Annales Academiæ Scientiarum Fennicæ
Series A. I. Mathematica
Volumen 2, 1976, 59 – 66

Commentationes in honorem
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LXXX annos nato

A NOTE ON FUNCTIONS WITH DEFICIENCY SUM TWO

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1. Introduction. We shall apply the Ahlfors theory of covering surfaces (cf. [1], [2] and [10], Ch. XIII) to analyse functions of order $\lambda < \infty$ having

$$\sum_{1}^{q} \delta(a_i, f) = 2 . \tag{1.1}$$

The problem of characterizing such functions was formally proposed by F. Nevanlinna in 1929 ([9], cf. also [6]). Nevanlinna conjectured in particular that $2\lambda - 1$ must be a positive integer and each $\delta(a_i)$ an integral multiple of λ^{-1}. This would imply

$$q \leqq 2\lambda \tag{1.2}$$

and Weitsman's remarkable proof of (1.2) ([13]) still remains an isolated step toward resolving this hypothesis (if f is entire very complete information is known [3], [4], [5], [11]).

A proof of the full F. Nevanlinna conjecture seems beyond the scope of the Ahlfors theory since this theory applies to very general exhaustions of the plane, in particular ones for which the conjecture is false.

That f has finite order implies that to each $K > 2$ may be associated $M < \infty$ and an unbounded R-set $\mathscr{P}$ where

$$\mathscr{P} = \{ R ; T(4KR) < M\,T(R) \} \tag{1.3}$$

(compare (1.3) with definition (3.3) of [7]; in [7] appear estimates on the size of $\mathscr{P}$).

In order to state our results, we recall a standard convention: if A is an open set of the plane, then

$$n(r, a, A)$$

[1] Research partially supported by National Science Foundation.

doi:10.5186/aasfm.1976.0208

is the cardinality of $\{f^{-1}(a)\} \cap A \cap \{ |z| < r\}$, and if $\Theta_A(r) = \{ \vartheta ; r e^{i\vartheta} \in A \}$,

$$m(r, a, A) = \frac{1}{2\pi} \int_{\Theta_A(r)} \log \left| \frac{1}{f(r e^{i\vartheta}) - a} \right| d\vartheta .$$

Theorem. *Let f satisfy (1.1) and be of finite order λ. Then in each disc $\{|z| < K R\}$ $(R \in \mathscr{P})$ exist unions of simply-connected components $A_i = A_i(R)$ $(1 \leqq i \leqq q)$ such that all components of A_i meet $\{|z| = R\}$,*

(1.4) *no components of A_i are compact subsets of $\{ |z| < K R \}$*

(1.5) $$|f(\zeta) - a_i| = \varepsilon \qquad (\zeta \in \partial A_i , \ |\zeta| < K R , \ R \in \mathscr{P})$$

and, as $R \to \infty$ in $\mathscr{P}$,

(1.6) $$m(R, a_i, A_i) \sim \delta(a_i) T(R) \qquad (i = 1, \dots, q)$$

(1.7) $$n(K R, a_i, A_i) + n(K R, \infty, A_i) = o(1) T(R) .$$

Conclusions (1.4) with (1.6) are implicit in [13], but (1.7), which implies that the A_i may be viewed asymptotically as coverings of the punctured disc $\{ 0 < |w - a_i| < \varepsilon \}$, is new. It is easy to see that error terms of the magnitude (1.7) can in fact occur, even in the presence of (1.1).

I thank Allen Weitsman for detecting a gap in the original version of § 3.

2. Preliminary lemmas. As was noted already, we use arguments from Ahlfors's theory. Let $\mathscr{F}$ be the covering which the meromorphic function f induces on the Riemann sphere and $\mathscr{F}(r)$ the covering obtained when f is restricted to $\{|z| < r\}$. The natural comparison function here is $S(r)$, the mean sheet number of $\mathscr{F}(r)$ (*cf.* [10], p. 327), and we have the Ahlfors–Shimizu formula

(2.1) $$T(r) \sim \int_0^r S(t)\, t^{-1}\, dt .$$

For $1 \leqq i \leqq q$ and (small) $\varepsilon > 0$, let γ_i be the curve on the (base surface) Riemann sphere which corresponds to $\{ |w - a_i| = \varepsilon \}$, and D_i the interior of γ_i. For a fixed r, the mean sheet number $S(\gamma_i)$ of all arcs on $\mathscr{F}$ $(= \mathscr{F}(r))$ [2] which are over each γ_i ([10], p. 327) satisfies

(2.2) $$|S - S(\gamma_i)| \leqq h L$$

where h here and below represents a constant which depends on the $\{a_i\}$ and ε, but not on r or the particular covering under consideration.

