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ON THE EXISTENCE OF AUTOMORPHIC QUASIMEROMORPHIC MAPPINGS IN R^n

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1. Introduction

Let G be a Möbius group in $\bar{R}^n = R^n \cup \{\infty\}$, $n \geqq 2$, and D a domain in R^n. A mapping $f\colon D \to \bar{R}^n$ is said to be *automorphic* with respect to G if f is continuous, open, discrete, sense-preserving, and $f \circ g = f$ for all $g \in G$. Note that if G has automorphic mappings $f\colon D \to \bar{R}^n$, then G is discrete and D is invariant under G. Discrete Möbius groups which have invariant domains are called *function groups*.

A mapping $f\colon D \to R^n$ is called *quasiregular*, abbreviated *qr*, if f is continuous, ACL^n, and

$$|f'(x)|^n \leqq KJ(x, f) \tag{1.1}$$

a.e. in D for some $K \in [1, \infty)$. If $f\colon D \to \bar{R}^n$, then the ACL^n condition and (1.1) can be checked at $f^{-1}(\infty)$ by means of auxiliary Möbius transformations. If these conditions hold the mapping is then said to be *quasimeromorphic*, abbreviated *qm*. If $n=2$ and (1.1) holds with $K=1$, then f is meromorphic.

The purpose of this note is to prove

1.2. Theorem. *Let G be a discrete Möbius group acting on B^n, $n \geqq 2$, with $V(B^n/G) < \infty$. Then G has qm automorphic mappings $f\colon B^n \to \bar{R}^n$.*

In the above theorem $V(B^n/G)$ denotes the hyperbolic volume of the orbit space B^n/G, see [4].

The proof is constructive. It is based on a modification of the method of Alexander [1], on basic properties of Möbius groups, see Chapter 3, and on the properties of radial strecthings, see Chapter 2. We shall not estimate the dilatations of f in terms of G. For the sake of simplicity we shall restrict ourselves to the case $n=3$. The same method applies to $n>3$ and to $n=2$.

It is known that every function group in R^2 has meromorphic automorphic mappings. We do not know whether function groups in R^n, $n>2$, have *qm* automorphic mappings, nor we know whether the condition $V(B^n/G) < \infty$ in Theorem 1.2 is essential. We have examples, see [4, 4.2], of *qm* automorphic mappings $f\colon B^n \to \bar{R}^n$ for infinite groups with $V(B^n/G) = \infty$.

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The notation and terminology will be as in [4]. In particular we denote $x=(x_1, \dots, x_n)=\sum x_i e_i$ for $x\in R^n$, $B^n(a, r)=\{x\in R^n\colon |x-a|<r\}$, $B^n(r)=B^n(0, r)$, $B^n=B^n(1)$, $S^{n-1}(a, r)=\partial B^n(a, r)$, $S^{n-1}(r)=S^{n-1}(0, r)$, $S^{n-1}=S^{n-1}(1)$, $H^n(h)=\{x\in R^n\colon x_n>h\}$, and $H^n=H^n(0)$. For Möbius groups G acting on B^n we let $\operatorname{Fix} G=\{x\in B^n\colon g(x)=x$ for some $g\in G\setminus\{\mathrm{id}\}\}$.

2. Radial stretchings

2.1. In this chapter we consider a special class of bi-lipschitzian mappings. A mapping $f\colon A\to R^n$, $A\subset R^n$, is called bi-lipschitzian if

(2.2) $$|x-y|/L \leqq |f(x)-f(y)| \leqq L|x-y|$$

for all x and y in A and for some $L\geqq 1$. The smallest L for which (2.2) holds will be denoted by $L(f)$.

