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CAUCHY–RIEMANN VECTOR FIELDS

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Introduction. It is the purpose of this paper to introduce the notions of (i) a Cauchy–Riemann vector field (CR-field) on a 2-dimensional Riemannian manifold, (ii) the index of a line field via mapping degree (cf. the definition in H. Poincaré [4], Ch. XIII) and to use these to obtain a modernized proof of a classical result of H. Hopf [3].

In § 1 we discuss the general properties of CR-fields. In particular, it is shown that a nonzero CR-field has only isolated zeros.

One of the main results of § 2 states that if Z is a CR-field with an isolated singularity at a point a and $\lim_{x\to a} |Z(x)| = \infty$, then the index of Z at a is negative.

The Gauss–Bonnet theorem for a line field on a compact oriented Riemannian 2-manifold is established in § 3.

Finally, in § 4 we apply the above notions and results to give a simple proof of a theorem of H. Hopf on immersions of 2-spheres in $\boldsymbol{R}^3$ with constant mean curvature.

1. Cauchy–Riemann vector fields

1.1. Cauchy–Riemann vector fields. Let M be a smooth orientable 2-manifold.

We shall denote the ring of smooth functions on M by $\mathfrak{S}(M)$ and the $\mathfrak{S}(M)$-module of vector fields on M by $\mathfrak{X}(M)$ (cf. [2] for details).

Recall that an almost complex structure on M is a tensor field J of type $(1, 1)$ such that $J^2 = -I$. In particular, a Riemannian metric g on an oriented 2-manifold determines an almost complex structure given by

$$g(J X , Y) = \Delta_M(X , Y) , \qquad X , Y \in \mathfrak{X}(M) ,$$

where Δ_M denotes the normed 2-form on M which represents the orientation.

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A vector field Z on M will be called a *Cauchy–Riemann field* (*CR-*field) if, for every $X \in \mathfrak{X}(M)$,

$$J([Z, X]) = [Z, J X] . \tag{1.1}$$

Example. Let $M = C$ (the complex plane) and consider a vector field

$$Z = u\, e_1 + v\, e_2 , \quad u, v \in \mathfrak{S}(C) ,$$

where e_1, e_2 is a positive orthonormal basis of C. Then it is easily checked that Z is a CR-field if and only if u and v satisfy the Cauchy–Riemann conditions

$$e_1(u) = e_2(v) , \quad e_1(v) = -e_2(u) .$$

Lemma I. *A vector field Z on M is a CR-field if and only if*

$$[Z, J Z] = 0 . \tag{1.2}$$

Proof. It is obvious that (1.2) follows from (1.1). Conversely, assume that (1.2) holds. Fix a point $a \in M$. We may assume that $a \in \operatorname{carr} Z$. Now we distinguish two cases:

Case I. $Z(a) \neq 0$. Then $Z(x) \neq 0$ in some neighbourhood U of a and so the vector fields Z and $J Z$ determine an orthogonal 2-frame at every point $x \in U$. Thus, if $X \in \mathfrak{X}(M)$, we have in U

$$X = \alpha Z + \beta J Z , \quad J X = -\beta Z + \alpha J Z , \quad \alpha, \beta \in \mathfrak{S}(U) ,$$

and so (1.2) implies that

$$[Z, X] = [Z, \alpha Z] + [Z, \beta J Z] = Z(\alpha) \cdot Z + Z(\beta) \cdot J Z ,$$

whence

$$J([Z, X]) = Z(\alpha) \cdot J Z - Z(\beta) \cdot Z = [Z, J X] , \quad \text{in } U .$$

In particular,

$$J[Z, X]\,(a) = [Z, J X]\,(a) .$$

Case II. $Z(a) = 0$. Then (1.1) follows from Case I via a continuity argument since $a \in \operatorname{carr} Z$.

Corollary. *If Z is a CR-field, then so is $J Z$.*

From now on we will assume that the almost complex structure on M is induced by a Riemannian metric g. Then we have $\nabla_X J = 0$, where ∇_X denotes covariant differentiation in the direction of the vector field X with respect to the corresponding Levi–Civita connection. Note that if g is replaced by $\lambda \cdot g$ (λ a positive function on M), then J is not changed and hence a CR-field in the g-metric is also a CR-field in the $\lambda \cdot g$-metric.

Lemma II. *With the above notation, a vector field* Z *is a CR-field if and only if*

$$\nabla_{JX} Z = J \nabla_X Z , \quad X \in \mathfrak{X}(M) . \tag{1.3}$$

In particular, a parallel vector field is a CR-field.

Proof. Since ∇ is torsion free we have for any two vector fields X and Z

$$\nabla_Z X - \nabla_X Z = [Z , X] .$$

This relation yields

$$\nabla_Z J X - J \nabla_X Z = J[Z , X]$$

and

$$\nabla_Z J X - \nabla_{JX} Z = [Z , J X] .$$

It follows that

$$\nabla_{JX} Z - J \nabla_X Z = J[Z , X] - [Z , J X] .$$

Thus Z is a CR-field if and only if (1.3) holds.