[2] To limit notation, references to r are often suppressed.

We will need to quote the following facts later; they all require that f be of finite order.

First, if $L(r)$ is the length of the relative boundary of $\mathscr{F}(r)$, J. Miles has proved [8]

$$\int_0^r L(t)\, t^{-1}\, dt = o(T(r)) . \tag{2.3}$$

In addition, we shall use the elementary fact (cf. [6], p. 3) that under hypothesis (1.1)

$$\lim_{r\to\infty} \frac{N(r, a)}{T(r)} = \begin{cases} 1, & a \notin \{a_i\}, \\ 1 - \delta(a_i), & a \in \{a_i\}; \end{cases} \tag{2.4}$$

i.e., lim sup may be replaced by limit in the definition of deficiency. Also, (1.1) implies

$$N(r, 1/f') = o(1)\, T(r, f'), \tag{2.5}$$

$$T(r, f') \sim (2 - \delta(\infty, f))\, T(r, f) \tag{2.6}$$

(cf. [13], Lemma A).

Finally, there is the following result of Collingwood–Selberg–Weitsman type (compare [12], pp. 200–201 and Lemma D of [14]).

Lemma A. *Let $f(z)$ be meromorphic with more than one deficient value, let $\mathscr{F}$ be the covering of the sphere induced by f, and let the $\{\gamma_i\}$, $\{D_i\}$ be as above. For a fixed $i > 0$, let A_i be a compact portion of $\mathscr{F}$ which lies above D_i, and has the property that each point of D_i is covered at most p times in each component of A_i. Identify A_i with its image in the z plane by f. Then*

$$m(r, a_i, A_i) \leqq O(p) \qquad (R \leqq r \leqq 2KR,\ R \in \mathscr{P}) \tag{2.7}$$

where the constant implicit in (2.7) *depends on* K, M (cf. (1.3)) *and the size of two of the positive deficiencies.*

3. A refinement of Ahlfors's estimates. Let D^i $(= D^i(r))$ be the components of $\mathscr{F}$ $(= \mathscr{F}(r))$ that lie over D_i, and introduce three subclasses C_i, C_i^*, C_i^{**} (which depend on r) as follows. The C_i are those D^i which are not compactly contained in $\mathscr{F}$; then those D^i which are not contained (in the topology of $\{|z| < r\}$) in some $c_j \in C_j$ $(1 \leqq j \leqq q)$ are divided into C_i^* (if D^i is not simply-connected) and C_i^{**} otherwise.

Let c_i be a component of C_i, let b_i be the union of c_i with the compact components of the complement of c_i relative to $\mathscr{F}(r)$ and $B_i = B_i(r)$

the union of all the b_i. We identify the C_i^* and B_i with their inverse images by f in the plane.

Lemma 1. *Let* $f(z)$ *be meromorphic in the plane and* $a_1, \ldots, a_q$ *finite complex numbers. Then*

$$(3.1)\qquad \sum_1^q n(r, a_i) \geqq (q-2)\, S(r) + n(r, \infty, \cup B_i) + \sum n(r, a_i, B_i) + \sum n(r, a_i, C_i^*) - h\, L(r)\,.$$

Proof. Define a bordered surface F by

$$F = F(r) = \left(\bigcup_1^q B_i\right)' \cap \{|z| < r\}\,,$$

so that

$$(3.2)\qquad \sum n(r, a_i) = \sum n(r, a_i, B_i) + \sum_i \sum_{j\neq i} n(r, a_i, B_j) + \sum n(r, a_i, F)\,.$$

Rouché's theorem implies that

$$(3.3)\qquad \sum_i \sum_{j \neq i} n(r, a_i, B_j) = (q-1)\, n(r, \infty, \cup B_i)$$

so (3.1) will follow at once from (3.2), (3.3) and

$$(3.4)\qquad \sum n(r, a_i, F) \geqq (q-2)\, S - (q-2)\, n(r, \infty, \cup B_i) + \sum n(r, a_i, C_i^*) - h\, L\,.$$

