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SOME PROPERTIES OF CERTAIN EXTREMAL DECOMPOSITIONS OF A RECTANGLE

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1. General notation. For two distinct points a, b of the complex plane $\boldsymbol{C}$ or of the two-dimensional Euclidean space $\boldsymbol{R}^2$, $[a, b]$ is the closed straight line segment joining these two points; we put $\langle a, b] = [a, b] - \{a\}$, $[a, b\rangle = [a, b] - \{b\}$, $\langle a, b\rangle = [a, b] - \{a, b\}$, $[a, a] = \{a\}$; for a subset S of $\boldsymbol{C}$ or of $\boldsymbol{R}^2$, $\bar{S}$ is the closure and ∂S is the boundary of S.

For fixed $M > 0$, Q denotes the rectangle $\langle 0, M\rangle \times \langle 0, i\rangle$. In the course of this investigation, 4 or 5 or 6 boundary points of Q will be distinguished, and Q will be considered as quadrilateral or pentagon or hexagon accordingly. Choosing $x \in \langle 0, M\rangle$ and singling out the distinct points 0, x, M, $M+i$, i on ∂Q, Q becomes a pentagon with sides $[0, x]$, $[x, M]$, $[M, M+i]$, $[M+i, i]$, $[i, 0]$. Selecting two non-adjacent sides s_1 and s_2 out of these five sides, the remaining sides s_3, s_4, s_5 are so numbered that s_3 and s_4 have a common point, denoted by P. We denote by $\mathfrak{K} = \mathfrak{K}(s_1, s_2; P)$ the class of continua K in $\bar{Q}$ with the properties

(i) K connects P and s_5,
(ii) K is disjoint from $s_1 \cup s_2$.

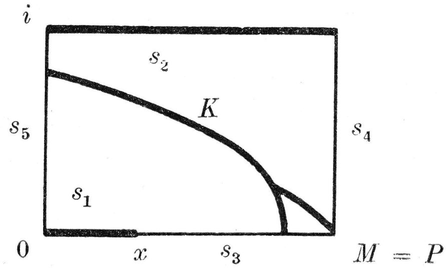

Figure 1

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K separates s_1 from s_2 in $\bar{Q}$ and decomposes Q into several simply connected components. The component of $(\bar{Q}-K) \cap Q$ which has s_j on its boundary $(j = 1, 2)$ we call $D_j = D_j(s_1, s_2; K)$. $K_j = K \cap \partial D_j$ is connected; $M_j = M_j(s_1, s_2; K)$ denotes the extremal distance between s_j and K_j in D_j; M_j is then the modulus of the quadrilateral D_j with respect to the pair (s_j, K_j) of sides. For a domain $D \subset C$ and two connected compact sets $\gamma_1, \gamma_2 \subset \partial D$ in general, $d(\gamma_1, \gamma_2; D)$ is the extremal distance between γ_1 and γ_2 in D. We note $d([0, i], [M, M+i]; Q) = M$ and the well known inequality

$$M_1(s_1, s_2; K) + M_2(s_1, s_2; K) \leqq d(s_1, s_2; Q). \tag{1.1}$$

Figure 1 gives an example in the case $s_1 = [0, x]$, $s_2 = [i, M+i]$.

2. We shall investigate the set of values of the pairs (M_1, M_2) obtained this way in its dependence of the chosen pair of sides and the choice of x. Here we shall find that certain "extremal" continua which are produced by quadratic differentials play a key role. We shall omit the case that the side $[x, M]$ is selected since a reflection in the line $\{z ; \operatorname{Re} z = M/2\}$ immediately leads to a situation in which $[0, M-x]$ is a selected side. So we are left essentially with the following three cases:

Case I. $s_1 = [0, i]$, $s_2 = [M, M+i]$, $P = x$;
Case II. $s_1 = [0, x]$, $s_2 = [M, M+i]$, $P = i$;
Case III. $s_1 = [0, x]$, $s_2 = [i, M+i]$, $P = M$.

Of these, we will study first case I giving detailed proofs of the relevant facts; since cases II and III exhibit many similar features, a briefer discussion seems appropriate.

