

RATIONAL APPROXIMATION IN THE COMPLEX PLANE AND ON THE LINE

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1. Introduction

Let K be a closed subset of the open set Ω in the extended complex plane and let $A = A(\Omega)$ be the set of functions holomorphic and bounded by 1 in Ω . This paper is concerned with the approximation on K of functions in A by rational functions. In particular we are interested in finding conditions on K , Ω and A , which make it possible to obtain precise estimates for the degree of uniform approximation in its dependence of the number of poles of the rationals in the extended plane.

A suitable starting-point for our discussion is a beautiful result of H. Widom [22]. Let R_n denote the set of rationals with n poles and put

$$(1.1) \quad r_n(f) = \inf_{g \in R_n} \sup_{z \in K} |f(z) - g(z)|.$$

If $r_n = \sup_{f \in A} r_n(f)$ and if $C(K, \Omega)$ is the Green capacity of K relative to Ω , then under very mild restrictions on the sets involved it holds that (see [22, p. 344])

$$\lim r_n^{1/n} = \exp(-1 / C(K, \Omega)).$$

Widom gives several related results among which a lower bound for the n -dimensional diameter d_n of A will be of interest for us. It is defined by

$$d_n = \inf_{E_n} \sup_{f \in A} \inf_{g \in E_n} \max_{z \in K} |f(z) - g(z)|,$$

where E_n runs through all n -dimensional subspaces of $\mathcal{C}(K)$. If Ω is a finite disjoint union of connected open sets with finitely many boundary components, Widom [22, Theorem 7] proves that

$$(1.2) \quad d_n \geq a \exp(-n / C(K, \Omega)).$$

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In many problems it is of considerable interest to have a more precise estimate for r_n than the limit of $r_n^{1/n}$. To obtain such estimates we certainly have to put strong restrictions on K and Ω . The main purpose of this paper is to present a method, which makes it possible to get precise estimates of r_n itself – if the functions and configurations are simple enough. Rational functions of a special type will be of importance and are constructed in section 3. They are generalizations of the classical Faber polynomials so they might be called Faber rationals. The construction has certain points in common with the deduction of a formula I used in [7] without giving a full proof. The formula is closely connected with the Faber expansion (cf. Andersson [3, p. 10]) and a proof is given in section 2.

In section 4 we use the Faber rationals to obtain estimates

$$r_n \leq C_0 \exp(-n / C(K, \Omega))$$

in the class A if the geometrical configuration is simple enough. The poles of the rationals can be chosen independently of f , so (1.2) shows that linear rational systems $\{(z - s_k)^{-1}\}_1^n$ are – within a bounded factor – as good as any other n -dimensional systems for approximation in A .

To test the precision and the applicability of the method we turn to two famous approximation problems. It is our hope that the flexibility of the method might elucidate the problem of rational approximation vs polynomial approximation. Rational approximation is better than polynomial when there is a need to distribute the poles depending on the singularities of the approximated function, and that is transparent in our method. One well-known case is Newman's [15] approximation of $|x|$ on $[-1, 1]$, where there is just one singularity, and good rational approximation comes by concentration of the poles close to that singularity. Another case occurs when there is an abundance of analyticity e.g. in the case of approximation of different constants on a finite number of closed sets. If we wish to apply the previous results with $C(K, \Omega)$ we must find the best choice of Ω and it is not evident, how that is done. This problem is also discussed in section 4 and solved in a restricted case.

However, the bound obtained, does not give the precise result if it is applied to the best-known special case, i.e. the approximation of $\operatorname{sgn} x$ on two intervals $[-1, b]$, $[b, 1]$, $0 < b < 1$, considered by Zolotarev close to hundred years ago (cf. Gončar [9]). I next show how the correct answer can be obtained by our method if $K = K_1 \cup K_2$, where K_1 and K_2 are disjoint, connected closed sets and the function to approximate is constant on each of them. In a forth-coming note I shall show how to settle the case of more than two components. – In passing I give some comments on the corresponding polynomial approximation problem, for which I know the solution from a communication by W. H. J. Fuchs.

In the last section we return to the problem of r_n instead of $r_n^{1/n}$, now in the case of piecewise holomorphic functions on intervals. The typical case is $|x|$ on $[-1, 1]$, where the important step recently was taken by Vjačeslavov [18] who proved that $r_n(|x|) \exp(\pi \sqrt{n})$ is bounded. That is bounded from below by a positive constant has been known a few years (see Bulanov [5]). By a closer study of certain Green potentials we obtain some results including an independent proof of Vjačeslavov's result.

2. A formula connected with the Faber expansion

Let K be a connected compact set in the complex plane, having connected complement K' and with a rectifiable boundary curve ∂K . We do not assume, however, that the boundary is a Jordan curve, since we want to include e.g. sets without interior like the interval. For $\zeta \in \partial K$ we define the function v_ζ by

$$v_\zeta(s) = \arg(\psi(e^{is}) - \zeta),$$

where ψ is the exterior mapping function. We assume that v_ζ is of bounded variation and that the variations have a bound independent of ζ .

