lambda) z_k^{d-j} - \sum_{j=0}^{d-1} P_j(\zeta, \lambda) z_k^{d-j}$$
$$= \sum_{j=1}^{d} \zeta P_{j-1}(\zeta, \lambda) z_k^{d-j} - \sum_{j=1}^{d} P_{j-1}(\zeta, \lambda) z_k^{d-j+1}$$
$$= \left(\sum_{j=1}^{d} P_{j-1}(\zeta, \lambda) z_k^{d-j}\right)(\zeta - z_k).$$

Adding $P_d(z_k, \lambda)$ to both sides of the identity we have obtained, viz.,

$$P_d(\zeta, \lambda) - P_d(z_k, \lambda) = \left(\sum_{j=1}^{d} P_{j-1}(\zeta, \lambda) z_k^{d-j}\right)(\zeta - z_k),$$

and dividing through by $P_d(\zeta, \lambda)(\zeta - z_k)$, we obtain

$$\frac{1}{\zeta - z_k} = \frac{P_d(z_k, \lambda)}{P_d(\zeta, \lambda)(\zeta - z_k)} + \sum_{j=1}^{d} \frac{P_{j-1}(\zeta, \lambda)}{P_d(\zeta, \lambda)} z_k^{d-j}. \tag{4.3}$$

Now choose $s > 0$ such that $\lambda \in \Delta_d(s)$ implies that the roots of $P_d(z_k, \lambda)$ are contained in $\Delta_1(\delta/2)$. If $z \in \bar{\Omega} \cap \Delta_k(\delta/2)$ and $\lambda \in \Delta_d(s)$, substitution of expression (4.3) for $1/(\zeta - z_k)$ into equation (4.2) yields

$$g(z) = \left[\frac{1}{2\pi i} \int_{|\zeta|=\delta} \frac{g(z', \zeta)}{P_d(\zeta, \lambda)(\zeta - z_k)} d\zeta\right] P_d(z_k, \lambda)$$
$$+ \sum_{j=1}^{d} \left[\frac{1}{2\pi i} \int_{|\zeta|=\delta} \frac{g(z', \zeta) P_{j-1}(\zeta, \lambda)}{P_d(\zeta, \lambda)} d\zeta\right] z_k^{d-j}.$$

Thus, we get an equation of the form (4.1) with

$$q(z,\lambda)=\frac{1}{2\pi i}\int\limits_{|\zeta|=\delta}\frac{g(z',\zeta)}{P_d(\zeta,\lambda)(\zeta-z_k)}\,d\zeta \tag{4.4}$$

and

$$r_j(z',\lambda)=\frac{1}{2\pi i}\int\limits_{|\zeta|=\delta}\frac{g(z',\zeta)P_{j-1}(\zeta,\lambda)}{P_d(\zeta,\lambda)}\,d\zeta,\quad 1\leqq j\leqq d. \tag{4.5}$$

Let $\varepsilon=\min(\delta/2, s)$, and note $|P_d(\zeta,\lambda)|\geqq C>0$ and $|\zeta-z_k|\geqq\varepsilon>0$ for $|\zeta|=\delta$ and $\lambda\in\Delta_d(\varepsilon)$. We may thus differentiate under the integral sign in equation (4.4) and obtain that $q(z,\lambda)$ is analytic in (z,λ) on $(\Omega\cap\Delta_k(\varepsilon))\times\Delta_d(\varepsilon)$. Also, since $g(z)$ represents an element of $\mathscr{A}_k(\{M_n\})$ on $\bar\Omega\cap\Delta_k(\delta)$, there exist $A_1>0$ and $B_1>0$ such that for all $n\in\mathbf{Z}^+$,

$$\sup_{\substack{\alpha\in(\mathbf{Z}^+)^k,\,|\alpha|\leqq n,\\ z\in\bar\Omega\cap\Delta_k(\delta)}}|D^\alpha g(z)|\leqq A_1B_1^nM_n.$$

Thus,

$$\sup_{\substack{\alpha\in(\mathbf{Z}^+)^{k-1},\,|\alpha|\leqq n,\\ (z',\zeta)\in\bar\Omega\cap\Delta_k(\delta)}}|D_{z'}^\alpha g(z',\zeta)|\leqq A_1B_1^nM_n.$$

Since $|P_d(\zeta,\lambda)|$ and $|\zeta-z_k|$ are bounded away from 0 for $|\zeta|=\delta$ and $(z,\lambda)\in\in(\bar\Omega\cap\Delta_k(\varepsilon))\times\Delta_d(\varepsilon)$, it follows that there exist $A>0$ and $B>0$, both independent of ζ with $|\zeta|=\delta$ and $\lambda\in\Delta_d(\varepsilon)$, such that

$$\sup_{\substack{\alpha\in(\mathbf{Z}^+)^k,\,|\alpha|\leqq n,\\ z\in\bar\Omega\cap\Delta_k(\varepsilon)}}|D_z^\alpha[g(z',\zeta)/P_d(\zeta,\lambda)(\zeta-z_k)]|\leqq AB^nM_n.$$