3. Let $M > 0$ and $Q = \langle 0, M \rangle \times \langle 0, i \rangle$. In this section, and up to section 8, we let $s_1 = [0, i]$ and $s_2 = [M, M+i]$. We put $S = \langle 0, M \rangle \times \langle 0, M \rangle$. With Weierstrass' p-function $p(z; 2M, 2i) = p(z)$ and $(x, y) \in \bar{S}$, we introduce the quadratic differential

$$\sigma_{xy} = -\frac{1 - p(z)/p(y)}{1 - p(z)/p(x)} dz^2, \quad z \in \bar{Q}, \tag{3.1}$$

where the formal expression $p(z)/p(0)$ is replaced by 0 for each $z \in \bar{Q}$. Denoting by K_{xy} the closure of the union of the trajectories $\sigma_{xy} > 0$ which have the limit point y, we describe first the set K_{xy}.

(a) $(x, y) \in S^+ = \{(x, y) ; 0 < y < x < M\}$. Then $\sigma_{xy} > 0$ has two trajectories with limit point y. One trajectory has as its closure the segment $[y, x]$; the other trajectory has as its closure the carrier K'_{xy} of an analytic arc with initial point y and terminal point on $\langle i, M+i \rangle$

which lies up to initial and terminal point in Q, forming at y with the direction of the positive real axis the angle $2\pi/3$.

(b) $(x, y) \in S^- = \{ (x, y) ; 0 < x < y < M \}$. Then again $\sigma_{xy} > 0$ has two trajectories with limit point y; $[x, y]$ is the closure of one trajectory, the closure K'_{xy} of the other trajectory has the same properties as mentioned in (a) except that the angle with the positive real axis at y is now $\pi/3$.

(c) (x, y) with $0 < x = y < M$. $\sigma_{xy} > 0$ has one trajectory with limit point y, which is already a closed set, namely $K'_{xy} = [y, y+i]$.

In the cases (a)–(c) thus $K_{xy} = [x, y] \cup K'_{xy}$; considering Q as a pentagon with sides s_1, s_2, and $s_3 = [0, x]$, $s_4 = [x, M]$, $s_5 = [i, i+M]$, we have $K_{xy} \in \mathfrak{K}(s_1, s_2; x)$, and s_1 as well as s_2 are trajectories of $\sigma_{xy} > 0$.

d$_1$) $x \in \langle 0, M \rangle$, $y = 0$. Then $K_{x0} = [0, x] \cup s_1$, and s_2 is a trajectory of $\sigma_{x0} > 0$.

d$_2$) $x \in \langle 0, M \rangle$, $y = M$. Then $K_{xM} = [x, M] \cup s_2$, and s_1 is a trajectory of $\sigma_{xM} > 0$.

e$_1$) $x = 0$, $y \in \langle 0, M \rangle$. Then $K_{0y} = [0, y] \cup K'_{0y}$, where K'_{0y} has the properties of K'_{xy} mentioned in b); s_2 is a trajectory of $\sigma_{0y} > 0$, and s_1 is the closure of a trajectory.

e$_2$) $x = M$, $y \in \langle 0, M \rangle$. Then $K_{My} = [y, M] \cup K'_{My}$, where K'_{My} has the properties of K'_{xy} mentioned in a); s_1 is a trajectory of $\sigma_{My} > 0$ and s_2 is the closure of a trajectory.

f) $K_{00} = s_1$, and s_2 is a trajectory of $\sigma_{00} > 0$;

$K_{M0} = s_1 \cup [0, M]$, and s_2 is the closure of a trajectory of $\sigma_{M0} > 0$;

$K_{0M} = s_2 \cup [0, M]$, and s_1 is the closure of a trajectory of $\sigma_{0M} > 0$;

$K_{MM} = s_2$, and s_1 is a trajectory of $\sigma_{MM} > 0$.