1.2. The 1-forms Φ_Z and Ψ_Z. Let Z be a vector field on M without zeros. Then we can write

$$\nabla_X Z = \Phi_Z(X) \cdot Z + \Psi_Z(X) \cdot J Z , \quad X \in \mathfrak{X}(M) , \tag{1.4}$$

where Φ_Z and Ψ_Z are 1-forms on M. They are given explicitly by

$$\Phi_Z(X) = X(\ln |Z|) \tag{1.5}$$

and

$$\Psi_Z(X) = \frac{1}{|Z|^2} \Delta_M(Z , \nabla_X Z) . \tag{1.6}$$

Proposition I. *The exterior derivatives of* Φ_Z *and* Ψ_Z *are given by*

$$\delta \Phi_Z = 0 \tag{1.7}$$

and

$$\delta \Psi_Z = -K \Delta_M , \tag{1.8}$$

where K *denotes the Gaussian curvature of* M.

Proof. Let R denote the curvature operator of M on $\mathfrak{X}(M)$:

$$R(X , Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X, Y]} , \quad X , Y \in \mathfrak{X}(M) ,$$

(cf. [2], p. 321). Then (1.4) yields after a short calculation,

$$R(X , Y) Z = \delta \Phi_Z(X , Y) \cdot Z + \delta \Psi_Z(X , Y) \cdot J Z . \tag{1.9}$$

On the other hand, the Gaussian curvature is determined by the equation

$$(1.10) \qquad R(X, Y) Z = -K \Delta_M(X, Y) \cdot J Z .$$

Now the proposition follows from (1.9) and (1.10).

Proposition II. *A vector field Z without zeros is a CR-field if and only if the 1-forms Φ_Z and Ψ_Z satisfy*

$$(1.11) \qquad * \Phi_Z = \Psi_Z ,$$

*where $*$ denotes the Hodge star operator (*cf. [5], p. 121*).*

Proof. First recall that, for any 1-form Φ on a Riemannian 2-manifold, $* \Phi(X) = -\Phi(J X)$, $X \in \mathfrak{X}(M)$. Now let $Z \in \mathfrak{X}(M)$. Then (1.4) yields

$$\nabla_{JX} Z - J \nabla_X Z = [\Phi_Z(J X) + \Psi_Z(X)] Z + [\Psi_Z(J X) - \Phi_Z(X)] J Z .$$

Thus Z is a CR-field if and only if $\Psi_Z(X) = -\Phi_Z(J X) = * \Phi_Z(X)$; i.e., $\Psi_Z = * \Phi_Z$.

Corollary I. *Let Z be a CR-field without zeros. Then*

$$\Delta \ln |Z| = K .$$

Proof. In fact, by definition of the (Hodge) Laplacian (cf. [5], p. 125)

$$\Delta \ln |Z| = - * \delta * \delta \ln |Z| = - * \delta * \Phi_Z .$$

If Z is a CR-field, Propositions I and II yield

$$\Delta \ln |Z| = - * \delta \Psi_Z = K * \Delta_M = K .$$

Corollary II. *If Z is a CR-field without zeros, then*

$$\Delta |Z|^p = p K |Z|^p - p^2 |Z|^{p-2} |\delta |Z||^2 , \qquad p = 1, 2, \dots .$$

Proof. Apply the formula $e^{-f} \Delta (e^f) = \Delta f - |\delta f|^2$, when $f = p \ln |Z|$.

Corollary III. *A CR-field, Z, with constant length is parallel.*

Proof. We may assume that the constant length is positive. Then, by (1.5), $\Phi_Z = 0$. Thus Proposition II implies that $\Psi_Z = 0$, whence $\nabla_X Z = 0$, $X \in \mathfrak{X}(M)$.

1.3. Existence of CR-vector fields. In this section we shall prove the local existence of non-trivial CR-fields.

Proposition III. *Let $a \in M$ and let $h \in T_a(M)$ be a nonzero tangent vector. Then there is a CR-field Z in some neighbourhood U of a such that $Z(a) = h$.*

Proof. Assume first that the Gaussian curvature K vanishes in a neighbourhood V of a. Choose a simply connected open subset $U \subset V$ containing a. Then there is precisely one parallel vector field Z in U such that $Z(a) = h$. By Lemma II, Z is a CR-field in U.

In the general case introduce a new Riemannian metric $\tilde{g}$ on M by setting

$$\tilde{g} = e^{2\lambda} g , \quad \lambda \in \mathfrak{S}(M) .$$

The corresponding Gaussian curvature $\tilde{K}$ is given by

$$e^{2\lambda} \tilde{K} = K + \Delta \lambda ,$$

where Δ denotes the Laplacian with respect to the metric g . Choose for λ a local solution of the elliptic differential equation

$$\Delta \lambda = -K$$

(cf. [5], p. 151). Then $\tilde{K} = 0$ and so there exists a local CR-field Z in the $\tilde{g}$-metric such that $Z(a) = h$. But this is also a CR-field in the g-metric. (Cf. the remark above Lemma II.)