Let F be continuous on K and holomorphic in $\text{int } K$. We use the same letter F to denote the harmonic extension of F to K' . By subtraction of a constant we may achieve that $F(\infty) = 0$. With $z = x + iy$ there are two holomorphic functions H and G such that $F(x, y) = H(z) + G(\bar{z})$ for $z \in K'$ and $H(\infty) = G(\infty) = 0$.

Take a $\zeta \in K'$ and let C_ζ be a rectifiable Jordan curve surrounding K and separating ζ and K . With the notation just introduced we find

$$\begin{aligned} \int_{C_\zeta} F(x, y) (z - \zeta)^{-1} dz &= \int_{C_\zeta} G(\bar{z}) (z - \zeta)^{-1} dz + \int_{C_\zeta} H(z) (z - \zeta)^{-1} dz \\ &= -2\pi i H(\zeta) + 2\pi i G(\bar{\zeta}) + \int_{C_\zeta} G(\bar{z}) (z - \zeta)^{-1} dz - \int_{C_\zeta} G(\bar{z}) (\bar{z} - \bar{\zeta})^{-1} d\bar{z}, \end{aligned}$$

and

$$(2\pi i)^{-1} \int_{C_\zeta} F(x, y) (z - \zeta)^{-1} dz = G(\bar{\zeta}) - H(\zeta) + \frac{1}{\pi} \int_{C_\zeta} G(\bar{z}) d \arg(z - \zeta).$$

Our assumptions on F imply that if the contour of integration on the left shrinks to ∂K and if G is continuous on the closure of K' (as it will be in our application), we obtain

$$\frac{1}{\pi} \int_{\partial K} G(\bar{z}) d \arg (z - \zeta) = H(\zeta) - G(\bar{\zeta}).$$

We next move ζ across ∂K so that $\arg (z - \zeta)$ henceforward is interpreted with ζ "inside" ∂K . That means that we have to add $2 G(\bar{\zeta})$ to the right side to maintain the equality. Hence

$$(2.1) \quad F(\zeta) = H(\zeta) + G(\bar{\zeta}) = \frac{1}{\pi} \int_{\partial K} G(\bar{z}) d \arg (z - \zeta), \quad \zeta \in \partial K.$$

We observe that the same formula is true if $\zeta \in \text{int } K$, since by the same method

$$\frac{1}{\pi} \int_{C_\varrho} G(\bar{z}) d \arg (z - \zeta) = (2 \pi i)^{-1} \int_{C_\varrho} F(x, y) (z - \zeta)^{-1} dz \rightarrow F(\zeta).$$

Formula (2.1) rewritten in the form

$$F(\zeta) = -\frac{1}{\pi} \int_{\varrho=1}^{\infty} d\varrho \int_{s=0}^{2\pi} (G \circ \bar{\psi})' (\varrho e^{-is}) e^{-is} dv_\zeta(s)$$

was applied in [7] to obtain a short proof of the following result (cf. Dzjadyk [6] and Kövari [13]).

Theorem 1. *Let K and F be as described above. If $F \circ \psi$ satisfies a Hölder condition of order α , $0 < \alpha \leq 1$, on the unit circle, then $E_n(f) \leq C n^{-\alpha}$, where $E_n(f)$ is the best approximation by polynomials of degree n .*

Several complements and extensions are given in Andersson [3], where also the connection with the Faber expansion is shown. The important thing is that the Faber polynomial P_k of order k is given (except for normalization) by (cf. Pommerenke [17, Lemma 1] and [14])

$$(2.2) \quad P_k(\zeta) = \int_0^{2\pi} e^{iks} dv_\zeta(s) = \int_{\partial K} \Phi(z)^k d \arg (z - \zeta),$$

where Φ is the inverse function of ψ .

When we now proceed to rational approximation we shall obtain our Faber rationals by a generalization of (2.2). In [7] the proof employed Schwarz's lemma and we shall see that similarly the Blaschke factors for the poles play an important role in rational approximation.

3. Faber rationals

3.1. Definition and properties. As mentioned in the introduction we shall strongly restrict the configurations we are dealing with. We shall assume that $K = \cup K_k$ is the union of p disjoint sets, each of the type described in the previous section. The important thing is that the variation of $\arg(z - \zeta)$ is uniformly bounded when z describes ∂K . We may e.g. assume that every ∂K_k has bounded rotation in the sense of Paatero [16]. Sets with these properties will be said to be of type N .