4. Defining

$$\left(\frac{1 - p(z)/p(y)}{1 - p(z)/p(x)}\right)^{1/2}$$

analytic in Q, continuous on $\bar{Q} - \{x\}$, and positive on $[i, M+i]$, we put

$$f(z) = \int_y^z (-\sigma_{xy})^{1/2}, \quad z \in Q. \tag{4.1}$$

For $(x, y) \in S^+ \cup S^-$, f maps Q conformally onto a hexagon bounded by straight line segments parallel to the real or the imaginary axis, with five right angles (corresponding to the points 0, x, M, $M+i$, i) and with one $3\pi/2$-angle (corresponding to y); for $(x, y) \in \bar{S} - (S^+ \cup S^-)$ various degeneracies appear. $(\bar{Q} - K_{xy}) \cap Q$ either consists of two components (in the cases (a), (b), (c), (e) i.e. when $y \neq 0, M$), then we call D_j that component ($j = 1, 2$) which has s_j on the boundary; or $(\bar{Q} - K_{xy}) \cap Q$ consists of just one component (in the remaining cases), which we call D_1 when $y = M$, and D_2 when $y = 0$ (they are obtained as limiting cases when $y \to 0, M$); for $(x, y) \in S$ we have in the notation of section 1, $D_j = D_j(s_1, s_2; K_{xy})$.

We put ($j = 1, 2$) for $(x, y) \in \bar{S}$

$$(4.2)\qquad \begin{aligned} M_j^*(x, y) &= d(s_j, K_{xy} \cap \partial D_j; D_j) && \text{when } D_j \text{ is defined,} \\ M_j^*(x, y) &= 0 && \text{when } D_j \text{ is not defined.} \end{aligned}$$

For $(x, y) \in S$ we have thus: $M_j^*(x, y) = M_j(s_1, s_2; K_{xy})$ is the modulus of the rectangle $f(D_j)$ with respect to its vertical sides. We put further for $(x, y) \in \bar{S}$

$$(4.3)\qquad \begin{aligned} a_1(x, y) &= \int_0^{\min(x, y)} |\sigma_{xy}|^{1/2}, & b_1(x, y) &= \int_0^{i} |\sigma_{xy}|^{1/2}, \\ a_2(x, y) &= \int_{\max(x, y)}^{M} |\sigma_{xy}|^{1/2}, & b_2(x, y) &= \int_M^{M+i} |\sigma_{xy}|^{1/2}, \end{aligned}$$

the integration being carried out on $[\alpha, \beta]$ when α is the initial and β the terminal point in any of the definite integrals (4.3). Simply checking the various cases, we obtain ($j = 1, 2$) from (4.3) now

$$(4.4)\qquad M_j^*(x, y) = \frac{a_j(x, y)}{b_j(x, y)}, \quad (x, y) \in \bar{S};$$

indeed (apart from the degenerate cases (d), (e), (f)) the function f maps D_j onto a rectangle with horizontal sides of length $a_j(x, y)$ and vertical sides of length $b_j(x, y)$. We observe further that in (4.3) a_1 and a_2 are always finite and non-negative on $\bar{S}$ while b_1 and b_2 are positive (possibly infinite) there.

5. While a_1, a_2, b_1, b_2 are not continuous on $\bar{S}$, we have

L e m m a 1. *Let* $h: \bar{S} \to \boldsymbol{R}^2$ *be defined by*

$$(5.1)\qquad h(x, y) = (M_1^*(x, y), M_2^*(x, y)), \quad (x, y) \in \bar{S}.$$

Then h *is continuous.*

We shall see later (Theorem 2) that the restriction of h to $\overline{S^+}$ or to $\overline{S^-}$ is actually a homeomorphism onto the triangle

$$\bar{\varDelta} = \{ (\mu_1, \mu_2) ; \ 0 \leqq \mu_1 + \mu_2 \leqq M , \ \mu_1 \geqq 0 , \ \mu_2 \geqq 0 \} . \tag{5.2}$$

Proof of Lemma 1. The continuity of h on $S \cup (\langle 0 , M \rangle \times \{0 , M\})$ follows from (4.4) since the functions a_1, b_1, a_2, b_2 are clearly continuous on this set, by (4.3) and (3.1). To prove the continuity of h on the remaining set $\{0 , M\} \times [0 , M]$, we observe that by the trajectory structure of the quadratic differentials σ_{xy}, $(x , y) \in \bar{S}$, the reflection of K_{xy} in the line $\{ z ; \ \mathrm{Re}\, z = M/2 \}$ yields $K_{M-x,\, M-y}$, whence

$$\begin{aligned} M_1^*(M-x , M-y) &= M_2^*(x , y) , \\ M_2^*(M-x , M-y) &= M_1^*(x , y) ; \end{aligned} \qquad (x , y) \in \bar{S} , \tag{5.3}$$

it suffices thus to prove the continuity of h just also on $\{0\} \times [0 , M]$.