1.4. Cauchy–Riemann frames. Let $e_1 \neq 0$ be a CR-field in a simply connected neighbourhood U of a point $a \in M$ and set $e_2 = J e_1$. Then e_2 is again a CR-field (cf. corollary to Lemma I). Moreover, we have the relations

$$|e_1| = |e_2| , \quad g(e_1 , e_2) = 0 \quad \text{and} \quad [e_1 , e_2] = 0 .$$

Thus e_1 , e_2 is an orthogonal frame field in U . It will be called a *Cauchy–Riemann frame* (CR-frame).

Now consider the dual frame e^{*1} , e^{*2} . Then

$$\delta e^{*1} = 0 \quad \text{and} \quad \delta e^{*2} = 0 .$$

Thus e^{*1} and e^{*2} are gradient fields,

$$e^{*i} = \delta x^i , \quad x^i \in \mathfrak{S}(U) , \quad i = 1, 2 .$$

Since the covectors $e^{*1}(x)$ and $e^{*2}(x)$ are linearly independent, it follows that the functions x^1 , x^2 are local coordinates in a neighbourhood $V \subset U$ of a . In this local coordinate system the metric tensor satisfies $g_{11} = g_{22}$, $g_{12} = 0$ and so (x^1 , x^2) is a system of isothermal parameters.

Now choose a covering $M = \cup_\alpha V_\alpha$ by such open sets and introduce, in each V_α , isothermal parameters. Then it is easy to check that corresponding indentification maps are conformal and so M becomes a 1-dimensional *complex analytic manifold.*

Finally, let e_1 , e_2 be a CR-frame in U and let

$$Z = u\, e_1 + v\, e_2$$

be a vector field. Then it is easy to check that Z is a CR-field if and only if the functions u and v satisfy

$$e_1(u) = e_2(v), \quad e_1(v) = -e_2(u).$$

Thus if Z is a CR-field, then

$$f = u + i\,v$$

is a complex analytic function in U.

In particular, a nonzero CR-field has only isolated zeros.

2. The index of a line field at an isolated singularity

2.1. The index of a vector field. Let X be a vector field on an oriented 2-manifold M with an isolated singularity at a point a. Recall that the index of X at a is defined as follows: Choose a local trivialization of the tangent bundle, $T_U \overset{\cong}{\to} U \times \boldsymbol{R}^2$. Then X determines a map $X_U : \dot{U} = U - \{a\} \to \dot{\boldsymbol{R}}^2 \, (= \boldsymbol{R}^2 - \{0\})$; the index of X at a is the local degree of X_U at a,

$$j_a(X) = \deg_a X_U.$$

It is well known that the index can be expressed in terms of a line integral. Regard $\boldsymbol{R}^2$ as the complex plane $\boldsymbol{C}$ and let Ω denote the 1-form in $\dot{\boldsymbol{C}} = \boldsymbol{C} - \{0\}$ given by

$$(2.1) \qquad \Omega(z\,;h) = \frac{1}{|z|^2}\,\Delta(z, h) = \frac{1}{|z|^2}\,\mathrm{Im}(\bar{z}\,h), \qquad z \in \dot{\boldsymbol{C}}, \ h \in \boldsymbol{C},$$

where Δ is the normed determinant function. Then

$$(2.2) \qquad j_a(X) = \frac{1}{2\pi}\int_{S_a} X_U^*\,\Omega,$$

where S_a is a positively oriented 1-sphere in U around a.

If e_1, e_2 is a positively oriented 2-frame in U, then the corresponding trivialization of T_U is given by $X(x) \mapsto (x, [e^{*1}(X) + i\,e^{*2}(X)]\,(x))$, $x \in U$, where e^{*1}, e^{*2} is the dual 2-frame. In particular, if $X = u\,e_1 + v\,e_2$, then $X_U = u + i\,v$.

Now formula (2.2) reads

$$(2.3) \qquad j_a(X) = \frac{1}{2\pi}\int_{S_a} \frac{u\,\delta v - v\,\delta u}{u^2 + v^2}.$$

Proposition I. *Let Z be a CR-field with an isolated singularity at a. Then*

$$j_a(Z) \begin{cases} > 0\,, & \text{if} \quad \lim_{x\to a} |Z| = 0\,; \\ = 0\,, & \text{if} \quad 0 < \lim_{x\to a} |Z| < \infty\,; \\ < 0\,, & \text{if} \quad \lim_{x\to a} |Z| = \infty\,. \end{cases}$$

Proof. Choose a CR-frame e_1, e_2 in a neighbourhood U of a and write

$$Z = u\,e_1 + v\,e_2\,.$$

Then

$$f = u + i\,v$$

is a complex analytic function in U (cf. section 1.4) and so we have, in view of (2.3),

$$j_a(Z) = \frac{1}{2\pi i}\int_{S_a}\frac{f'(z)}{f(z)}\,dz\,.$$

Now the proposition follows from a standard theorem on complex analytic functions.