We next prescribe the set S_n of poles, writing $S_n = \{s_k\}_1^n$ where the points $s_k \in K'$ are not necessarily distinct. The Faber rational $F(z; K, S_n)$ will be a rational function of degree at most $n + p - 1$. The Green function for K' singular at s is denoted $g(\cdot, s)$ and a (multiplevalued) holomorphic function with real part $g(\cdot, s)$ is called $G(\cdot, s)$ so that

$$G(\cdot, s) = g(\cdot, s) + i h(\cdot, s).$$

Now it is well-known (Grunsky [10, 11], cf. also Ahlfors [1] and Widom [20]) that we can (in several ways) pick $(p-1)$ points $\{s_k\}_{n+1}^{n+p-1}$ in K' so that

$$\Phi_n(z) = \exp\left(-\sum_1^{n+p-1} G(z, s_k)\right)$$

is a single-valued holomorphic function in K' , bounded by 1, with zeros in $S_n^* = \{s_k\}_1^{n+p-1}$ and having absolute value 1 on $\partial K' = \partial K$.

Around every point $s_k \in S_n^*$ we take a negatively oriented circle C_k and choose the radii so small that the corresponding discs U_k are disjoint (except for coinciding $s_k = s_i$) and all in K' . With $\zeta \in (\cup U_k)'$ we define the Faber rational by

$$\begin{aligned} (3.1.1) \quad F_n(\zeta) &= F(\zeta; K, S_n) \\ &= (2\pi i)^{-1} \sum_1^{n+p-1} \int_{C_k} \Phi_n(z)^{-1} (z - \zeta)^{-1} dz + \Phi_n(\infty)^{-1}. \end{aligned}$$

We collect the important properties of F_n in

Theorem 2. *The function F_n defined by (3.1.1) is a rational function with poles in S_n^* . If K is of type N , then uniformly*

$$(3.1.2) \quad \begin{aligned} F_n(\zeta) &= O(1), \quad \zeta \in K, \\ F_n(\zeta) &= \Phi_n(\zeta)^{-1} + O(1), \quad \zeta \in K'. \end{aligned}$$

Proof. It is evident that F_n is holomorphic in $(\cup U_k)'$ and hence in $S_n^{*'}$. On U_k we can write $\Phi_n(z)^{-1} = (z - s_k)^{-m} \psi(z)$ with $\psi \in H(U_k)$ and $\psi(s_k) \neq 0$. Hence

$$\begin{aligned}
& (2\pi i)^{-1} (\zeta - s_k)^m \int_{C_k} \Phi_n(z)^{-1} (z - \zeta)^{-1} dz \\
&= (2\pi i)^{-1} (\zeta - s_k)^m \int_{C_k} \psi(z) (z - s_k)^{-m} (z - \zeta)^{-1} dz \\
&= -\frac{(\zeta - s_k)^m}{(m-1)!} (D^{m-1} \psi(z) (z - \zeta)^{-1})_{z=s_k} \rightarrow (-1)^m \psi(s_k)
\end{aligned}$$

as $\zeta \rightarrow s_k$. We conclude that F_n has a pole of the appropriate order at s_k .

We next move the contours from the neighbourhoods of the poles and let them tend to ∂K . We get one contribution from ζ if $\zeta \in K'$ and one from infinity, the latter cancelling the last term in (3.1.1). Hence

$$F_n(\zeta) = \Phi_n(\zeta)^{-1} + (2\pi i)^{-1} \sum_1^p \int_{\partial K_k} \Phi_n(z)^{-1} (z - \zeta)^{-1} dz$$

if $\zeta \in K'$, and

$$F_n(\zeta) = (2\pi i)^{-1} \sum_1^p \int_{\partial K_k} \Phi_n(z)^{-1} (z - \zeta)^{-1} dz$$

if $\zeta \in K$. The integrations are in the positive direction.

It is now time to use the fact that Φ_n has absolute value 1 on ∂K , so that

$$\Phi_n(z)^{-1} = \overline{\Phi_n(z)} \quad \text{if } z \in \partial K.$$

If $\zeta \in K'$ we get

$$(2\pi i)^{-1} \int_{\partial K} \overline{\Phi_n(z)} (\bar{z} - \bar{\zeta})^{-1} d\bar{z} = \overline{\Phi_n(\zeta)} + \overline{\Phi_n(\infty)},$$

and thus

$$(3.1.3) \quad F_n(\zeta) = \Phi_n(\zeta)^{-1} + \overline{\Phi_n(\zeta)} + \overline{\Phi_n(\infty)} + \frac{1}{\pi} \int_{\partial K} \Phi_n(z)^{-1} d \arg (z - \zeta).$$

If $\zeta \in K$ we find

$$(3.1.4) \quad F_n(\zeta) = \overline{\Phi_n(\infty)} + \frac{1}{\pi} \int_{\partial K} \Phi_n(z)^{-1} d \arg (z - \zeta).$$

From (3.1.3) and (3.1.4) we can immediately read off the remaining statement (3.1.2) in the theorem, which hence is proved.