To prove (3.4), let $\mathscr{F}_1 = \mathscr{F}_1(r)$ be the subsurface of $\mathscr{F}$ which corresponds to F. Each component of $\mathscr{F}_1(r)$ is simply-connected and $\mathscr{F}_1(r) \subset \mathscr{F}_1(r')$ $(r < r')$; thus the $\mathscr{F}_1(r)$ exhaust a covering of the sphere in the sense of [10], p. 341. If all C_i^*, C_i^{**} are now deleted from $\mathscr{F}_1$, we obtain $\mathscr{F}^* = \mathscr{F}^*(r) = \cup F^*$ which now is a covering of F_0, the Riemann sphere with the D_i deleted. It follows as in [10], pp. 342–3, that

$$(3.5)\qquad -\sum_{\mathscr{F}_1} \varrho(D^i) = -\sum_{C_i^* \cup C_i^{**}} \varrho(D^i) \geqq \sum_{\mathscr{F}^*} \varrho^+(F^*)$$

where ϱ is Euler's characteristic and $\varrho^+ = \max(\varrho, 0)$. The right side of (3.5) is estimated by the Main Theorem ([10], p. 332):

$$(3.6)\qquad \sum \varrho^+(F^*) \geqq \varrho(F_0)\, S^*(r) - h^*\, L^*(r) = (q-2)\, S^*(r) - h^*\, L^*(r)\,;$$

here S^* is the mean sheet number of $\mathscr{F}^*$, h^* is a positive constant depending only on F_0 and L^* is the length of the relative boundary of $\mathscr{F}^*(r)$. Clearly

(3.7) $$L^* \leqq L$$

where L is as in § 2, so (3.5)–(3.7) yield a positive constant h as in § 2 with

(3.8) $$-\sum_{\mathscr{F}_1} \varrho(D^i) \geqq (q-2)\, S^*(r) - h\, L(r)\,.$$

Let β_i join γ_i to γ_{i+1} on the sphere (indexing mod q) and be disjoint from $\cup D_i$. Then if $S^*(\beta_i)$ is the mean covering number of β_i in $\mathscr{F}^*$ we have (cf. (2.2), (3.7))

(3.9) $$\left|\sum S^*(\beta_i) - q\, S^*\right| < h\, L\,,$$

(3.10) $$\left|\sum S(\beta_i) - q\, S\right| < h\, L\,.$$

It is straightforward from Rouché's theorem to see that if $S_i^*(\beta_j)$ is the mean covering number of β_j from curves interior to B_i then

$$\sum S_i^*(\beta_j) = q\, n(r\,,\infty\,,B_i)$$

and so

$$\sum \{ S(\beta_i) - S^*(\beta_i) \} = \sum_{i,j} S_i^*(\beta_j) = q\, n(r\,,\infty\,,\cup B_i)\,.$$

It follows from this, (3.9) and (3.10) that

$$\begin{aligned} S^* &\geqq q^{-1} \sum S^*(\beta_i) - h\, L \\ &= q^{-1} \sum S(\beta_i) - q^{-1} \sum \{ S(\beta_i) - S^*(\beta_i) \} - h\, L \\ &\geqq S - n(r\,,\infty\,,\cup B_i) - h\, L\,. \end{aligned}$$

Also $\varrho(D^i) \geqq 0$ unless D^i is in C_i^{**}. Thus (3.8) now becomes

(3.11) $$\begin{aligned} \sum_{C_i^{**}} n(r\,,a_i) &\geqq -\sum_{\mathscr{F}_1} \varrho(D^i) \\ &\geqq (q-2)\, S - (q-2)\, n(r\,,\infty\,,\cup B_i) - h\, L\,, \end{aligned}$$

and (3.4) is proved.

4. Proof of the Theorem. Choose $a_1\,,\ldots\,,a_q$ ($q \geqq 2$) to exhaust the deficient values of f; it is no loss of generality to suppose $\infty \notin \{a_i\}$.

Let $R \in \mathscr{P}$, and for each i consider the components D^i over D_i in $\mathscr{F}(2\,K\,R)$. In order to define A_i, it is first necessary to delete three subclasses of the D^i.

First, let D_1^i be all compact components of D^i in $\mathscr{F}(2\,K\,R)$ which are simply-connected and also are $1-1$ coverings of D_i; as usual, we identify the D_1^i with their inverse image in the plane by f. It then follows from (2.7) that

$$(4.1) \qquad m(r\,,a_i\,,D_1{}^i) \;=\; O(1) \qquad (R \leqq r < 2\,K\,R\,,\ R \in \mathscr{P})\,.$$