2.2. Line fields. Let M be a smooth oriented 2-manifold with tangent bundle $\tau_M = (T_M\,, p\,, M\,, R^2)$ and consider the corresponding projective bundle $\pi_M = (P_M\,, q\,, M\,, \boldsymbol{R}\,P^1)$ whose fibre at x consists of the 1-dimensional subspaces of $T_x(M)$. A *line field* on M is a smooth cross-section in π_M.

Suppose now that σ is a line field on M with an isolated singularity at a point a. To define the index of σ at a, choose a local trivialization $P_U \xrightarrow{\cong} U \times \boldsymbol{R}\,P^1$ of π_M and consider the map $\sigma_U : \dot{U} \to \boldsymbol{R}\,P^1$ determined by σ. The index of σ at a is defined as the mapping degree

$$j_a(\sigma) = \deg_a \sigma_U\,.$$

In particular, if the line field σ is induced from a unit vector field X with an isolated singularity at a, then $\sigma_U = \varrho \circ X_U$, where $\varrho : S^1 \to \boldsymbol{R}\,P^1$ is the double covering. Thus,

$$j_a(\sigma) = 2\,j_a(X)\,. \tag{2.4}$$

Let $\varphi : M \to N$ be a smooth map between 2-manifolds such that the induced map $(d\varphi)_x$ is a linear isomorphism for each $x \in M$. Then φ induces a bundle map $\tilde{\varphi} : P_M \to P_N$ which restricts to diffeomorphisms on the fibres. Thus every line field τ on N determines a line field σ on M given by

$$\sigma(x) = \tilde{\varphi}(x)^{-1}\,\tau(\varphi(x))\,, \qquad x \in M\,.$$

Moreover, if τ has an isolated singularity at $b = \varphi(a)$, then σ has an isolated singularity at a and

$$j_a(\sigma) = \deg_a \varphi \cdot j_b(\tau),$$

as follows from standard properties of the mapping degree.

As in the case of vector fields there is an integral formula for $j_a(\sigma)$. In fact, consider the double covering $\varrho : S^1 \to \boldsymbol{R}P^1$ and let $\Omega_{\boldsymbol{R}P^1}$ be the unique 1-form on $\boldsymbol{R}P^1$ satisfying

$$\varrho^* \Omega_{\boldsymbol{R}P^1} = \Omega.$$

Since ϱ has degree 2, it follows that

$$\int_{\boldsymbol{R}P^1} \Omega_{\boldsymbol{R}P^1} = \pi.$$

Thus

$$\deg_a \sigma_U = \frac{1}{\pi} \int_{S_a} \sigma_U^* \Omega_{\boldsymbol{R}P^1}$$

and we obtain the formula

$$j_a(\sigma) = \frac{1}{\pi} \int_{S_a} \sigma_U^* \Omega_{\boldsymbol{R}P^1}. \tag{2.5}$$

2.3. Proposition II. *Let φ be a smooth map from a connected 2-manifold M to S^1. Assume that the induced map in homology takes all elements of $H_1(M; \boldsymbol{Z})$ into even multiples of the generator of $H_1(S^1; \boldsymbol{Z})$. Then there is a smooth map $\psi : M \to S^1$ such that*

$$\varphi(x) = \psi(x)^2, \quad x \in M.$$

Proof. Choose a base point x_0 on M. Without loss of generality we may assume that $\varphi(x_0) = 1$. Let Ω be the 1-form on S^1 given by (2.1). The 1-form Ω determines a 1-form $\Omega_\varphi = \varphi^* \Omega$ on M. If α is a loop on M we have, in view of the hypothesis,

$$\int_\alpha \Omega_\varphi = \int_\alpha \varphi^* \Omega = \int_{\varphi(\alpha)} \Omega = 2k \int_{S^1} \Omega = 4k\pi, \quad k \in \boldsymbol{Z}.$$

Thus a smooth map $\psi : M \to S^1$ is well defined by

$$\psi(x) = \exp\left(\frac{i}{2} \int_{x_0}^{x} \Omega_\varphi\right), \quad x \in M.$$

To show that $\psi(x)^2 = \varphi(x)$, consider the map $\chi : M \to S^1$ given by

$$\chi(x) = \psi(x)^2 = \exp\left(i \int_{x_0}^{x} \Omega_\varphi\right).$$

It satisfies

$$(2.6) \qquad (d\chi)_x h = \Omega_\varphi(x ; h)\, i\, \chi(x), \qquad x \in M, \quad h \in T_x(M).$$