Consider now the continuity at $(0 , \eta)$, $\eta \in [0 , M]$. By (4.3), (4.4) we have

$$M_1^*(0 , \eta) = 0 , \quad M_2^*(0 , \eta) = \frac{a_2(0 , \eta)}{b_2(0 , \eta)} . \tag{5.4}$$

The continuity of M_2^* at $(0 , \eta)$ follows since a_2 and b_2 are easily seen to be continuous at that point by (4.3), (3.1). The continuity of M_1^* at $(0 , \eta)$ may either be inferred directly (considering the quotient $a_1(x , y) / b_1(x , y)$ near $(0 , \eta)$ which is, however, a little cumbersome when $\eta = 0$) or by using $M_1^*(x , M) \to 0$ when $x \to 0$ together with the inequality $M_1^*(x , y) \leqq M_1^*(x , M)$ for each $y \in [0 , M]$, which inequality is immediate from the structures of K_{xy} and K_{xM}.

6. Lemma 2. *Let $x \in \langle 0 , M \rangle$ and $K \in \mathfrak{K}(s_1 , s_2 ; x)$. Then for any $y \in [0 , M]$ we have the inequality*

$$\begin{aligned} &b_1^2(x , y)\, M_1(s_1 , s_2 ; K) + b_2^2(x , y)\, M_2(s_1 , s_2 ; K) \\ &\quad \leqq b_1^2(x , y)\, M_1^*(x , y) + b_2^2(x , y)\, M_2^*(x , y) ; \end{aligned} \tag{6.1}$$

for $y = 0$ or M, the inequality (6.1) is strict ; for $y \in \langle 0 , M \rangle$ we have equality in (6.1) if and only if $K = K_{xy}$.[1]

The proof is accomplished in a standard way by the extremal-length-method, using in Q the metric $|\sigma_{xy}|^{1/2}$.

Using the fact that $K_{xy} \in \mathfrak{K}(s_1 , s_2 ; x)$ once $(x , y) \in S$, we conclude from (1.1) that $h(S)$ lies in the closed triangle $\bar{\varDelta}$ of (5.2), and Lemma 1 yields

[1] Note that (6.1) yields (1.1) for $y = x$.

(6.2) $$h(\bar{S}) \subset \bar{\Delta} .$$

Writing

(6.3) $$\Delta = \{ (\mu_1 , \mu_2) ; \; 0 < \mu_1 + \mu_2 < M , \; \mu_1 > 0 , \; \mu_2 > 0 \}$$

for the interior of $\bar{\Delta}$ we have

Theorem 1. *Let* $x \in \langle 0 , M \rangle$ *and let* $h_x : [0 , M] \to \boldsymbol{R}^2$ *denote the section of* h *at* x *given by*

$$h_x(y) = (M_1^*(x , y) , M_2^*(x , y)) , \quad y \in [0 , M] .$$

Then

(i) h_x *is injective;*

(ii) $\Gamma_x = h_x([0 , M])$ *is a smooth curve in the closed triangle* $\bar{\Delta}$ *which has at* $h_x(y)$ *the slope*

(6.4) $$\lambda_x(y) = -b_1^2(x , y) \,/\, b_2^2(x , y) < 0 ;$$

(iii) *the slope* λ_x *is strictly decreasing on* $[0 , M]$ *from* $\lambda_x(0)$ *at* $h_x(0) = (0 , M_2^*(x , 0)) \in \partial\Delta$ *over* $\lambda_x(x) = -1$ *at* $h_x(x) = (x , M-x) \in \partial\Delta$ *to* $\lambda_x(M)$ *at* $h_x(M) = (M_1^*(x , M) , 0) \in \partial\Delta$;

(iv) *the component* Δ_x *of* $(\bar{\Delta} - \Gamma_x) \cap \Delta$ *which has the point* $(0 , 0)$ *on the boundary is a convex domain touching the side* $[(M , 0) , (0 , M)]$ *of* Δ *at just one point, namely* $(x , M-x)$.