2.3. A bounded domain $D\subset R^n$ is said to be *strictly star shaped* if each ray L from 0 meets ∂D at exactly one point. It follows that $0\in D$ and that the mapping $\varphi^*\colon \partial D\to S^{n-1}$ which sends $L\cap\partial D$ to $L\cap S^{n-1}$ is a homeomorphism. We let $\varphi\colon R^n\to R^n$ denote the radial linear extension of φ^*, i.e. $\varphi(x)=x\varphi^*(x^*)/|x^*|$, $x\neq 0$, and $\varphi(0)=0$ where $\{x^*\}=\partial D\cap\{tx\colon t>0\}$. This mapping φ which is an automorphism of R^n and maps D onto B^n will be called the *radial linear stretching* defined by D.

2.4. Lemma. *Suppose that D is strictly star shaped and $\varphi^*\colon \partial D\to S^{n-1}$ is bi-lipschitzian. Then φ is bi-lipschitzian.*

Proof. Let $M=\sup\{|x|\colon x\in\partial D\}$ and $m=\inf\{|x|\colon x\in\partial D\}$. Let $x, y\in R^n$. We may assume that $x\neq 0$ and that $\varphi(x)=x$ i.e. $\varphi^*(x^*)=x^*$ since otherwise we consider the mapping $\varphi\circ F_x$ where $F_x(z)=|x^*|z$. Then F_x is bi-lipschitzian with $L(F_x)\leqq\max(M, 1/m)$, $\varphi\circ F_x(x)=x$, and φ is bi-lipschitzian if and only if $\varphi\circ F_x$ is. Let $\alpha\in[0, \pi]$ denote the angle between the vectors x and y. If $y=0$ we set $\alpha=0$. We claim that

(2.5) $$|\varphi(y)-y| \leqq |y|\big(L(\varphi^*)+1\big)\alpha/m.$$

If $y=0$ then (2.5) is trivial. Suppose $y\neq 0$. Now

$$\begin{aligned} |\varphi(y)-y| &= |y|\,|\varphi^*(y^*)-y^*|/|y^*| \\ &\leqq |y|\big(|\varphi^*(y^*)-x^*|+|y^*-x^*|\big)/m \\ &\leqq |y|\big(|\varphi^*(y^*)-\varphi^*(x^*)|+L(\varphi^*)\,|\varphi^*(y^*)-\varphi^*(x^*)|\big)/m \\ &\leqq |y|\big(\alpha+L(\varphi^*)\alpha\big)/m \end{aligned}$$

and (2.5) follows.

To prove that $|\varphi(x)-\varphi(y)| \leqq K|x-y|$ suppose first that $\alpha \geqq \pi/2$. Then $|x-y| \geqq |y|$ and (2.5) yields

$$|\varphi(x)-\varphi(y)|/|x-y| \leqq 1+\pi\big(L(\varphi^*)+1\big)/m.$$

If $\alpha<\pi/2$, then $|x-y| \geqq |y| \sin\alpha$ and (2.5) implies

$$\begin{aligned}|\varphi(x)-\varphi(y)|/|x-y| &\leqq 1+\alpha\big(L(\varphi^*)+1\big)/(m \sin\alpha)\\ &< 1+\pi\big(L(\varphi^*)+1\big)/m.\end{aligned}$$

To prove the opposite inequality let $x, y \in R^n \setminus \{0\}$. Then

$$|\varphi(x)-\varphi(y)| = \big||x|\varphi^*(x^*)/|x^*|-|y|\varphi^*(y^*)/|y^*|\big| \geqq |x-y|/M$$

and since this inequality is trivial when $x=0$ or $y=0$, the lemma follows.

2.6. Let D be a bounded domain in R^n with $0 \in D$ and let $\beta \in (0, \pi/4]$. We say that D satisfies the *β-cone condition* if the open cone

$$C(x, \beta) = \{z \in R^n : |z-x| < |x|,\ (x-z)\cdot x > |x-z|\,|x| \cos\beta\}$$

with vertex x and central angle β lies in D whenever $x \in \partial D$. Note that if D satisfies the β-cone condition, then D is strictly star shaped.