On the other hand, the relation $(\varphi(x), \varphi(x)) = 1$, $x \in M$, implies that

$$(2.7) \qquad (d\varphi)_x h = \Omega(\varphi(x) ; (d\varphi)_x h)\, i\, \varphi(x) = \Omega_\varphi(x ; h)\, i\, \varphi(x).$$

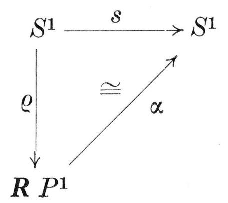
Equations (2.6) and (2.7) show that

$$\chi = \lambda \cdot \varphi, \qquad \lambda \in \boldsymbol{C}.$$

Since $\chi(x_0) = \varphi(x_0) = 1$, it follows that $\chi = \varphi$.

C o r o l l a r y. *Let* $\varphi : M \to \boldsymbol{R}P^1$ *(M connected) be a smooth map and assume that the induced map in homology takes all the elements of* $H_1(M : \boldsymbol{Z})$ *into even multiples of the generator of* $H_1(\boldsymbol{R}P^1 ; \boldsymbol{Z})$. *Then* φ *lifts to a smooth map* $\tilde{\varphi} : M \to S^1$.

Proof. Consider the diffeomorphism $\alpha : S^1 \overset{\cong}{\leftarrow} \boldsymbol{R}P^1$ which is determined by the commutative diagram



where $s(z) = z^2$, $z \in S^1$. Then the composite map

$$M \xrightarrow{\varphi} \boldsymbol{R}P^1 \xrightarrow{\alpha} S^1$$

satisfies the hypothesis of Proposition II. Thus there is a smooth map $\tilde{\varphi} : M \to S^1$ such that

$$\alpha \circ \varphi = s \circ \tilde{\varphi} = \alpha \circ \varrho \circ \tilde{\varphi}.$$

It follows that

$$\varphi = \varrho \circ \tilde{\varphi}.$$

P r o p o s i t i o n III. *Let* σ *be a line field in a neighbourhood* U *of* a *with an isolated singularity at* a. *Then* σ *lifts to a vector field if and only if* $j_a(\sigma)$ *is even.*

Proof. If σ lifts to a vector field, formula (2.4) shows that $j_a(\sigma)$ is even. Conversely, if $j_a(\sigma)$ is even, choose a trivialization of τ_U and apply the corollary of Proposition II to σ_U.

3. The Gauss–Bonnet theorem for line fields

3.1. Line fields on Riemannian manifolds. In section 3.3 it will be shown that the index sum of a line field σ with finitely many singularities on a compact oriented Riemannian 2-manifold M is given by

$$j_\sigma = \frac{1}{\pi}\int_M K \Delta_M ,$$

where K denotes the Gaussian curvature of M. The proof is essentially based on a 1-form Ψ_σ associated with σ.

Consider the circle bundle (S_M, r, M, S^1) associated with τ_M via the metric and observe that S_M is a double covering manifold of P_M. Choose an open covering $M = \bigcup_\alpha U_\alpha$ such that the covering projection $\varrho : S_M \to P_M$ admits a cross-section over each U_α. Then there are precisely two unit vector fields X_α and $-X_\alpha$ in U_α such that

$$\varrho \circ X_\alpha = \sigma .$$

Now set

$$\Psi_\alpha(x ; h) = \Delta_M(x ; X_\alpha(x), \nabla X_\alpha(x ; h)), \qquad x \in U_\alpha, \quad h \in T_x(U_\alpha) .$$

Since Ψ_α is not changed if X_α is replaced by $-X_\alpha$, it is a well-defined 1-form in U_α. Moreover, if $U_\alpha \cap U_\beta \neq \emptyset$, we have

$$\Psi_\alpha = \Psi_\beta \quad \text{in } U_\alpha \cap U_\beta$$

and so the local 1-forms Ψ_α determine a 1-form Ψ_σ on M.

P r o p o s i t i o n I. *The 1-form Ψ_σ has the following properties:*

(i) $\delta \Psi_\sigma = -K \Delta_M$;

(ii) *if σ has an isolated singularity at a and if the metric is flat in a neighbourhood of a, then*

$$j_a(\sigma) = \frac{1}{\pi}\int_{S^1} \Psi_\sigma .$$

Proof. (i) follows directly from the definition of Ψ_σ and formula (1.8).

To establish (ii) observe that, since the metric is flat near a, we can choose a trivialization $T_U \xrightarrow{\cong} U \times \boldsymbol{R}^2$ which induces isometries between the tangent spaces. It is easily checked that for such a trivialization

$$\sigma_U^* \Omega_{RP^1} = \Psi_\sigma$$

and so (ii) follows from formula (2.5).

3.2. The Gauss–Bonnet theorem. Suppose now that M is oriented and compact. Let σ be a line field with finitely many singularities and denote the index sum of σ by j_σ.