Proof. To prove (i) we observe $h_x(0) \in \langle (0 , 0) , (0 , M) \rangle$ and $h_x(M) \in \langle (0 , 0) , (M , 0) \rangle$ whence $h_x(0) \neq h_x(M)$, while for $y_1 \in \langle 0 , M \rangle$ and a different point $y_2 \in [0 , M]$ we obtain $h_x(y_1) \neq h_x(y_2)$ at once from Lemma 2 choosing there $K = K_{xy_1}$ and $y = y_2$, since $K_{xy_1} \neq K_{xy_2}$.

We show now that for fixed $x \in \langle 0 , M \rangle$,

(6.5)
$$\begin{aligned} &M_1^*(x , y) \text{ is strictly increasing in } y \in [0 , M] , \\ &M_2^*(x , y) \text{ is strictly decreasing in } y \in [0 , M] . \end{aligned}$$

First consider $(\partial/\partial y)\, M_1^*(x , y)$ for $y \in \langle x , M \rangle$. We have

(6.6)
$$\begin{aligned} &\operatorname{sign} \frac{\partial}{\partial y} M_1^*(x , y) \\ &= \operatorname{sign}\left[b_1(x , y) \frac{\partial}{\partial y} a_1(x , y) - a_1(x , y) \frac{\partial}{\partial y} b_1(x , y) \right] , \end{aligned}$$

and the expression in parentheses in (6.6) equals

(6.7) $$\int_0^i |\sigma_{xy}|^{1/2} \cdot \frac{\partial}{\partial y} \int_0^x |\sigma_{xy}|^{1/2} - \int_0^x |\sigma_{xy}|^{1/2} \cdot \frac{\partial}{\partial y} \int_0^i |\sigma_{xy}|^{1/2} .$$

A little consideration shows that the differentiation can be taken under the integral sign which makes (6.7) equal to

$$(6.8)\qquad \frac{1}{2}\frac{p'(y)}{p^2(y)}\Big[\int_0^i |\sigma_{xy}|^{1/2} \cdot \int_0^x \frac{p(z)}{1-p(z)/p(y)}\,|\sigma_{xy}|^{1/2} - \int_0^x |\sigma_{xy}|^{1/2} \cdot \int_0^i \frac{p(z)}{1-p(z)/p(y)}\,|\sigma_{xy}|^{1/2}\Big].$$

Since for $t \in \langle 0\,,x\rangle$ and $\tau \in \langle 0\,,1]$ we have

$$\frac{p(t)}{1-p(t)/p(y)} < -p(y) < \frac{p(i\,\tau)}{1-p(i\,\tau)/p(y)},$$

the bracketed expression in (6.8) is negative, and since $p'(y) < 0$, (6.8) is positive, whence

$$(6.9)\qquad \frac{\partial}{\partial y} M_1^*(x\,,y) > 0 \qquad \text{for } y \in \langle x\,,M\rangle\,.$$

Thus $M_1^*(x\,,y)$ is strictly increasing for $y \in [x\,,M]$.

Let now $x \leqq y_1 < y_2 < M$. Writing l_{xy} for the right hand side of (6.1), the point $h_x(y_1) = (M_1^*(x\,,y_1)\,,M_2^*(x\,,y_1))$, lies trivially on the line

$$(6.10)\qquad L_{xy_1} = \{\,(\mu_1\,,\mu_2)\,;\ b_1^2(x\,,y_1)\,\mu_1 + b_2^2(x\,,y_1)\,\mu_2 = l_{xy_1}\,\}\,,$$

which line has the negative slope

$$(6.11)\qquad \lambda_x(y_1) = -b_1^2(x\,,y_1)\,/\,b_2^2(x\,,y_1)\,.$$

By Lemma 2, on the other hand, the point $h_x(y_2)$ lies strictly below the line (6.10). Since for the first component of these points we have $M_1^*(x\,,y_1) < M_1^*(x\,,y_2)$, the slope $\lambda_x(y_1\,,y_2)$ of $[h_x(y_1)\,,h_x(y_2)]$ satisfies

$$(6.12)\qquad \lambda_x(y_1\,,y_2) < \lambda_x(y_1)\,, \qquad x \leqq y_1 < y_2 < M\,;$$

and in particular the second components of $h_x(y_1)$ and $h_x(y_2)$ fulfill by (6.11) the opposite inequality $M_2^*(x\,,y_1) > M_2^*(x\,,y_2)$, whence $M_2^*(x\,,y)$ is strictly decreasing for $y \in [x\,,M]$. A similar procedure applied to the interval $[0\,,x]$ then gives (6.5).