2.7. Lemma. *Suppose that D satisfies the β-cone condition for some $\beta>0$. Then φ^* is bi-lipschitzian.*

Proof. Let $x, y \in \partial D$. Then

$$|\varphi^*(x)-\varphi^*(y)| \leqq |x-y|/m$$

where $m=\inf\{|z| : z \in \partial D\}$.

To prove the other inequality we may assume that $\varphi^*(x)=x$ and $|y| \leqq |x|$. Let $\alpha \in (0, \pi]$ denote the angle between x and y. Suppose first that $\alpha \geqq \pi/2$. Then

$$|\varphi^*(x)-\varphi^*(y)| \geqq \sqrt{2} \geqq |x-y|/\sqrt{2}.$$

Suppose now that $\alpha<\pi/2$. Since y is outside the cone $C(x, \beta)$, elementary trigonometry yields

$$|x-y| \leqq |y| \sin\alpha/\sin\beta \leqq |\varphi^*(x)-\varphi^*(y)|/\sin\beta,$$

and the lemma follows.

Since every bi-lipschitzian mapping of R^n is quasiconformal, see [3], the above lemmas imply:

2.8. Corollary. *Suppose that D satisfies the β-cone condition for some $\beta>0$. Then the radial linear stretching $\varphi\colon R^n \to R^n$ defined by D is quasiconformal.*

2.9. Remarks. (a) We shall mainly use Corollary 2.8 to show that the homeomorphism $\varphi|D$ onto B^n is quasiconformal.

(b) It is easy to see that if D is strictly star shaped, then φ is bi-lipschitzian if and only if D satisfies the β-cone condition for some $\beta>0$. Moreover, for $n=3$ the cone condition is equivalent to the boundary condition of [3, 5.3].

(c) We shall later use the following elementary property of linear stretchings: Let D_1 and D_2 be strictly star shaped domains and φ_1, φ_2 the corresponding linear stretchings. Define $\varphi=\varphi_2^{-1}\circ\varphi_1$. Suppose that $x, y\in\partial D_1$ and $E(x)=y$ for some $E\in O(n)$. If $|\varphi(x)|=|\varphi(y)|$, then $E\circ\varphi(x)=\varphi\circ E(x)$.

3. Fundamental polyhedra for Möbius groups

3.1. *Normal fundamental polyhedra.* Let G be a discrete Möbius group acting on B^n. Then G is countable and thus $B^n\setminus \operatorname{Fix} G\neq\emptyset$. The *normal fundamental polyhedron* P centered at a point $x_0\in B^n\setminus\operatorname{Fix} G$ is defined by

$$P=\{x\in B^n : d(x, x_0)<d(x, g(x_0)) \text{ for all } g\in G\setminus\{\mathrm{id}\}\}.$$

Here d denotes the hyperbolic distance in B^n. P is a convex polyhedron in the hyperbolic sense, possibly with infinite number of faces. Each $(n-1)$-face, considering only $\partial P\cap B^n$, lies in a hyperbolic $(n-1)$-plane

$$H(A, x_0)=\{x\in B^n : d(x, x_0)=d(x, A(x_0))\}$$

for some $A\in G\setminus\{\mathrm{id}\}$. Since $AH(A^{-1}, x_0)=H(A, x_0)$, the $(n-1)$-faces of P are pairwise G-equivalent. Note also that $H(A^{-1}, 0)$ is contained in the isometric sphere $I(A)=\{x\in R^n\colon |A'(x)|=1\}$ of A and $A=E\circ I$ where I is the reflection in $H(A^{-1}, 0)$ and E is an orthogonal transformation in R^n. Indeed, $(I\circ A^{-1})B^n=B^n$ and $(I\circ A^{-1})(0)=0$ and so $I\circ A^{-1}\in O(n)$. Therefore $A=E\circ I$ for some $E\in O(n)$, and $|A'(x)|=1$ for all $x\in H(A^{-1}, 0)$. Finally, recall that if G is discrete and the hyperbolic measure $V(B^n/G)$ is finite, then, see [2], [5], or [6], every normal fundamental polyhedron P has finitely many faces and $\bar{P}\cap S^{n-1}$ is either empty, which happens only when B^n/G is compact, or consists of finitely many points, called *boundary vertices*. We summarize the above facts:

3.2. Lemma. *Let G be a discrete Möbius group acting on B^n with $V(B^n/G)<\infty$. Suppose that $0\notin\operatorname{Fix} G$ and let P be a normal fundamental polyhedron centered at* 0. *Then P is of the form*

$$P=B^n\setminus\bigcup_{i=1}^{2k}\bar{B}^n(x_i, r_i)$$

where each $S_i=S^{n-1}(x_i, r_i)$, $i=1, \dots, 2k$, is orthogonal to S^{n-1}, $r_i=r_{i+k}$, and $T_iS_i=S_{i+k}$ for some $T_1, \dots, T_k\in G$. Furthermore, each T_i, $i=1, \dots, k$, is of the form $T_i=E_i\circ I_i$ where I_i denotes the reflection in S_i and $E_i\in O(n)$.

3.3. *Simple fundamental polyhedron.* Let G be a discrete Möbius group acting on B^n with $V(B^n/G)<\infty$. A normal fundamental polyhedron P for G is said to be *simple* if no two boundary vertices of P are G-equivalent. In other words, P is simple if and only if for each boundary vertex $p\in\bar{P}\cap S^{n-1}$ all the $(n-1)$-faces of P which meet at p are pairwise G-equivalent. By [4, Lemma 3.5], G has always simple fundamental polyhedra. To understand the action of G near a boundary vertex p of a simple fundamental polyhedron P centered at $x_0\in B$, choose a Möbius transformation A with $AB^n=H^n$, $A(p)=\infty$, and $A(x_0)=e_n$. Then $P_1=AP$ is a simple fundamental polyhedron centered at e_n for the group $G_1=AGA^{-1}$ with a boundary vertex at ∞. The $(n-1)$-faces of P_1 which meet at ∞ are pairwise equivalent via elements of G_1 which generate the stabilizer $G_\infty=\{g\in G_1\colon g(\infty)=\infty\}$. Each g in $G_\infty\setminus\{\mathrm{id}\}$ is a similarity in R^n with a unique fixed point at ∞ and acts on each $(n-1)$-plane $\partial H^n(h)$, $h>0$, in the same manner, see [4]. The normal fundamental polyhedron P_2 for G_∞ centered at e_n is of the form $P_2=Q\times(0,\infty)$ where Q is a finite bounded convex euclidean $(n-1)$-dimensional polyhedron. There exists $h_0>0$ such that $P_1\cap H^n(h)=P_2\cap H^n(h)$ for all $h\geqq h_0$.

For the sake of notational simplicity we shall from now on restrict our considerations to the case $n=3$. The extension to the general case $n>3$ and $n=2$ is quite straightforward.

3.4. Lemma. *Let G be a discrete Möbius group acting on B^3 with $V(B^3/G)<\infty$. Suppose that G has a simple fundamental polyhedron P centered at* 0. *Then there exist a finite convex euclidean* 3*-dimensional polyhedron $Q\subset B^3$ with all its vertices in S^2 and a homeomorphism $h\colon\bar{P}\to\bar{Q}$ such that*

(i) *$h|P$ is quasiconformal,*

(ii) *$h\circ T_i(x)=E_i\circ h(x)$ for all $x\in S_i\cap\partial P$, $i=1,\dots,k$, where S_i, T_i, and E_i are as in Lemma* 3.2.