3.2. Some comments on generalized Faber polynomials. It is more intricate to construct useful Faber polynomials by the method just employed. One has to choose points $\{t_k\}_1^{p-1}$ in K' so that $\exp(-nG(\cdot, \infty) + \sum G(\cdot, t_k))$ is single-valued. By the methods in the literature quoted, that can be done e.g. in such a way that $\sum g(a, t_k)$ is minimal at a point $a \in K'$ and the minimal value has bounds independent of the periods, i.e. independent of n . The polynomials are defined by

$$P_n(\zeta) = (2\pi i)^{-1} \int_C \Phi_n(z)^{-1} (z - \zeta)^{-1} dz,$$

where C is a large circle in the positive direction and

$$(3.2.1) \quad \Phi_n(z) = \exp(-nG(z, \infty) + \sum G(z, t_k)).$$

As before we get

$$(3.2.2) \quad P_n(\zeta) = \Phi_n(\zeta)^{-1} + (2\pi i)^{-1} \int_{\partial K} \Phi_n(z)^{-1} (z - \zeta)^{-1} dz,$$

if $\zeta \in K'$.

This formula is good enough for most purposes. The problems come when we turn to $(2\pi i)^{-1} \int_{\partial K} \overline{\Phi_n(z)} (\bar{z} - \bar{\zeta})^{-1} d\bar{z}$, with $\zeta \in K$, since we get contributions from the poles t_k . With $\zeta \in K$ we find that

$$P_n(\zeta) = \frac{1}{\pi} \int_{\partial K} \Phi_n(z)^{-1} d \arg(z - \zeta) - \sum_1^{p-1} a_k (\bar{\zeta} - \bar{t}_k)^{-1},$$

where a_k is the residue of $\overline{\Phi_n(z)}$ at $\bar{z} = \bar{t}_k$. To get the boundedness on K we have to know that the t_k 's stay away from ∂K . That is not generally true with the choice above but the doubly connected case is easy to handle. Grunsky [11] gives further information about the possible choices of $\{t_k\}$.

4. The approximation method

4.1. Let us consider the problem to approximate $f \in A(\Omega)$ by rationals on a set K of type N in Ω . It is now natural to consider compact sets in the extended plane so we assume without restriction that ∞ belongs to K . Hence $\partial\Omega$ has finite maximal distance from the origin. We take a Faber rational F_n and consider

$$(4.1.1) \quad Q_n(\zeta) = (2\pi i)^{-1} F_n(\zeta) \int_{\Gamma} F_n(z)^{-1} (z - \zeta)^{-1} f(z) dz$$

for $\zeta \in K$ and a curve (a system of curves) $\Gamma \subset \Omega$ surrounding K in the positive direction. If the zeros of F_n inside Γ are called $\{a_k\}$ we have

$$Q_n(\zeta) = f(\zeta) - F_n(\zeta) \sum F'_n(a_k) (\zeta - a_k)^{-1} = f(\zeta) - G_n(\zeta),$$

where G_n is a rational function. Hence

$$(4.1.2) \quad r_{n+p-1}(f) \leq \max_{\zeta \in K} |Q_n(\zeta)|,$$

and if we take $\Gamma = \partial\Omega$ we find

$$r_{n+p-1} \leq (2\pi)^{-1} \max_{z \in \partial\Omega, \zeta \in K} |F_n(\zeta) (z - \zeta)^{-1}| \int_{\partial\Omega} |F'_n(z)|^{-1} |dz|.$$

The factor in front of the integral sign only depends on the geometry. We now apply 3.2. We presume that the poles are chosen so that $\min |\Phi_n(z)|^{-1}$ dominates the bounded term on $\partial\Omega$, and we get

$$(4.1.2) \quad r_{n+p-1} \leq C \int_{\partial\Omega} |\Phi_n(z)| dz \leq C \int \exp\left(-\sum_1^n g(z, s_k)\right) |dz|.$$

We now have to choose the poles. We first note that the estimate

$$\overline{\lim} r_n^{1/n} \leq \exp(-1/C(\Omega', K'))$$

is almost immediate in our simple case. It is well-known that $C(\Omega', K') = C(K, \Omega)$ (cf. Widom [22, p. 350]).

Formula (4.1.2) enables us to get more precise results. Assuming a sufficient differentiability of $\partial\Omega$ we can choose the set S so that

$$(4.1.3) \quad \sum g(z, s_k) \geq n/C(\Omega', K') - C_1,$$

where C_1 depends on the geometrical configuration. When this can be done we obtain a linear set $\{(z - s_k)^{-1}\}$ with a degree of approximation smaller than a constant times the n -dimensional diameter (formula (1.2)).

I shall not pursue this matter in any generality at this place but turn to some interesting special cases. In section 5 we shall also give one example of how to take the step from continuous to discrete distribution.