Considering instead of the line (6.10) the line L_{xy_2}, we obtain instead of (6.12)

$$(6.13)\qquad \lambda_x(y_2) < \lambda_x(y_1\,,y_2)\,, \qquad x \leqq y_1 < y_2 < M\,;$$

both (6.12) and (6.13) are easily verified also when $0 < y_1 < y_2 \leqq x$, and the continuity of (6.4) for $y \in [0\,,M]$ finally yields

(6.14) $\lambda_x(y_2) < \lambda_x(y_1, y_2) < \lambda_x(y_1)$, when $0 \leqq y_1 < y_2 \leqq M$.

$\Gamma_x \subset \bar{\Delta}$ follows from (6.2) while the remainder of statement (ii) in Theorem 1 is obtained by letting $y_1, y_2 \to y \in [0, M]$ in formula (6.14) where $y_1 \leqq y \leqq y_2$, and using the continuity of (6.11) in $y_1 \in [0, M]$.

With (6.4), (6.14) as well as the relation $b_1(x, x) = b_2(x, x)$, obvious from (3.1) and (4.3), statement (iii) now follows; statement (iv) is easily obtained combining the statements (ii) and (iii).

Remark. Though Γ_x is a smooth curve with slope -1 at $(x, M-x)$ the partial derivatives of $M_1^*(x, y)$, $M_2^*(x, y)$ with respect to y at $y = x$ are infinite.

7. For $(x, y) \in S$, the function f of (4.1) maps Q onto a hexagon H (degenerate to a rectangle when $x = y$) which is bounded by straight line segments parallel to the real or the imaginary axis. The imaginary axis decomposes $H = f(Q)$ into two rectangles $H_1 = f(D_1)$ in the left half plane and $H_2 = f(D_2)$ in the right half plane. $M_j^*(x, y)$, $(j = 1, 2)$, is the extremal distance of the vertical sides of H_j, while M is the extremal distance of the "outer" vertical sides $f([0, i])$ and $f([M, M+i])$ of H in H. The question arises: to what extent does a pair (M_1^*, M_2^*), obtained that way, characterize the pair $(x, y) \in S$. In order to answer that question we need some information about the change of the extremal distance of the outer vertical sides of such a hexagon when one of the rectangles H_j is subject to a homothetic transformation.

L e m m a 3. *Let $M > 0$ and $\xi \in \langle 0, M\rangle$; let $R = \langle -\xi, M-\xi\rangle \times \langle 0, i\rangle$, $R_1 = \langle -\xi, 0\rangle \times \langle 0, i\rangle$, $R_2 = \langle 0, M-\xi\rangle \times \langle 0, i\rangle$; for $\vartheta > 0$ let $R_2(\vartheta) = \langle 0, \vartheta(M-\xi)\rangle \times \langle 0, \vartheta i\rangle$ be the rectangle obtained from R_2 through homothetic stretching by the factor ϑ, keeping 0 fixed; let*

(7.1) $$R(\vartheta) = \text{interior of } (\overline{R_1} \cup \overline{R_2(\vartheta)}) = R(M, \xi, \vartheta),$$

(7.2) $$\mu(\vartheta) = d([-\xi, -\xi+i], [\vartheta(M-\xi), \vartheta(M-\xi+i)]; R(\vartheta)).$$

Then[2] $\mu : \langle 0, \infty\rangle \to \mathbf{R}$ *is strictly decreasing on* $\langle 0, 1]$, *strictly increasing on* $[1, \infty\rangle$, *and*

(7.3) $$\lim_{\vartheta\to 0} \mu(\vartheta) = \infty = \lim_{\vartheta\to\infty} \mu(\vartheta).$$

We have $R = R(1)$; $R(\vartheta)$ is a hexagon of a type similar to that of H above.