Proof. Case 1: B^3/G is compact. Let $z_1,\dots,z_m$ be the vertices of P and let φ_1 and φ_2 be the radial linear stretchings defined by P and the euclidean polyhedron Q which is spanned by $\varphi_1(z_1),\dots,\varphi_1(z_m)$, respectively. Then $h=\varphi_2^{-1}\circ\varphi_1$ is the required mapping. Indeed, h maps $\bar{P}$ homeomorphically onto $\bar{Q}$, and since P and Q satisfy the β-cone condition for some $\beta>0$, φ_1 and φ_2 are quasiconformal by Corollary 2.8 and consequently so is $h|P$. For (ii) let $x\in S_i\cap\partial P$. Then by Lemma 3.2

$$h\circ T_i(x)=h\circ E_i\circ I_i(x)=h\circ E_i(x).$$

Since $|x_i|=|x_{i+k}|$ and $r_i=r_{i+k}$, it follows by the nature of $h|\partial P$ and E_i, see 2.9 (c), that $h\circ E_i(x)=E_i\circ h(x)$ and so (ii) follows.

Case 2: B^3/G is non-compact. Since now P does not satisfy the β-cone condition for any $\beta>0$, we first map P quasiconformally onto a domain $R\subset B^3$ which satisfies the β-cone condition and then proceed as in Case 1.

Let $p_1,\dots,p_q$ be the boundary vertices of P. For each $j=1,\dots,q$ choose a Möbius transformation A_j with $A_jB^3=H^3$, $A_j(p_j)=\infty$, and $A_j(0)=e_3$. Pick

$h>0$ such that $A_j^{-1}(H^3(h))\cap A_l^{-1}(H^3(h))=\emptyset$ for $j\neq l$ and so that $\cup A_j^{-1}(H^3(h))$ does not contain any vertex of P. By 3.3 each set $C_j=A_j(P)\cap H^3(h)$ is of the form $Q_j\times(h,\infty)$ where Q_j is a convex, bounded, and finite 2-dimensional euclidean polyhedron containing 0. Let $\psi_j^*\colon R^2\to R^2$ be the radial linear stretching defined by Q_j and let $\psi_j\colon R^2\times[h,\infty)\to H^3$ be the mapping $\psi_j(x,s)=(\psi_j^*(x), s-h)$. Then ψ_j is bi-lipschitzian and hence quasiconformal and maps C_j onto the semi-infinite cylinder $Z=B^2\times(0,\infty)$. Next we construct a quasiconformal mapping of Z onto a finite cylinder.

Let (r,φ,x_3) and (t,φ,Θ) denote the cylinder and spherical coordinates of R^3, respectively. The polar angle is measured from the positive half of the x_3-axis. The mapping $f_1\colon Z\to B^3\cap H^3$ which is defined by $t=e^{-x_3}$, $\varphi=\varphi$, and $\Theta=\pi r/2$ is quasiconformal, see [3, p. 47], surjective, and maps the discs $B^2\times\{x_3\}$, $x_3>0$, onto the hemispheres $S^2(e^{-x_3})\cap H^3$. Let f_2 denote the reflection in the sphere $S^2(e_2/3, 2/3)$. Then f_2 is anti-conformal and maps $B^3\cap H^3$ onto $H^3\setminus\bar{B}^3(e_2/2, 1/2)$, $B^3(1/3)\cap H^3$ onto $\{x\in H^3\colon x_2<-1/3\}$ and the 2-planes through the x_3-axis onto the spheres centered on the line $L=\{x\in\partial H^3\colon x_2=-1/3\}$ and passing through the points $e_2/3$ and $-e_2$. Using cylinder coordinates (r,φ,x_1) with L as the axis of symmetry and φ measured from the direction of the vector $-e_2$ to the direction of e_3, we define a mapping $f_3\colon H^3\to H^3$ by

$$f_3(r,\varphi,x_1)=(r,\pi/2+\varphi/4,x_1),\quad 0\leqq\varphi<2\pi/3,$$
$$=(r,\varphi,x_1),\quad 2\pi/3\leqq\varphi<\pi.$$

Then f_3 is quasiconformal, maps $H^3\setminus\bar{B}^3(e_2/2, 1/2)$ onto $\{x\in H^3\colon x_2>-1/3\}\setminus\bar{B}^3(e_2/2, 1/2)$, maps each sphere which is centered on L and which passes through the points $e_2/3$ and $-e_2$ into itself, and maps half-planes through L rigidly into half-planes through L.