4.2. The case of excessive holomorphy. A case of practical importance is the approximation by rationals of functions taking different constant values on the sets K_k . If we want to apply Widom's results or the method in 4.1 we have to choose Ω so that $C(K, \Omega)$ becomes as small as possible. In the general case that leads us to non-trivial problems connected with extremal length (where the methods in Jenkins [12] seem to be useful).

Let us study the special case where f equals 1 on $K^* = \bigcup_1^s K_k$ and -1 on $K_* = \bigcup_{s+1}^p K_k$, assuming $K = K_* \cup K^*$ to be of type N .

It can in fact be seen by applying the method in Gončar [9] in combination with estimates given by Widom [21, 22] that the best rational approximation r_n satisfies

$$(4.2.1) \quad \lim r_n^{1/n} = \exp(-\pi d)$$

and also $r_n \geq a \exp(-\pi d n)$, where d is the extremal distance of K_* and K^* , the inverse of the capacity of the condenser formed by K_* and K^* .

I have found it interesting to compute how small $C(K, \Omega)$ can be made in this case by a good choice of Ω . It turns out that we cannot do better than

$$C(K, \Omega)^{-1} = \frac{1}{2} \pi d.$$

The formulas for extremal length we use can be found in Ahlfors [2, Ch. 4].

We are looking for an $\Omega_* \supset K_*$ and an $\Omega^* \supset K^*$ and they shall be as large as possible so that $\Omega' = \partial\Omega_* = \partial\Omega^*$. Let u be the Green potential of the equilibrium distribution on $\partial\Omega_* = \partial\Omega^*$ and let v be the harmonic function taking the values 0 on K_* and 1 on K^* .

If $\partial\Omega_* = \partial\Omega^*$ is taken as the level line $v = 1/2$, we have

$$u = 2C^{-1}v \quad \text{in } \Omega_*, \quad u = 2C^{-1}(1-v) \quad \text{in } \Omega^*.$$

With the appropriate orientation of the integration the Green formula gives

$$(4.2.2) \quad \begin{aligned} 2\pi &= C \int_{\partial\Omega_*} u \frac{\partial u}{\partial n} ds + C \int_{\partial\Omega^*} u \frac{\partial u}{\partial n} ds = C \int_{\Omega \setminus K} (\nabla u)^2 dx dy \\ &= 4C^{-1} \int_{\Omega} \int (\nabla v)^2 dx dy, \end{aligned}$$

where we have used the fact that $u = 0$ on ∂K . The Dirichlet integral on the right is known to be d^{-1} , so we have proved that with this choice of Ω the capacity $C(K, \Omega)^{-1} = (\pi d)/2$. If we take another boundary between Ω_* and Ω^* we can proceed as in formula (4.2.2) to obtain

$$2\pi = C \int_{\Omega_* \setminus K_*} (\nabla u)^2 dx dy + C \int_{\Omega^* \setminus K^*} (\nabla u)^2 dx dy.$$

If we introduce the extremal distance d_* from ∂K_* to $\partial\Omega_*$ and d^* from ∂K^* to $\partial\Omega^*$ we get

$$2\pi = C^{-1}(d_*^{-1} + d^{*-1}).$$

The harmonic mean is smaller than the arithmetic mean so it follows that

$$C^{-1} = \pi \cdot 2(d_*^{-1} + d^{*-1})^{-1} \leq \frac{\pi}{2} (d_* + d^{**}) \leq \frac{1}{2} \pi d$$

by the composition law. Since the level line of v is smooth enough, the best approximation by rationals of our function f satisfies

$$r_{n+p-1}(f) \leq C_0 \exp\left(-\frac{\pi}{2} dn\right).$$

In 4.3 it will be shown how the special type of the function may be used to obtain the better estimate

$$r_n \leq C_0 \exp(-\pi dn)$$

in the case of two disjoint connected sets. The method does not generalize naturally to configurations with more sets. One way to get sharper estimates than (4.2.1) should be to introduce Faber rationals in the method applied in [9].

4.3. Rational approximation of functions constant on two disjoint sets. Gončar [9] proved that the best rational approximation r_n of a function equal to $+1$ and -1 on two continua bounding a doubly connected domain satisfies $\lim r_n^{1/n} = (R_1/R_2)^{1/2}$, if the domain is conformally equivalent to an annulus $\{w : R_1 < |w| < R_2\}$. The special case when the continua are intervals $[-1, -b]$, $[b, 1]$ with $0 < b < 1$ was already solved by Zolotarev, who found that $(R_2/R_1)^{n/2} r_n$ is bounded above and below independently of n . I shall now state a theorem that gives a result of Zolotarev's type for a class of sets which of course is smaller than the one considered by Gončar.

