Annales Academiæ Scientiarum Fennicæ
Series A. I. Mathematica
Volumen 3, 1977, 131—134

ON NEVANLINNA'S CHARACTERISTIC FUNCTIONS OF ENTIRE FUNCTIONS AND THEIR DERIVATIVES

SAKARI TOPPILA

We use the usual notation of the Nevanlinna theory. We shall consider the following problem of Nevanlinna ([3], p. 104, [4], p. 239, Hayman [1] and [2], Problem 1.21): Does there exist a function f transcendental and meromorphic in the plane such that $\liminf_{r\to\infty} T(r,f)/T(r,f')>1$? We prove the following

Theorem. *There exists an integral function F of order 1 such that*

$$\liminf_{r\to\infty} T(r, F)/T(r, F') \geqq 1+7/10^7.$$

Proof. Let k and m be positive integers. We denote $r_k=k$ if $3^{2m}+1\leqq k\leqq \leqq 3^{2m+1}$, and if $3^{2m-1}+1\leqq k\leqq 3^{2m}$ then $r_k=-k$. We set

$$g_p(z) = \prod_{k=3^p+1}^{3^{p+1}} (1-z/r_k)e^{z/r_k}. \tag{1}$$

We denote

$$s_p = \sum_{k=3^p+1}^{3^{p+1}} 1/k.$$

The sequence s_p is increasing and $\lim_{p\to\infty} s_p=\log 3$. We choose a positive odd integer $p\geqq 5$ such that

$$\log 3-s_p < 1/1000 \tag{2}$$

and set

$$f(z) = 3^{z/2} \prod_{k=p}^{\infty} g_k(z). \tag{3}$$

Let $n>p$. We write $t=3^n$ and

$$f(z) = H_n(z)S_n(z)A_n(z) \prod_{k=n+1}^{\infty} g_k(z) \tag{4}$$

where $H_n(z) = \prod_{k=3^p+1}^{t} (1-z/r_k)$, $S_n(z) = 3^{z/2} \prod_{k=3^p+1}^{3t} e^{z/r_k}$ and $A_n(z) = = \prod_{k=t+1}^{3t} (1-z/r_k)$. Let $2t\leqq|z|\leqq 2t+1$. We have

$$\log S_n(z) = z\left((\log 3)/2+ \sum_{k=p}^{n} (-1)^k s_k\right) \tag{5}$$

doi:10.5186/aasfm.1977.0326

and get from (2)

(6) $$|\log S_n(z)-(-1)^n(z\log 3)/2| \leqq |-z|/1000.$$

We have

$$|H_n(z)| = \prod_{k=3^p+1}^{t} |(z-r_k)/r_k|$$

$$\leqq |z|^{-3} \prod_{k=1}^{t} (2t+1+k)/k$$

$$= (3t+1)!/(|z|^3(2t+1)!\,t!),$$

and it follows from Stirling's formula that

(7) $$\log|H_n(z)| \leqq 2t\log 3 - t\log(4/3).$$

Let $k>3t$. Then

(8) $$\log((1-z/r_k)e^{z/r_k}) = -(1/2)(z/r_k)^2-(1/3)(z/r_k)^3-\dots$$

and therefore

$$\log|(1-z/r_k)e^{z/r_k}| \leqq (1/2)(|z/k|^2+|z/k|^3+\dots) \leqq 2|z/k|^2.$$

This implies that

(9) $$\log\left|\prod_{k=n+1}^{\infty} g_k(z)\right| \leqq 2|z|^2 \sum_{k=3t+1}^{\infty} (1/k)^2 \leqq 3t.$$

If $t+1\leqq k\leqq 3t$ then $|1-z/r_k|\leqq 3$ and we get

(10) $$\log|A_n(z)| \leqq 2t\log 3.$$

Combining the inequalities (6), (7), (9) and (10), we see that

(11) $$\log|f(z)| \leqq 8.28t$$

on $2t\leqq|z|\leqq 2t+1$. It follows from the maximum principle that $\log|f(z)|\leqq 8.28t$ on $|z|\leqq 2t$. This implies that $\log|f(z)|\leqq 13|z|$ on $2t/3\leqq|z|\leqq 2t$ and therefore

(12) $$\log|f(z)| \leqq 13|z|$$

for all large values of $|z|$.

Let $t=3^n$. If n is even then $J_n=\{x+iy\colon y=0,\ 2t+1/4\leqq x\leqq 2t+3/4\}$, and if n is odd then $J_n=\{x+iy\colon y=0,\ -2t-3/4\leqq x\leqq -2t-1/4\}$. Let $z\in J_n$. It follows from (8) that

(13) $$\left|\prod_{k=n+1}^{\infty} g_k(z)\right| < 1.$$

We have $|A_n(z)|=\prod_{k=t+1}^{3t}|(z-r_k)/r_k|\leqq (t!)^3/(3t)!$, and it follows from Stirling's formula that

(14) $$\log|A_n(z)| \leqq -3t\log 3+2\log t.$$

Combining the inequalities (6), (7), (13) and (14), we see that

(15) $$\log|f(z)| \leqq -t(\log(4/3)-2/1000)+O(\log t).$$

This implies that

(16) $$\log|f(z)| \leqq -0.135|z|+O(\log|z|)$$

on J_n.