Finally, let $\psi=f_1^{-1}\circ f_2^{-1}\circ f_3\circ f_2\circ f_1$. Then ψ maps the semi-infinite cylinder Z quasiconformally onto the finite cylinder $Z'=B^2\times(0,\log 3)$ and has the following properties:

(a) ψ has a homeomorphic extension to $\bar{Z}$ denoted again by ψ,

(b) $\psi(x)=x$ for $x\in\partial Z\cap\partial H^3$, and

(c) each plane through the x_3-axis is mapped into itself in the same manner.

To map P onto a domain R in B^3 which satisfies the β-cone condition for some $\beta>0$, define $\varphi\colon P\to B^3$ by $\varphi(x)=A_j^{-1}\circ\psi_j^{-1}\circ\psi\circ\psi_j\circ A_j(x)$ for $x\in P\cap A_j^{-1}(H^3(h))$, $j=1,\dots,q$, and $\varphi(x)=x$ otherwise. Then φ is quasiconformal with a homeomorphic extension, denoted by φ, to $\bar{P}$. The image domain $R=\varphi P$ is obtained from P by cutting away balls tangent to S^2 at the boundary vertices and hence R satisfies the β-cone condition for some $\beta>0$. Because of (c),

$$\varphi\circ E_i(x)=E_i\circ\varphi(x) \tag{3.5}$$

for every $x\in\partial P\cap S_i$.

Let $\{z_1, \dots, z_m\}$ be the set of all vertices of P and let φ_1 and φ_2 be the radial linear stretchings defined by R and by the euclidean polyhedron Q which is spanned by $\{\varphi_1(z_1), \dots, \varphi_1(z_m), p_1, \dots, p_q\}$, respectively. Finally, let $h=\varphi_2^{-1}\circ\varphi_1\circ\varphi$. Then h maps P quasiconformally onto Q with a homeomorphic extension to $\bar{P}$.

To prove (ii) note that, by Lemma 3.2 and by the symmetry of Q, see 2.9 (c),

$$h\circ T_i(x) = h\circ E_i\circ I_i(x) = h\circ E_i(x) = \varphi_2^{-1}\circ\varphi_1\circ E_i(x) = E_i\circ\varphi_2^{-1}\circ\varphi_1(x) = E_i\circ h(x)$$

for $x\in\partial P\setminus\bigcup A_j^{-1}(H^3(h))$. Suppose now that $x\in\partial P\cap A_j^{-1}(H^3(h))\cap S_i$ for some j and i. Then $h(x)=\varphi_2^{-1}\circ\varphi_1\circ\varphi(x)$ and hence by (3.5) and Lemma 3.2, $h\circ T_i(x)=$ $=\varphi_2^{-1}\circ\varphi_1\circ E_i\circ\varphi(x)$. By the same reason as above we can now interchange E_i and $\varphi_2^{-1}\circ\varphi_1$. This yields (ii) and the proof is complete.

4. Construction of a QM automorphic mapping in B^3

4.1. *Simplices in* R^3. Given a simplex $\sigma=(x^0, x^1, x^2, x^3)$ in R^3 we let $|\sigma|$ denote the closed tetrahedron in R^3 which is spanned by the vertices x^0, x^1, x^2, x^3 and let $\mathbf{C}|\sigma|=\bar{R}^3\setminus\text{int}\,|\sigma|$. All 3-simplices in R^3 will be oriented by the sign of the standard determinant function associated with the basis e_1, e_2, e_3.

