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ASYMPTOTIC PATHS FOR SUBHARMONIC FUNCTIONS IN R^n

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1. The purpose of this note is to prove the following theorem.

T h e o r e m. *Let* $u(x)$ *be subharmonic in* R^n *and assume* $\sup u(x) = +\infty$. *Then there is polygonal path* γ *to* ∞ *so that*

$$\lim_{\gamma} u(x) = \infty .$$

The theorem is a generalization of Iversen's theorem. It was inspired by a recent manuscript of B. Fuglede where, among other results, the theorem was proved for a continuous path. Fuglede used finely harmonic functions and probability and the above result was obtained in an effort to find a classical proof. See also the work of M. N. M. Talpur (W. K. Hayman: Einige Verallgemeinerungen des Iversenschen Satzes auf subharmonische Funktionen. – Jber. Deutsch. Math.-Verein. 71, 1969, 115–122).

2. We first assume $u(x)$ continuous. Let O_n be the open set where $u(x) > n$. There are two cases.

a) O_n has only one component for every n . We then choose x_n with $u(x_n) = n$ and connect x_n to x_{n+1} inside O_{n-1} with a polygon. This gives the desired path.

b) Some O_n has two components (or more). Let A and B be two components. By the maximum principle both are unbounded. We say that A has the Phragmén–Lindelöf property if every harmonic function in A which is bounded and $\leqq 0$ on ∂A is $\leqq 0$. The following criterion is easy to prove.

L e m m a . *A has the Phragmén–Lindelöf property iff the complement of* $A^{-1} = \{ x \mid x\, |x|^{-2} \in A \}$ *is thin at* $x = 0$.

C o r o l l a r y. *At least one of A and B has the Phragmén–Lindelöf property.*

This follows e.g. from the Wiener criterion since

$$(R^n \setminus A^{-1}) \cup (R^n \setminus B^{-1}) = R^n \setminus \{0\} .$$

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To complete the proof choose $A = A_n$ as above. Then $u(x)$ has to be unbounded in A. Choose $x_{n+1} \in A_n$ and let A_{n+1} be the corresponding component of O_{n+1}. A_{n+1} also has the Phragmén–Lindelöf property and we can choose $x_{n+2} \in A_{n+1}$ etc. – The proof in this case is complete.

3. In the general case we have to find a method of constructing γ inside the set where the potential representing $u(x)$ converges uniformly. We do this by approximating $u(x)$ by smooth subharmonic functions which are negative on the set where $u(x)$ misbehaves. The construction is quite explicit and depends on a dyadic subdivision which we are next going to describe.

We may assume that $u(x) \geqq 0$. Let K_ν be the symmetric cube of side $2^{\nu+1}$ and centre at $x = 0$ and set $R_\nu = K_{\nu+1} \setminus K_\nu$. We write ($n \geqq 3$)

$$u(x) = H_\nu(x) - \int_{K_{\nu+1}} \frac{d\mu(y)}{|x-y|^{n-2}}, \quad x \in K_{\nu+1}, \tag{3.1}$$

where $H_\nu(x)$ is harmonic in $K_{\nu+1}$. Set

$$M_\nu = \operatorname*{Max}_{K_\nu} H_\nu(x) + \mu(K_{\nu+1}). \tag{3.2}$$

C will denote constants only depending on the dimension n.

We are now going to describe a subdivision of R^n into a grid G of dyadic cubes of sizes tending to zero at ∞. The construction depends on a given sequence of numbers $\delta_\nu \searrow 0$ and the sides $s(Q)$ of a cube $Q \subset K_\nu$ will be $< \delta_\nu$.

We may assume that $\delta_\nu = 2^{-N_\nu}$, N_ν integers. For $Q \subset K_1$ choose G so that $s(Q) = 2^{-N_1}$. Assume that G is constructed in K_ν. We choose

$$K_\nu \subset K_\nu^{(1)} \subset K_\nu^{(2)} \subset \dots \subset K_\nu^{(N_{\nu+1}-N_\nu)} \subset K_{\nu+1}$$

so that the cube $K_\nu^{(i)}$ has side $2^{\nu+1}(1 + 1/2 + 1/4 + \dots + 1/2^i)$. In $K_\nu^{(i+1)} \setminus K_\nu^{(i)}$ we construct $Q \in G$ with sides $2^{-N_\nu - i}$ where we set $K_\nu^{(N_{\nu+1}-N_\nu+1)} = K_{\nu+1}$. This defines G completely. It is important that $s(Q)$ changes slowly in the following sense. If $Q \in G$, $Q \subset R_\nu$, then $s(Q') \leqq 2\, s(Q)$ for all $Q' \in G$ with distance $< 2^{N_\nu} s(Q)$ from Q.