Proof of Lemma 3. $\mu(\vartheta) \to \infty$ for $\vartheta \to 0$ is obvious. Writing for the moment $\mu(M, \xi, \vartheta)$ instead of $\mu(\vartheta)$ to take into account also the

[2] Here the notation $\langle 0, \infty\rangle$ etc. for subsets of the real axis is self-explanatory.

dependence on M and ξ, we have $\mu(M, \xi, \vartheta) = \mu(M, M-\xi, 1/\vartheta)$; $\lim_{\vartheta \to \infty} \mu(\vartheta) = \infty$ now follows, and we conclude further that the Lemma is proved once it is shown that μ is just strictly increasing for $\vartheta \in [1, \infty\rangle$. Let now $\vartheta > 1$. Abbreviating the vertices of $R(\vartheta)$ by

$$(7.4)\qquad \begin{aligned} A &= -\xi + i, \quad B = -\xi, \quad C = \vartheta(M-\xi), \\ D &= \vartheta(M-\xi+i), \quad E = \vartheta i, \quad F = i, \end{aligned}$$

we have for the extremal distances M_1, M_2 of the vertical sides of the two rectangles R_1, $R_2(\vartheta)$ obviously

$$\begin{aligned} M_1 &= d([A, B], [0, F]; R_1) = \xi, \\ M_2 &= d([0, E], [C, D]; R_2(\vartheta)) = M-\xi, \end{aligned}$$

whence (observing $\vartheta > 1$) immediately

$$(7.5)\qquad \mu(\vartheta) = d([A, B], [C, D]; R(\vartheta)) > M_1 + M_2 = M = \mu(1).$$

To prove now $\mu(\vartheta_1) > \mu(\vartheta)$ when $1 < \vartheta < \vartheta_1$, we determine first an extremal metric ϱ for the extremal distance $\mu(\vartheta)$. We select $t_3 > t_2 > t_1 > 1$ so that the upper half plane U can be mapped conformally onto $R(\vartheta)$ in such a way that the boundary points $\infty, 0, 1, t_1, t_2, t_3$ of U correspond in turn to the vertices A, B, C, D, E, F of $R(\vartheta)$. With suitable $\lambda > 0$ the mapping function Φ is then given by

$$(7.6)\qquad \Phi(z) = \lambda \int_0^z \left(\frac{t-t_3}{t\,(t-1)\,(t-t_1)\,(t-t_2)} \right)^{1/2} dt - \xi, \quad z \in U,$$

where that branch of the square root is taken which has positive boundary values for $t \in \langle 0, 1 \rangle$. The function φ given by

$$(7.7)\qquad \varphi(z) = \int_0^z \frac{dt}{[t\,(t-1)\,(t-t_1)]^{1/2}}, \quad z \in U,$$

again with that branch of the square root which has positive boundary values on $\langle 0, 1 \rangle$, maps U onto a rectangle $\langle 0, a \rangle \times \langle 0, i\,b \rangle$ with $a > 0$, $b > 0$. Having $w \in R(\vartheta)$ related to $z \in U$ by $w = \Phi(z)$, an extremal metric $\varrho(w)\,|dw|$ in $R(\vartheta)$ is given by

$$\frac{|dz|}{|z\,(z-1)\,(z-t_1)|^{1/2}} = \frac{1}{|z\,(z-1)\,(z-t_1)|^{1/2}} \frac{|dw|}{|\Phi'(z)|},$$

thus

$$(7.8)\qquad \varrho(w) = \frac{1}{\lambda} \left| \frac{t_2 - z}{t_3 - z} \right|^{1/2}, \quad z = \Phi^{-1}(w).$$

We have then ($w = u + i\,v$) for the family Γ of curves joining $[A\,,B]$ and $[C\,,D]$ in $R(\vartheta)$

$$\inf_{\gamma\in\Gamma} \int_\gamma \varrho(w)\,|dw| = a\,, \qquad \iint_{R(\vartheta)} \varrho^2(w)\,du\,dv = a\,b\,,$$

whence

$$\mu(\vartheta) = \frac{a}{b}\,. \tag{7.9}$$

For small $\varepsilon > 0$ we put $\vartheta_1 = (1+\varepsilon)\,\vartheta$ and denote the corresponding vertices of $R(\vartheta_1)$ by

$$\begin{aligned} A_1 &= -\xi+i\,, \quad B_1 = -\xi\,, \quad C_1 = \vartheta_1(M-\xi)\,,\\ D_1 &= \vartheta_1(M-\xi+i)\,, \quad E_1 = \vartheta_1\,i\,, \quad F_1 = i\,. \end{aligned} \tag{7.10}$$