Theorem 3. *Let K_1 and K_2 be disjoint sets of type N . If ψ is a holomorphic mapping of $B = \{w : q^{1/2} < |w| < q^{-1/2}\}$ on the doubly connected domain bounded by K_1 and K_2 , we further assume that to every $\omega \in \partial B$ there is a constant $p(\omega)$ so that*

$$\psi'(w) / (\psi(w) - \psi(\omega)) - p(\omega) / (w - \omega)$$

is integrable along the boundary circles (or more generally that the condition following (4.3.8) is satisfied). Under these conditions

$$(4.3.1) \quad r_{n+1} \leq C q^{n/2}.$$

We note that if d is the extremal distance of K_1 and K_2 then $q = \exp(-2\pi d)$.

For the proof we define $m = [(1/2)n]$. As set S^* we take $\{\psi(\varepsilon_k)\}_0^{2m-1}$ where $\varepsilon_k = \exp(-i\pi k/m)$. I claim that

$$\Phi_n(z) = \exp\left(-\sum_0^{2m-1} G(z, \psi(\varepsilon_k))\right)$$

is single-valued and regular in $K' = (K_1 \cup K_2)'$. That can easily be checked since $\chi_n = \Phi_n \circ \psi$ is explicitly known, the Green function for the annulus being characterized by

$$\begin{aligned} & \exp(-G(\psi(w), \psi(\varepsilon_k))) \\ &= q^{1/4} v^{-1/2} (v-1) \prod_{s=1}^{\infty} (1 - q^{2s} v) (1 - q^{2s-1} v)^{-1} (v - q^{2s}) (v - q^{2s-1})^{-1}, \end{aligned}$$

where $v = w/\varepsilon_k$. Hence we get a single-valued function χ_n by

$$\begin{aligned} (4.3.2) \quad \chi_n(w) &= \Phi_n \circ \psi(w) \\ &= q^{m/2} w^{-m} (w^{2m} - 1) \prod_{s=1}^{\infty} \frac{(1 - q^{4sm} w^{2m}) (w^{2m} - q^{4sm})}{(1 - q^{2(2s-1)m} w^{2m}) (w^{2m} - q^{2(2s-1)m})}. \end{aligned}$$

From the theory of elliptic functions we need the definition of sn by

$$\begin{aligned} (4.3.3) \quad sn(2Kx/\pi) \\ &= 2q^{1/4} k^{-1/2} \sin x \prod_{s=1}^{\infty} (1 - 2q^{2s} \cos 2x + q^{4s}) (1 - 2q^{2s-1} \cos 2x + q^{4s})^{-1}, \end{aligned}$$

where k and K are well-known parameters determined by q (see e.g. Whittaker–Watson [19, p. 479]), and also the Fourier series ([19, p. 510])

$$\begin{aligned} (4.3.4) \quad sn(2Kx/\pi) \\ &= 2\pi(Kk)^{-1} \left\{ \frac{q^{1/2} \sin x}{1-q} + \frac{q^{3/2} \sin 3x}{1-q^3} + \frac{q^{5/2} \sin 5x}{1-q^5} + \dots \right\}. \end{aligned}$$

From now on sn denotes the elliptic function defined by $q^{2m} = q^{2[n/2]}$. To stress that we write k_n and K_n for the corresponding quantities. With this convention (4.3.2) and (4.3.3) give

$$(4.3.5) \quad \chi_n(e^{ix}) = k_n^{1/2} sn(2K_n m x / \pi).$$

We are now ready to apply the method of 4.1. Let F_n be the Faber rational corresponding to the Φ_n we just constructed. In formula (4.1.1) we take Γ to be the image of $\{w : |w| = 1\}$ described in one direction for K_1 and in the other for K_2 . Since the function takes values with opposite signs we get

$$|Q_n(\zeta)| \leq \pi^{-1} \max_K F_n \cdot \left| \int_{\Gamma} F_n(z)^{-1} (z - \zeta)^{-1} dz \right|.$$

We now change variable by $z = \psi(w)$ and find

$$(4.3.6) \quad |Q_n(\zeta)| \leq C \left| \int_{|w|=1} F_n \circ \psi(w)^{-1} (\psi(w) - \zeta)^{-1} \psi'(w) dw \right|.$$

By (3.1.2) we have

$$F_n \circ \psi(w) = \chi_n(w)^{-1} + O(1) \quad \text{on } |w| = 1$$

and (4.3.5) shows that the first term dominates, for n larger than some n_0 since sn is bounded by 1. Hence

$$F_n \circ \psi(w)^{-1} = \chi_n(w) + O(1) |\chi_n(w)|^2.$$

Introduction in (4.3.6) gives

$$(4.3.7) \quad |Q_n(\zeta)| \leq C \left| \int_{|w|=1} \chi_n(w) \frac{w \psi'(w)}{\psi(w) - \zeta} \frac{dw}{w} \right| + C \max |\chi_n(w)|^2.$$