We may thus differentiate under the integral sign in equation (4.4), estimate in a straightforward manner, and obtain that if $\lambda\in\Delta_d(\varepsilon)$, then $q=q(z,\lambda)$ represents an element of $\mathscr{A}_k(\{M_n\})$ on $\bar\Omega\cap\Delta_k(\varepsilon)$. A similar argument shows that if $\lambda\in\Delta_d(\varepsilon)$, then $r_j=r_j(z',\lambda)$ in equation (4.5) represents an element of $\mathscr{A}_{k-1}(\{M_n\})$ on $\bar\Omega\cap\Delta_{k-1}(\varepsilon)$ which is analytic in (z',λ) on $(\Omega\cap\Delta_{k-1}(\varepsilon))\times\Delta_d(\varepsilon)$ for $1\leqq j\leqq d$.

Finally, to prove uniqueness, suppose that

$$g=P_dq+\sum_{j=1}^d r_jz_k^{d-j}$$

$$=P_d\tilde q+\sum_{j=1}^d \tilde r_jz_k^{d-j},$$

where $q=q(z,\lambda)$, $\tilde q=\tilde q(z,\lambda)\in\mathscr{A}_k(\{M_n\})$ and $r_j=r_j(z',\lambda)$, $\tilde r_j=\tilde r_j(z',\lambda)\in\mathscr{A}_{k-1}(\{M_n\})$ for $1\leqq j\leqq d$ and for each $\lambda\in\mathbf{C}^d$ which is sufficiently small. Then for some $\varepsilon>0$ and all $(z,\lambda)\in(\bar\Omega\cap\Delta_k(\varepsilon))\times\Delta_d(\varepsilon)$,

$$\sum_{j=1}^d\big(r_j(z',\lambda)-\tilde r_j(z',\lambda)\big)z_k^{d-j}=P_d(z_k,\lambda)\big(\tilde q(z,\lambda)-q(z,\lambda)\big).$$

$P_d(z_k, \lambda)$ has exactly d zeros, while $\sum_{j=1}^{d} (r_j(z', \lambda) - \tilde{r}_j(z', \lambda)) z_k^{d-j}$ is a polynomial in z_k of degree at most $d-1$. Thus $r_j = \tilde{r}_j$ for $1 \leqq j \leqq d$ and $q = \tilde{q}$. □

Theorem 4.2. *Let $f = f(z) \in \mathcal{O}_k$, $k \geqq 2$, be regular in z_k of order d. Then we may divide by f in $\mathcal{A}_k(\{M_n\})$.*

Proof. Since $f \in \mathcal{O}_k$ is regular in z_k of order d, we may apply the Weierstrass preparation theorem in $\mathcal{O}_k$ to write

$$f = uP,$$

where $u \in \mathcal{O}_k$ is a unit and $P \in \mathcal{O}_{k-1}[z_k]$ is a Weierstrass polynomial in z_k of degree d. Let $g \in \mathcal{A}_k(\{M_n\})$. If we can perform the division

$$g = Pq' + r',$$

where $q' \in \mathcal{A}_k(\{M_n\})$ and $r' \in \mathcal{A}_{k-1}(\{M_n\})[z_k]$, then we can obtain the division

$$g = fq + r$$

by taking $q = u^{-1} q' \in \mathcal{A}_k(\{M_n\})$ and $r = r' \in \mathcal{A}_{k-1}(\{M_n\})[z_k]$. Thus, we may assume $f = P$.

Choose a polydisc $\Delta_k(r)$ such that the germ P is defined on $\Delta_k(r)$ and the germ g is defined for $z \in \bar{\Omega} \cap \Delta_k(r)$. Since P is a Weierstrass polynomial in z_k, we can find numbers δ_j with $0 < \delta_j < r$, $1 \leqq j \leqq k$, such that $P(z) \neq 0$ if $|z_k| = \delta_k$ and $|z_j| \leqq \delta_j$, $1 \leqq j \leqq k-1$. Let $\Delta_k(\delta) = \{z \in \mathbf{C}^k \colon |z_j| < \delta_j \text{ for } 1 \leqq j \leqq k\}$. For $z \in \bar{\Omega} \cap \Delta_k(\delta)$, define

$$q(z) = \frac{1}{2\pi i} \int_{|\zeta| = \delta_k} \frac{g(z', \zeta)}{P(z', \zeta)} \frac{d\zeta}{\zeta - z_k} \tag{4.6}$$

and

$$r(z) = \frac{1}{2\pi i} \int_{|\zeta| = \delta_k} \frac{g(z', \zeta)}{P(z', \zeta)} \frac{P(z', \zeta) - P(z', z_k)}{\zeta - z_k} \, d\zeta.$$