4.2. Lemma. *Given any two simplices* $\sigma=(x^0, x^1, x^2, x^3)$ *and* $\tau=(y^0, y^1, y^2, y^3)$ *in* R^3 *there exists a sense-preserving homeomorphism* $h=h_{\sigma\tau}$ *from* $|\sigma|$ *into* $\bar{R}^3$ *such that*

(i) $h(|\sigma|)=|\tau|$ *if* σ *and* τ *have the same orientation and* $h(|\sigma|)=\mathbf{C}|\tau|$ *otherwise,*

(ii) $h(x^i)=y^i$, $i=0, 1, 2, 3$,

(iii) $h|\partial|\sigma|$ *is a piecewise linear homeomorphism,*

(iv) $h|\text{int}\,|\sigma|$ *is quasiconformal.*

Proof. If σ and τ have the same orientations, then the piecewise linear map of $|\sigma|$ onto $|\tau|$ which is defined by (ii) satisfies (i), (iii), and (iv).

Suppose now that σ and τ have different orientations. We may assume that $0\in\text{int}\,|\tau|$. The radial linear stretching φ defined by int $|\tau|$ is quasiconformal in R^3 and has a quasiconformal extension to $\bar{R}^3$ with $\varphi(\infty)=\infty$. Let I denote the reflection in S^2 and $h_1\colon |\sigma|\to|\tau|$ the sense-reversing piecewise linear map which satisfies $h_1(x^i)=y^i$, $i=0, 1, 2, 3$. Then it is easy to check that $h=\varphi^{-1}\circ I\circ\varphi\circ h_1$ is the required map.

4.3. *Proof of Theorem* 1.2 *for* $n=3$. Let P be a simple fundamental polyhedron for G centered at $x_0\in B^3$. We may assume that $x_0=0$, otherwise consider first AGA^{-1} for some Möbius transformation A with $AB^3=B^3$ and $A(x_0)=0$.

Let T_i and E_i, $i=1, \dots, k$, be as in Lemma 3.2 and Q and $h\colon \bar{P}\to\bar{Q}$ as in Lemma 3.4. Using planes through 0 triangulate Q so that any two E_i-equivalent faces in ∂Q have E_i-equivalent sub-triangulations. In this triangulation, call it K, identify E_i-equivalent faces in ∂Q and thus denote vertices of K which are E_i-equivalent, $i=1, \ldots, k$, by the same symbols. Now K can be made fine enough so that all four vertices of any 3-simplex are distinct. Let $S=\{\sigma_1, \ldots, \sigma_\nu\}$ be the set of all 3-simplices in K and $\{0, x^2, \ldots, x^N\}$ the set of all vertices in K. Then each σ_i is of the form $(0, x^{i_1}, x^{i_2}, x^{i_3})$. Choose N points $0, y^2, \ldots, y^N$ in general position in R^3, i.e. no four points are coplanar, and associate with each 3-simplex $\sigma_i=(0, x^{i_1}, x^{i_2}, x^{i_3})$ in S the simplex $\tau_i=(0, y^{i_1}, y^{i_2}, y^{i_3})$. Now define $g\colon \bar{Q}\to\bar{R}^3$ by $g||\sigma_i|=h_{\sigma_i\tau_i}$ for $i=1, \ldots, \nu$. Here $h_{\sigma_i\tau_i}$ is as in Lemma 4.2.

Finally let $f\colon B^3\to\bar{R}^3$ be defined for $x\in T^{-1}(\bar{P})\cap B^3$, $T\in G$, by $f(x)=g\circ h\circ T(x)$. Then f is continuous by Lemma 3.4 (ii) and by the construction of K and g. f is automorphic with respect to G, and since $g\circ h$ is qm in P by Lemma 3.4 and Lemma 4.2, it follows that so is f in B^3.

4.4. Remark. The automorphic mapping f constructed above has the property $N(A,f)<\infty$ where A is any fundamental set for G and $N(A,f)=\sup \operatorname{card}(f^{-1}(y)\cap A)$ over all $y\in\bar{R}^3$.

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