Next, let $D_2{}^i$ be the remaining compact simply-connected components of D^i in $\mathscr{F}(2\,K\,R)$. Let D_0 be a component of $D_2{}^i$. Since D_0 is compact, each $w \in D_i$ is covered by the same number, p_0, of times by f for $z \in D_0$ (with due account of multiplicity). But D_0 is (connected and) simply-connected, so f' must have $(p_0 - 1)$ zeros in D_0, and since $D_0 \notin D_1{}^i$, it follows that $p_0 \geqq 1$. However (1.3), (2.5) and (2.6) readily give

$$\begin{aligned} n(2\,K\,R\,,0\,,1/f') &\leqq (\log 2)^{-1} N(4\,K\,R\,,0\,,1/f') \\ &= o(1)\,T(4\,K\,R\,,f') \;=\; o(1)\,T(4\,K\,R\,,f) \\ &= o(1)\,T(R) \qquad (R \to \infty\,,\ R \in \mathscr{P})\,, \end{aligned}$$

from which we deduce that the total number of times each point $w \in D_i$ is covered from all $D_2{}^i$ in $\{\,|z| < 2\,K\,R\,\}$ is $o(1)\,T(R)$ and, from (2.7), that

$$(4.2) \quad m(r\,,a_i\,,D_2{}^i) \;=\; o(1)\,T(R) \qquad (R \leqq r < K\,R\,,\ R \to \infty\,,\ R \in \mathscr{P})\,.$$

The third subclass of D^i to be eliminated is $D_3{}^i$; the D^i which are compactly contained in $\mathscr{F}(2\,K\,R)$. Thus, $D_3{}^i$ consists of the compact components of $\mathscr{F}(2\,K\,R)$ over D^i which are not in $D_1{}^i$ or $D_2{}^i$.

L e m m a 2. *We have*

$$(4.3) \quad m(r\,,a_i\,,D_3{}^i) \;=\; o(1)\,T(R) \qquad (R \leqq r < K\,R\,,\ R \to \infty\,,\ R \in \mathscr{P})\,.$$

Proof. Consider (3.1) for $2\,K\,R \leqq r \leqq 4\,K\,R$. Then the $D_3{}^i$ are a subset of the $C_i{}^*(r)$ in $\mathscr{F}(r)$. When (3.1) is integrated from $2\,K\,R$ to $4\,K\,R$, (1.1), (2.1), (2.3), (2.4) yield

$$\begin{aligned} (4.4) \qquad &\{\,(q-2) + o(1)\,\}\{\,T(4\,K\,R) - T(2\,K\,R)\,\} \\ &\geqq \{\,(q-2) + o(1)\,\}\{\,T(4\,K\,R) - T(2\,K\,R)\,\} \\ &\quad + \log 2 \sum n(2\,K\,R\,,a_i\,,D_3{}^i) \\ &\quad - o(1)\,T(4\,K\,R) \qquad (R \to \infty\,,\ R \in \mathscr{P})\,. \end{aligned}$$

Thus

$$(4.5) \qquad \sum n(2\,K\,R\,,a_i\,,D_3{}^i) \;=\; o(1)\,T(R) \qquad (R \to \infty\,,\ R \in \mathscr{P})$$

and since the $D_3{}^i$ are compact, (4.3) follows from (2.7) and (4.5).

The set A_i demanded by the theorem consists of those components $D_4{}^i$ of D^i which are not in $\bigcup_{k=1}^{3} D_k{}^i$ and which meet $\{|z| = R\}$, together with any compact components of their complements relative to $\{\,|z| < K\,R\,\}$. Thus, the A_i are simply-connected and since no $D_4{}^i$ is compact in $\mathscr{F}(K\,R)$, (1.4) and (1.5) are obvious. Now, any contribution to $m(R, a_i)$ must come from D^i which meet $\{|z| = R\}$, but the contribution from those D^i which meet $\{|z| = R\}$ but are not in the $D_4{}^i$ satisfies (4.1) – (4.3). Thus

$$(4.6)\quad m(R, a_i, D_4{}^i) \sim m(R, a_i, \bigcup_1^4 D_k^i) \sim \delta(a_i) T(R) \qquad (R \to \infty,\ R \in \mathscr{P}).$$

In $A_i - \bigcup_1^4 D_k^i$, $|f - a_i| > \varepsilon$, so the assumption that $\delta(\infty, f) = 0$ with (2.4) gives that $|m(R, a_i, A_i - \bigcup_1^4 D_k{}^i)| = o(1) T(R)$ which with (4.6) yields (1.6).

Finally, (1.7) is obtained from (3.1) as in the proof of (4.5) since now $A_i \subset B_i = B_i(r)$ for all $KR < r < 2KR$.

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Received 6 October 1975