By the Cauchy integral theorem, if $z \in \Omega \cap \Delta_k(\delta)$, then

$$P(z) q(z) + r(z) = \frac{1}{2\pi i} \int_{|\zeta| = \delta_k} \frac{g(z', \zeta)}{\zeta - z_k} \, d\zeta$$
$$= g(z).$$

Since P is a Weierstrass polynomial in z_k of degree d, r is a polynomial in z_k of degree at most $d-1$. We may differentiate under the integral signs in equations (4.6) and (4.7) and see that q and r are analytic in z on $\Omega \cap \Delta_k(\delta)$. Since g represents an element of $\mathcal{A}_k(\{M_n\})$ on $\bar{\Omega} \cap \Delta_k(\delta)$, there exist $A_1 > 0$ and $B_1 > 0$ such that for all $n \in \mathbf{Z}^+$,

$$\sup_{\substack{\alpha \in (\mathbf{Z}^+)^k, |\alpha| \leqq n, \\ z \in \bar{\Omega} \cap \Delta_k(\delta)}} |D^\alpha g(z)| \leqq A_1 B_1^n M_n.$$

Thus for all ζ with $|\zeta|=\delta_k$ and all $n\in\mathbf{Z}^+$,

$$\sup_{\substack{\alpha\in(\mathbf{Z}^+)^{k-1},\,|\alpha|\leqq n,\\ z'\in\Omega\cap\Delta_{k-1}(\delta)}} |D^{\alpha}_{z'}g(z',\zeta)| \leqq A_1B_1^nM_n.$$

Since $P(z',\zeta)\neq 0$ for $|\zeta|=\delta_k$ and $|z_j|<\delta_j$, $1\leqq j\leqq k-1$, $|P(z',\zeta)|\geqq C>0$ for $|\zeta|=\delta_k$ and $|z_j|<\delta_j/2$, $1\leqq j\leqq k-1$. Also $|\zeta-z_k|\geqq\delta_k/2$ for $|\zeta|=\delta_k$ and $|z_k|<<\delta_k/2$. Let $\varepsilon=\min_{1\leqq j\leqq k}\delta_j/2$. It follows that there exist $A>0$ and $B>0$, both independent of ζ with $|\zeta|=\delta_k$, such that for all $n\in\mathbf{Z}^+$,

$$\sup_{\substack{\alpha\in(\mathbf{Z}^+)^{k},\,|\alpha|\leqq n,\\ z\in\Omega\cap\Delta_{k}(\varepsilon)}} |D^{\alpha}_{z}[g(z',\zeta)/P(z',\zeta)(\zeta-z_k)]| \leqq AB^nM_n.$$

We may now differentiate under the integral sign in equation (4.6) and estimate in a straightforward manner to see that g represents an element of $\mathscr{A}_k(\{M_n\})$ on $\bar\Omega\cap\Delta_k(\varepsilon)$. A similar argument shows that r, given by equation (4.7), represents an element of $\mathscr{A}_{k-1}(\{M_n\})[z_k]$ on $\bar\Omega\cap\Delta_k(\varepsilon)$.

To prove uniqueness, suppose

$$g = Pq+r = P\tilde q+\tilde r.$$

Then for some polydisc Δ_k and all $z\in\bar\Omega\cap\Delta_k$,

$$r(z)-\tilde r(z) = P(z)(\tilde q(z)-q(z)).$$

For z' sufficiently small, $P(z',z_k)$ has exactly d zeros, while $r(z)-\tilde r(z)$ is a polynomial in z_k of degree at most $d-1$. Hence $\tilde r=r$ and $\tilde q=q$. □

Corollary 4.3. *Let* $k\geqq 2$, $P\in\mathcal{O}_{k-1}[z_k]$ *be a Weierstrass polynomial, and* $f\in\mathscr{A}_k(\{M_n\})$. *If* $fP\in\mathscr{A}_{k-1}(\{M_n\})[z_k]$ *is a polynomial, then* f *is a polynomial,* $f\in\mathscr{A}_{k-1}(\{M_n\})[z_k]$.

Proof. Since fP and P are polynomials in z_k over $\mathscr{A}_{k-1}(\{M_n\})$, we may apply the algebraic division theorem for polynomials to write

$$fP = Pq+r,$$

where $q, r\in\mathscr{A}_{k-1}(\{M_n\})[z_k]$ and r has degree less than the degree of P. By the uniqueness part of Theorem 4.2, $r=0$ and $f=q$. Thus $f\in\mathscr{A}_{k-1}(\{M_n\})[z_k]$, as desired. □

In closing, we mention one final application of Theorem 4.2. Let R be a commutative ring with unit and M be an R-module. M is a flat R-module if for every exact sequence of R-modules $A\to B\to C$, the tensored sequence

$$A\otimes_R M \to B\otimes_R M \to C\otimes_R M$$

is also exact. It is possible to use Theorem 4.2 to establish that $\mathscr{A}_k(\{M_n\})$ is a flat ring extension of $\mathcal{O}_k$, $k\geqq 2$. The details are so similar to those found in Nagel [7], however, that we choose to omit them.

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