It follows from our assumptions on the mapping function that the coefficients (a_s) in the Laurent series

$$(4.3.8) \quad \frac{w \psi'(w)}{\psi(w) - \zeta} = \sum_{-\infty}^{\infty} a_s w^s$$

satisfy $|a_s| \leq C q^{|s|/2}$. We now recall the Fourier series (4.3.4) for χ_n as given by (4.3.5) and insertion in the integral of (4.3.7) gives

$$|Q_n(\zeta)| \leq C K_n^{-1} k_n^{-1/2} q^{3m/2} (1 - q^{2m})^{-2} + C k_n \leq C q^m (1 - q^{2m})^{-2} \leq C q^{n/2}.$$

4.4. A comment on the corresponding polynomial case. The best polynomial approximation of a function equal to $+1$ and -1 on disjoint compact sets K_1 and K_2 was exactly determined by S. Bernstein in the case $K_1 = [-1, -b]$, $K_2 = [b, 1]$, $0 < b < 1$. (An equivalent result is given on p. 120 in [4].)

Let us take a $K = K_1 \cup K_2$ of type N and a function f coinciding on K_1 with one entire function and on K_2 with another. The Green function $g(\cdot, \infty)$ for K' singular at ∞ has one critical point z_0 , where g takes a value $w_0 = g(z_0, \infty)$ such that w_0 is the smallest maximum of $g(\cdot, \infty)$ on any curve connecting K_1 and K_2 . We consider the eight-shaped level-line $g(z, \infty) = w_0$, where the two branches form an angle of $\pi/2$ at z_0 . Like in section 4.1 we consider

$$(4.4.1) \quad Q_n(\zeta) = (2\pi i)^{-1} P_n(\zeta) \int_{\Gamma} (z - \zeta)^{-1} P_n(z)^{-1} f(z) dz,$$

where now P_n is a Faber polynomial discussed in 3.2 and $\Gamma = \Gamma_1 + \Gamma_2$ consists of two curves in $\{z : g(z, \infty) \geq w_0\}$, both described in the positive direction, Γ_1 around K_1 and Γ_2 around K_2 , and having a line segment in common that bisects the angle of the level lines at z_0 .

In a similar way as in section 4.1 we find that $\max_{\zeta \in K} |Q_n(\zeta)|$ is an upper bound for the best polynomial approximation $E_n(f)$. According to (3.2.1) the dominant contribution to the integral in (4.4.1) comes from the immediate neighbourhood of z_0 . Introducing $w = G(z, \infty)$ as new variable in the integral in (4.4.1) we recall that it follows from the orthogonality mentioned above that locally $w - w_0 = C(z - z_0)^2$ so that

$$|E_n(f)| \leq C_0 \int_{w_0}^{\infty} (w - w_0)^{-1/2} \exp(-n w) dw \leq C_0 n^{-1/2} \exp(-n w_0).$$

W. H. J. Fuchs has proved that a corresponding formula gives the correct estimate for any number of sets (w_0 being the minimal value of $g(\cdot, \infty)$ at the critical points).

If we wish to compare polynomial and rational approximation for two sets, it is relatively easy to express w_0 by elliptic functions in the symmetric case. If the doubly connected domain is conformally equivalent to an annulus with $R_1/R_2 = q$, then $n^{1/2} k^{-n/2} E_n(f)$ is bounded, to be compared with $q^{-n/2} r_n(f)$ for rational approximation. Here k is the modulus connected with q .

There is a considerable difference, since for small k we have $k \sim 4 q^{1/2}$ and in any case $q^{1/2} \leq k \leq 4 q^{1/2}$. We do still worse by polynomials in the unsymmetric case while the rational approximation only depends on q .

5. A remark on the best rational approximation of $|x|$

5.1. The discussion in the previous section indicates that the good information about the poles we get by our method must be paid for by rather intricate computations. The alternative we mentioned is based on the fact that for very general sets K_1 and K_2 we have good estimates (Gončar [8], Widom [21, 22]) for

$$(5.1.1) \quad \varrho_n = \sup_{h \in R_n} \max_{K_2} |h| / \min_{K_1} |h|,$$

in particular $\varrho_n^{1/n} \rightarrow \exp(-2\pi d)$. (Cf. section 4.2.)

As far as I know the only way known to prove that we have positive upper and lower bounds for $r_n(|x|) \exp(\pi \sqrt{n})$ is by similar methods (Vjačeslavov [18]). Since we are dealing with the intervals $[-1, 0]$ and

$[0, 1]$, which are not disjoint, we have to consider a more complicated quantity. My only aim is to show the connection with the problems discussed in 4.1. After the publication of [18] there is nothing really new in this remark, and what I should like to know is a way to carry through the computations by the methods of 4.3.