Theorem. *With the notation and hypotheses above,*

$$j_\sigma = \frac{1}{\pi} \int_M K \Delta_M .$$

Thus $j_\sigma = 2\,\chi(M)$, *where* $\chi(M)$ *denotes the Euler characteristic of* M.

For the proof we establish first

Lemma I. *Let* g_1 *and* g_2 *be Riemannian metrics on an oriented 2-manifold which agree in a neighbourhood* U *of a point* a. *Denote the corresponding normed 2-forms and Gaussian curvatures by* Δ_i *and* K_i ($i = 1, 2$). *Then the 2-form*

$$K_1 \Delta_1 - K_2 \Delta_2$$

is exact on M.

Proof. Denote the Levi-Civita connections corresponding to g_i by ∇_i ($i = 1, 2$). Choose a vector field, X, with a single singularity at a and consider the 1-forms

$$\Psi_i(x; h) = \frac{\Delta_i(x; X(x), \nabla_i X(x; h))}{g_i(x; X(x), X(x))}, \qquad x \in M - \{a\}, \ i = 1, 2 .$$

Then by (1.8)

$$\delta \Psi_i = -K_i \Delta_i, \qquad i = 1, 2,$$

whence

$$\delta(\Psi_2 - \Psi_1) = K_1 \Delta_1 - K_2 \Delta_2 . \tag{3.1}$$

But in U we have $g_1 = g_2$ and thus (3.1) holds on the whole manifold M.

3.3. Proof of the theorem. We first reduce to the case that g is flat in a neighbourhood of the singularities a_i ($i = 1, \dots, r$). Choose neighbourhoods U_i of a_i such that $U_i \cap U_j = \emptyset$ for $i \neq j$ and let $V_i \subset U_i$ be open subsets diffeomorphic to the unit disk such that $\overline{V}_i \subset U_i$. Set

$$U = M - \bigcup_{i=1}^{r} \overline{V}_i .$$

Then

$$M = U_1 \cup \cdots \cup U_r \cup U$$

is an open covering of M. Choose a partition of unity (cf. [1], p. 32) $f_1, \dots, f_r, f$ subordinate to this covering. Then

$$\tilde{g} = f \cdot g + \sum_{i=1}^{n} f_i \cdot g_i$$

is again a Riemannian metric where g_i is a flat metric in U_i. Since $\tilde{g} = g_i$ in V_i, it follows that $\tilde{g}$ is flat in V_i. On the other hand, $\tilde{g} = g$ in $M - \bigcup_{i=1}^{r} \bar{U}_i$. Thus, by Lemma I,

$$\int_M K \Delta = \int_M \tilde{K} \tilde{\Delta}.$$

Hence we may assume that g is flat in V_i ($i = 1, \dots, r$).

Now consider the 1-form Ψ_σ (cf. section 3.1) in the complement of the singularities $\{a_1, \dots, a_r\}$. Set $V = \bigcup_{i=1}^{r} V_i$. Then Stokes' theorem yields, in view of Proposition I, (i)

$$\int_{M-V} K \Delta_M = \sum_{i=1}^{r} \int_{S_{a_i}} \Psi_\sigma, \qquad S_{a_i} = \partial V_i, \; i = 1, \dots, r.$$

On the other hand, by part (ii) of that proposition,

$$\int_{S_{a_i}} \Psi_\sigma = \pi j_{a_i}(\sigma), \qquad i = 1, \dots, r.$$

Finally, since the metric is flat in V,

$$\int_V K \Delta_M = 0.$$

These equations yield

$$\int_M K \Delta_M = \pi \sum_{i=1}^{r} j_{a_i}(\sigma) = \pi j_\sigma.$$

4. Immersions into R^3 with constant mean curvature

4.1. Let M be an oriented Riemannian 2-manifold and let $\varphi : M \to R^3$ be an isometric immersion. Let F_x denote the oriented plane in R^3 given by

$$F_x = (d\varphi)_x T_x(M), \qquad x \in M.$$

Then there is a unique unit vector $n(x) \in R^3$ orthogonal to F_x such that the oriented plane F_x together with $n(x)$ induces the given orientation of

$\boldsymbol{R}^3$. The correspondence $x \mapsto n(x)$ determines a smooth map $n : M \to \boldsymbol{R}^3$ called the *normal field of the immersion* φ .