The definition of f implies that

$$\pi z f(z) f(-z) \prod_{k=1}^{3^p} (1-(z/k)^2) = \sin(\pi z)$$

and therefore

(17) $$f(z)f(-z) = \sin(\pi z)/P(z)$$

where P is a polynomial. Now it follows from (16) that

(18) $$\log|f(z)| \geqq 0.135|z|+O(\log|z|)$$

if $-z\in J_n$.

We denote $F(z)=\int_0^z f^2(w)\,dw$. Note that $f^2(z)\geqq 0$ on the real axis. Let n be even and $t=3^n$. It follows from (18) that $\log f^2(w)\geqq 0.27(2t/3)+O(\log t)$ for $-w\in J_{n-1}$ and therefore

(19) $$\log F(z) \geqq 3z/100+O(\log z)$$

on the segment $2t/3+1\leqq z\leqq 6t+1$. Then (19) holds on the whole positive real axis. Similarly, we see that $\log(-F(z))\geqq 3|z|/100+O(\log|z|)$ on the negative real axis we conclude that

(20) $$\log|F(z)| \geqq 3|z|/100+O(\log|z|)$$

on the real axis.

Let $\alpha=1/300$, $-\alpha\leqq\varphi\leqq\alpha$, and $z=re^{i\varphi}=x+iy$. It follows from (17) that $|f(z)f(-z)|\leqq|\sin(\pi z)|$ for large values of r. Then either $\log|f^2(z)|\leqq\pi r|\sin\varphi|$ or $\log|f^2(-z)|\leqq\pi r|\sin\varphi|$. Let us suppose that $\log|f^2(z)|\leqq\pi r|\sin\varphi|$. It follows from the definition of f that $|f(\overline{w})|=|f(w)|$ and $|f(x+iy)|\geqq|f(x+is)|$ if $-|y|\leqq s\leqq|y|$. Therefore $\log|f^2(w)|\leqq\pi r|\sin\varphi|$ on the segment $\{w=x+is\colon -|y|\leqq s\leqq|y|\}$, and we see that

(21) $$|F(z)| \geqq |F(x)|-|y|\exp\{\pi r|\sin\varphi|\}.$$

We denote $G(w)=|F(w)|/(1+|F'(w)|)$. It follows from (21) and (20) that $\log G(z)\geqq 1.9|z|/100+O(\log|z|)$ if $\log|F'(z)| = \log|f^2(z)| \leqq \pi r|\sin\varphi|$. Similarly, if $\log|f^2(-z)|\leqq\pi r|\sin\varphi|$, then we get $\log G(-z)\geqq 1.9|z|/100+O(\log|z|)$. Therefore

$$\log^+ G(re^{i\varphi})+\log^+ G(-re^{i\varphi}) \geqq 1.9r/100+O(\log r)$$

if $|\varphi|\leqq 1/300$. This implies that

(22) $$B(r, F) = (2\pi)^{-1}\int_0^{2\pi} \log^+ G(re^{i\varphi})\,d\varphi \geqq 1.9r/(3\pi\, 10^4)+O(\log r).$$

It follows from the identity $|F|=(1+|F'|)(|F|/(1+|F'|))$ that

$$\log^+|F|-\log^+|1/F| \geqq \log^+ G-\log^+((1+|F'|)/|F|)+\log^+|F'|.$$

Since $\log^+ ((1+|F'|)/|F|) \leqq \log^+ |1/F| + \log^+ |F'/F| + \log 2$, we get

$$\log^+|F| \geqq \log^+ G - \log^+|F'/F| + \log^+|F'| - \log 2.$$

This implies that

$$m(r, F) \geqq B(r, F) - m(r, F'/F) + m(r, F') - \log 2.$$

Here $m(r, F'/F) = o(1)T(r, F)$ because F is of order 1. Therefore we get

$$T(r, F) \geqq (1+o(1))(B(r, F)+T(r, F')) \tag{23}$$

where $o(1)\to 0$ as $r\to\infty$. It follows from (12) that $T(r, F')\leqq 26r$, and we see from (22) that

$$B(r, F) \geqq 7T(r, F')/10^7 + O(\log r).$$

Therefore we get from (23)

$$\liminf_{r\to\infty} T(r, F)/T(r, F') \geqq 1+7/10^7,$$

and the theorem is proved.

References

[1] HAYMAN, W. K.: On the characteristic of functions meromorphic in the plane and of their integrals. - Proc. London Math. Soc. (3) 14a, 1965, 93—128.

[2] HAYMAN, W. K.: Research problems in function theory. - Athlone Press, London, 1967.

[3] NEVANLINNA, R.: Le théorème de Picard—Borel et la théorie des fonctions méromorphes. - Gauthier-Villars, Paris, 1929.

[4] NEVANLINNA, R.: Eindeutige analytische Funktionen. - Springer-Verlag, Berlin—Göttingen—Heidelberg, 1953.

University of Helsinki
Department of Mathematics
SF-00100 Helsinki 10
Finland

Received 15 March 1976