5.2. A Green potential approximating $\log x$. Our reformulation of the crucial inequality reads as follows.

Theorem 4. *Let $g(\cdot, a)$ be the Green function for the right half-plane singular at a . There is a positive measure μ of total mass ω on $[0, 1]$, such that*

$$(5.2.1) \quad \int g(x, y) d\mu(y) \geq \log x + \pi \sqrt{\omega} - C,$$

for x on $[0, 1]$ and with a numerical C . If $\omega = n$ we can take μ as a discrete measure with integer masses.

It is rather easy to see that

$$dv(y) = (\log y + \pi \sqrt{\omega}) y^{-1} dy \quad \text{on } [\exp(-\pi \sqrt{\omega}), 1]$$

is a good candidate. If we put $\varphi(u) = \exp(\pi \sqrt{u})$ and take $y_k = \varphi(k)/\varphi(n)$, we evidently have $v(y_k, y_{k+1}) = 1$ and an approximating discrete potential is thus given by

$$\sum_{k=0}^n g(x, y_k) = \sum_{k=0}^n \log \left| \frac{x + y_k}{x - y_k} \right| = -v(x).$$

The inequality (5.2.1) is trivial if $0 < x < \exp(-\pi \sqrt{n})$, so we suppose that $y_k < x \leq y_{k+1}$ for some k . Fixing x in this way we see that $g(x, \varphi(\cdot))$ has two convex parts; on each of them the trapezoid method gives an upper approximation. Hence with $y = \varphi(u) = \exp(\pi \sqrt{u})$ and $\xi = x \exp(\pi \sqrt{n}) \geq 1$,

$$(5.2.2) \quad v(x) - C \leq \int_0^n \log \left| \frac{\xi - \varphi(u)}{\xi + \varphi(u)} \right| du = \frac{2}{\pi^2} \int_1^{\varphi(n)} \log \left| \frac{\xi - y}{\xi + y} \right| \log y \frac{dy}{y},$$

the only tricky part being to show that

$$(5.2.3) \quad \frac{2}{\pi^2} \int_{\varphi(k)}^{\varphi(k+1)} \log \left| \frac{\xi - y}{\xi + y} \right| \log y \frac{dy}{y} + \frac{1}{2} g(x, \varphi(k)) + \frac{1}{2} g(x, \varphi(k+1)) \geq -C.$$

We observe that substitution of $\xi^{-1} \log \varphi(k)$ for $y^{-1} \log y$ in the integral only gives a bounded difference. In the resulting integral we put $y = \xi t$ and with $1 + \delta = \varphi(k+1)/\xi$, $1 - \varepsilon = \varphi(k)/\xi$, it remains to prove that

$$(\delta + \varepsilon)^{-1} \int_{1-\varepsilon}^{1+\delta} \log \left| \frac{1-t}{1+t} \right| dt - \frac{1}{2} \log \frac{\delta}{2+\delta} - \frac{1}{2} \frac{\varepsilon}{1+\varepsilon} \geq -C.$$

Since the integral can be explicitly calculated that is completely elementary and (5.2.3) follows.

We now turn to the right member of (5.2.2). The substitution $y = \xi u$ gives

$$\begin{aligned} v(x) - C &\leq \frac{2}{\pi^2} \int_{\xi^{-1}}^{x^{-1}} (\log u + \log \xi) \log \left| \frac{1-u}{1+u} \right| \frac{du}{u} \\ &\leq \frac{1}{2} + \frac{2}{\pi^2} \log \xi \int_0^\infty \log \left| \frac{1-u}{1+u} \right| \frac{du}{u} + \frac{2}{\pi^2} \log \xi \left[\int_{x^{-1}}^\infty + \int_0^{\xi^{-1}} \log \left| \frac{1+u}{1-u} \right| \frac{du}{u} \right] \end{aligned}$$

so that by aid of the value $-\pi^2/2$ of the first integral we get

$$v(x) + \log \xi - C \leq \frac{2}{\pi^2} \log \xi [4x + 4\xi^{-1}] \leq C + x\pi\sqrt{n}.$$

Since

$$\sqrt{n} \log \left| \frac{1+x}{1-x} \right| \geq 2x\sqrt{n},$$

we find that

$$\begin{aligned} u(x) &= \sum_{k=0}^n \log \left| \frac{x+y_k}{x-y_k} \right| + 2[\sqrt{n}] \log \left| \frac{x+1}{x-1} \right| \\ &\geq \log \xi - C = \log x + \pi\sqrt{n} - C. \end{aligned}$$

Putting $\omega = n + 2[\sqrt{n}]$ and observing that $\sqrt{n} \geq \sqrt{\omega} - 1$ we have proved the estimate (5.2.1) in the theorem.

That means that we have a rational function g of degree n such that

$$x |g(x)| \leq C \exp(-\pi\sqrt{n})$$

for $0 < x \leq 1$, and $g(-x)g(x) = 1$. As in the original work [15] we immediately see that

$$h(x) = x \frac{1-g(x)}{1+g(x)}$$

approximates $|x|$ better than $C \exp(-\pi\sqrt{n})$.

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