Recall that the second fundamental form for φ is the symmetric tensor field of degree two on M given by

$$\varLambda(x ; h , k) = - \langle (d\varphi)_x h , (dn)_x k \rangle , \qquad x \in M , \ h , k \in T_x(M) .$$

Thus $\varLambda$ determines a selfadjoint tensor field $\varGamma$ of type $(1 , 1)$ such that

$$(4.1) \qquad g(x ; \varGamma(x) h , k) = \varLambda(x ; h , k) , \qquad x \in M , \ h , k \in T_x(M) ,$$

where g denotes the Riemannian metric. Recall further that the mean curvature of φ is defined by

$$H = \tfrac{1}{2} \operatorname{tr} \varGamma .$$

In this section we shall prove the following

T h e o r e m (Hopf). *Let M be a Riemannian 2-manifold which is diffeomorphic to S^2 and let $\varphi : M \to \boldsymbol{R}^3$ be an isometric immersion with constant mean curvature. Then φ is a diffeomorphism from M onto a Euclidean 2-sphere in $\boldsymbol{R}^3$.*

4.2. Gauss–Godazzi fields. A selfadjoint tensor field $\varTheta$ of type $(1 , 1)$ on a Riemannian 2-manifold will be called a *Gauss–Codazzi field,* if

$$(4.2) \qquad \nabla_X(\varTheta(Y)) - \nabla_Y(\varTheta(X)) = \varTheta([X , Y]) , \qquad X , Y \in \mathfrak{X}(M) .$$

In particular, the tensor field $\varGamma$ determined by (4.1) is a Gauss–Codazzi field.

L e m m a I. *A nonzero Gauss–Codazzi field, $\varTheta$, with vanishing trace has only isolated zeros.*

Proof. Set

$$\hat{\varTheta} = e^{\mu} \varTheta ,$$

where μ is a smooth function. Then $\hat{\varTheta}$ satisfies the relation

$$(4.3) \quad \nabla_X(\hat{\varTheta} Y) - \nabla_Y(\hat{\varTheta} X) = X(\mu) \hat{\varTheta}(Y) - Y(\mu) \hat{\varTheta}(X) + \hat{\varTheta}([X , Y]) .$$

Now choose a local Cauchy–Riemann frame e_1 , e_2 and write

$$\hat{\varTheta} e_1 = u e_1 + v e_2 ,$$
$$\hat{\varTheta} e_2 = v e_1 - u e_2 .$$

Then (4.2) implies that

$$e_1(v) - e_2(u) = e_1(\mu - f) v - e_2(\mu - f) u$$

and

$$e_1(u) + e_2(v) = e_1(\mu - f)\, u + e_2(\mu - f)\, v ,$$

where $f = \ln |e_1|^2 = \ln |e_2|^2$, as is easily checked by using 1.4 and 1.5. Now set

$$\mu = f .$$

Then these equations become

$$e_1(v) = e_2(u) ,$$
$$e_2(v) = -e_1(u) .$$

It follows that $v + i\,u$ is a complex analytic function and so it can have only isolated zeros. Thus the same is true for $\hat{\Theta}$ and hence for Θ.

Corollary. *If Θ is a nonzero Gauss–Codazzi field with vanishing trace, then the function* $\det \Theta$ *has only isolated zeros.*

Lemma II. *Let Θ be a trace free Gauss–Codazzi field and suppose that $\Theta(x) \neq 0$ for $x \in U$ (an open subset of M). Denote the positive eigenvalue of $\Theta(x)$ by $\lambda(x)$ ($x \in U$). Let Z be a smooth eigenvector field of Θ in $V \subset U$ such that $|Z|^2 = \lambda^{-1}$. Then Z is a CR-field.*

Proof. Consider the 1-forms Φ_Z and Ψ_Z (cf. section 1.2). We have to show that

$$* \Phi_Z = \Psi_Z . \tag{4.4}$$

Now the unit vector field

$$X = \frac{1}{|Z|} Z = \sqrt{\lambda}\, Z$$

satisfies

$$\Theta(X) = \lambda\, X .$$

X determines a 1-form Ψ_X (cf. section 1.2) such that

$$\nabla_Y X = \Psi_X(Y) \cdot J(X) , \qquad Y \in \mathfrak{X}(M) .$$

Since $\operatorname{tr} \Theta = 0$, $J \circ \Theta + \Theta \circ J = 0$. Thus $\Theta\, J X = -\lambda \cdot J X$. Putting $Y = J X$ in (4.2) we find that these relations imply that

$$[J X(\lambda) - 2\lambda \cdot \Psi_X(X)]\, X + [X(\lambda) + 2\lambda \cdot \Psi_X(J X)]\, J X = 0 .$$

Thus (since X, $J X$ form a frame on V)

$$Y(\lambda) = -2\lambda \cdot \Psi_X(J(Y)) , \qquad Y \in \mathfrak{X}(V) . \tag{4.5}$$

Finally, observe that, by (1.5) and (1.6)

$$\Phi_Z = -\frac{1}{2\lambda}\, \delta \lambda \quad \text{and} \quad \Psi_Z = \Psi_X$$

and thus (4.5) implies (4.4).