In the formula (3.1) we now replace the measure μ by the following continuous measure μ':

$$d\mu' = \frac{\mu(Q)}{m(Q)} dx, \quad x \in Q \in G.$$

More precisely, we define

$$u'(x) = u(x) + \int_{R^n} \frac{d\mu(y) - d\mu'(y)}{|x-y|^{n-2}} ;$$

$u'(x)$ is continuous and subharmonic if the integral converges in a suitable sense. We have for $x \in K_\nu$

$$u'(x) = H_\nu(x) - \int_{K_{\nu+1}} \frac{d\mu'(y)}{|x-y|^{n-2}} + \int_{R^n \setminus K_{\nu+1}} \frac{d(\mu-\mu')(y)}{|x-y|^{n-2}} .$$

The last term can be estimated (in K_ν) by

$$(3.3) \qquad \left| \int_{R^n \setminus K_{\nu+1}} \frac{d\mu - d\mu'}{|x-y|^{n-2}} \right| \leq C \sum_{\nu}^{\infty} M_i \, \delta_i .$$

We can also show that $u(x) - u'(x)$ is small in general. Let $x \in K_\nu$ and let Q^* be the union of all Q's in G with distance $< M_\nu \, \delta_\nu$ from x. We find

$$(3.4) \qquad |u(x) - u'(x)|$$

$$\leqq \int_{Q^*} \frac{d\mu(y) + d\mu'(y)}{|x-y|^{n-2}} + C \int_{K_{\nu+1} \setminus Q^*} \frac{\delta_\nu}{|x-y|^{n-1}} d\mu(y) + \sum O(M_j \, \delta_j)$$

$$\leqq \int_{Q^*} \frac{d\mu + d\mu'}{|x-y|^{n-2}} + \frac{C}{M_\nu} \int_{K_{\nu+1}} \frac{d\mu(y)}{|x-y|^{n-2}} + \sum O(M_j \, \delta_j)$$

$$\leqq \int_{Q^*} \frac{d\mu(y) + d\mu'(y)}{|x-y|^{n-2}} + O(1)$$

if we assume $\sum M_j \, \delta_j < \infty$ and observe the definition of M_ν and $u(x) \geqq 0$.

If now $u(x_\nu) \to \infty$ it follows that if we choose δ_ν small enough $u'(x)$ is a subharmonic continuous function so that $u'(x)$ is unbounded. Hence γ exists for $u'(x)$ and if we could make the estimate (3.4) uniformly by controling $\int_{Q^*}$ we would have solved our problem. This however is not possible, so an additional construction is needed to make γ avoid these bad cubes.

4. We fix some grid G and consider the set of cubes $Q \in G$ in R_ν. We increase each such Q in the scale M_ν and denote the resulting cubes Q^*. They cover R_ν M_ν^n times ($\delta_\nu < M_\nu^{-1}$). Denote by Δ_ν^* the set of such cubes such that

$$\Delta_\nu^* : \qquad \mu(Q^*) \geqq s(Q)^{n-2},$$

and set

$$E_\nu = \bigcup_{Q^* \in \Delta^*_\nu} Q^* .$$

By Egorov's theorem

$$\int_{|x-y|<M_\nu \delta_\nu} \frac{d\mu(y)}{|x-y|^{n-2}} \leqq M_\nu^{-2n}$$

except for $x \in E_\nu'$ in R_ν such that $\mu(E_\nu') \leqq M_\nu^{-n}$ provided δ_ν is small enough. Clearly $E_\nu \subset E_\nu'$.

We can now finally fix our grid G so that all conditions above are satisfied. We replace $u'(x)$ considered above by

$$U(x) = u'(x) - \sum_1^\infty M_\nu^{n-1} \int_{E_\nu} \frac{d\mu'(y)}{|x-y|^{n-2}} = u' - p' .$$

Since $p'(x)$ is the potential of a bounded measure ($\sum 1/M_\nu$ is supposed finite) and $u'(x)$ is an unbounded subharmonic function, it follows easily that $U(x)$ is also unbounded. Hence there is a continuous path γ so that $U(x) \to \infty$ along γ. The important improvement is that if $\mu(Q^*) > s(Q)^{n-2}$ then $\gamma \cap Q = \emptyset$. This is clear since $u'(x) \leqq M_\nu$ and for $x \in Q$ $p'(x) > M_\nu$ so $U(x) \leqq 0$.

5. It is clear that $u'(x) \to \infty$ along γ. However $u(x)$ may not but we do have the estimate (3.4).

Let $Q_1 \in G$ be the "first" cube intersected by γ and let x_2 be the last point on γ in Q_1. Then $x_2 \in Q_2$ also. Let x_3 be the last point in Q_2 etc. x_2 belongs to the face F_1 of Q_1 and F_2' of Q_2. $x_3 \in F_2$ in Q_2 and F_3' in Q_3 etc. Observe that $F_i'' = F_i \cap F_{i+1}' = F_i$ or F_{i+1}' and that each face includes at least $100 \cdot 2^{-n}$ % of the other.

We now join F_i'' to F_{i+1}'' by a line-segment l_i in Q_i. Since $\mu(Q_i^*) \leqq s(Q_i)^{n-2}$ it is a well-known property of Newtonian potentials that except for a small fraction of endpoints in F_i'' and F_{i+1}'' (with respect to normalized $(n-1)$-dimensional measure)

$$\int_{Q^*} \frac{d\mu(y)}{|x-y|^{n-2}} \leqq C$$

along l_i.

We now modify x_i in the following way. Consider faces F_i'' with even index i. By Fubini, for every $\xi_i \in F_i''$ except a set of small relative $(n-1)$-dimensional measure there is an l_i going forward to every ξ_{i+1}

except a small exceptional set and one l_{i-1} going backward to a corresponding set of ξ_{i-1}'s. We choose these ξ_{2i}'s in this manner. Then clearly they can be joined via $\xi_{2i\pm 1}$ to each other. This now gives the desired polygon.

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