4.3. The operator $\bar{\Theta}$. Let Θ be a nonzero trace free Gauss–Codazzi field and set $\dot{M} = M - \{a_1, \dots, a_r\}$, where the a_i are the zeros of Θ (cf. Lemma I). Then Θ induces a strong bundle map $\bar{\Theta} : P_{\dot{M}} \to P_{\dot{M}}$. Since $\Theta(x) : T_x(\dot{M}) \to T_x(\dot{M})$ is a nonzero selfadjoint linear map with trace zero, there are precisely two (orthogonal) straight lines $\sigma_1(x)$ and $\sigma_2(x)$ in $T_x(\dot{M})$ spanned by its eigenvectors. These lines define cross-sections σ_1 and σ_2 in $P_{\dot{M}}$ which satisfy

$$\bar{\Theta}(\sigma_i) = \sigma_i \qquad (i = 1, 2).$$

L e m m a III. *Let σ be a cross-section in $P_{\dot{M}}$ such that $\bar{\Theta}(\sigma) = \sigma$ and let a be one of the zeros of Θ. Then*

$$j_a(\sigma) < 0.$$

Proof. Assume first that $j_a(\sigma)$ is even. Then σ lifts to a vector field X in a neighbourhood U of a (cf. Proposition III, section 2.3). We may assume that X is a unit vector field. Since

$$\varrho(\Theta(X(x))) = \bar{\Theta}(\sigma(x)) = \sigma(x) = \varrho(X(x)),$$

it follows that X is a unit eigenvector field of Θ. Denote the corresponding eigenvalue at x by $\lambda(x)$. Then, by Lemma II,

$$Z = \frac{1}{\sqrt{\lambda}} X$$

is a CR-field. Since $\lambda^2 = -\det \Theta$, it follows that

$$\lim_{x \to a} |Z| = \infty$$

and so Proposition 1, section 2.1, implies that $j_a(Z) < 0$. Thus, by formula (2.4),

$$j_a(\sigma) < 0.$$

If $j_a(\sigma)$ is odd, choose a diffeomorphism φ from U onto the unit disk in the complex plane such that $\varphi(a) = 0$ and set $\dot{U} = U - \{a\}$. Let $s : U \to U$ be the map which corresponds to the map $z \mapsto z^2$, $z \in C$, under φ. Then $(ds)_x : T_x(\dot{U}) \to T_{s(x)}(\dot{U})$ is a linear isomorphism for $x \in \dot{U}$. Thus s induces a bundle map $\tilde{s} : P_{\dot{U}} \to P_{\dot{U}}$ and so σ determines a cross-section σ_1 in $\dot{U}$ by (cf. section 2.2)

$$\sigma_1(x) = \tilde{s}^{-1} \sigma(s(x)).$$

Finally, introduce a new Riemannian metric g_1 in $\dot{U}$ by $g_1 = s^* g$ and consider the tensor field Θ_1 given by

$$\Theta_1(x) = (ds)_x^{-1}\, \Theta(s(x))\, (ds)_x\,, \qquad x \in \dot{U}\,.$$

Then Θ_1 is a trace free Gauss–Codazzi field with respect to g_1 and

$$\bar{\Theta}_1(\sigma_1) = \sigma_1\,.$$

Since (cf. section 2.2)

$$j_a(\sigma_1) = \deg_a s \cdot j_a(\sigma) = 2\, j_a(\sigma)$$

it follows from the first part of the proof that

$$j_a(\sigma_1) < 0\,.$$

Thus

$$j_a(\sigma) < 0\,.$$

4.4. Proof of the Hopf theorem. Let Γ be the tensor field of type $(1, 1)$ corresponding to the second fundamental form and set

$$\Theta = \Gamma - H \cdot I \tag{4.6}$$

(I the unit tensor field). Then, since H is constant, Θ is again a Gauss–Codazzi field. Moreover,

$$\operatorname{tr} \Theta = 0\,.$$

We shall show that

$$\Theta = 0\,. \tag{4.7}$$

In fact, assume that $\Theta \neq 0$. Then, by Lemma 4.1, Θ has only finitely many zeros $a_1, \ldots, a_r$ $(r \geqq 1)$. Set $\dot{M} = M - \{a_1, \ldots, a_r\}$. In view of section 4.3 there is a cross-section σ in $P_{\dot{M}}$ such that

$$\bar{\Theta}(\sigma) = \sigma\,.$$

Hence, by Lemma III,

$$j_{a_i}(\sigma) < 0\,.$$

It follows that

$$j_\sigma < 0\,.$$

On the other hand, since M is diffeomorphic to S^2, by the Gauss–Bonnet theorem,

$$j_\sigma = 2\,\chi(M) = 4\,.$$

Thus we have a contradiction and (4.7) follows.

Now relation (4.6) implies via a standard result (cf. [6], p. 99) that φ maps M into a Euclidean sphere S^2 in $\boldsymbol{R}^3$. Since M is compact, this must be an onto map and hence a covering projection. Since S^2 is simply connected, it follows that φ is a diffeomorphism.

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