The choice of a suitable metric ϱ_1 in $R(\vartheta_1)$ will then give $\mu(\vartheta_1) > \mu(\vartheta)$, and the Lemma is proved. [3]

We put $S' = \langle 0\,, C] \times [E\,, E_1\rangle$, $S'' = [C\,, C_1\rangle \times \langle 0\,, E_1\rangle$. Noticing that ϱ has a continuous extension to $[E\,,D] \cup [C\,,D]$, again denoted by ϱ, we put (see Figure 2)

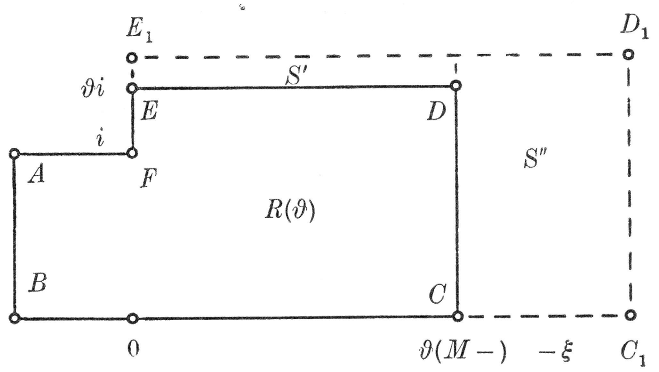

Figure 2

(i) $\varrho_1(w) = \varrho(w)$ for $w \in R(\vartheta)$;
(ii) $\varrho_1(w) = \varrho(u+i\,\vartheta)$ for $w = u+i\,v \in S'$ where $u \in \langle 0\,, C]$ and $v \in [\vartheta\,, \vartheta_1\rangle$, i.e. $i\,v \in [E\,, E_1\rangle$,
(iii) $\varrho_1(w) = \varrho(D)$ for $w \in S''$,

observing

[3] A more subtle reasoning, not needed here, gives in fact for the partial derivatives of $d([A\,,B]\,,[C\,,D]\,;R(\vartheta))$ with respect to each of the variables ξ, M, ϑ a not too complicated expression.

$$(7.11)\qquad \varrho(D) = \frac{1}{\lambda}\left|\frac{t_2-t_1}{t_3-t_1}\right|^{1/2} = \inf_{w\in[C,D]} \varrho(w)\,.$$

If Γ_1 is the family of curves joining $[A_1, B_1] = [A, B]$ and $[C_1, D_1]$ in $R(\vartheta_1)$, we have

$$(7.12)\qquad \mu(\vartheta_1) \geqq \frac{L^2(\Gamma_1, \varrho_1)}{A_1(\varrho_1)}\,,$$

where

$$(7.13)\qquad L(\Gamma_1, \varrho_1) = \inf_{\gamma_1\in\Gamma_1} \int_{\gamma_1} \varrho_1(w)\,|dw| = a + \varrho(D)\,(C_1 - C)\,,$$

and

$$(7.14)\qquad A_1(\varrho_1) = \iint_{R(\vartheta_1)} \varrho_1^2(w)\,du\,dv = a\,b + \iint_{S'\cup S''} \varrho_1^2(w)\,du\,dv\,.$$

Putting $(1 \leqq t \leqq t_2)$

$$(7.15)\qquad f(t) = |t\,(t-1)\,(t-t_1)|^{-1/2}\,,\quad g(t) = |(t_2-t)\,/\,(t_3-t)|^{1/2}\,,$$

the relations $C_1 - C = (\vartheta_1-\vartheta)\,(M-\xi) = \varepsilon\,\vartheta(M-\xi) = \varepsilon\,(D-E)$ and (7.11), (7.13), (7.15) yield

$$(7.16)\qquad L(\Gamma_1, \varrho_1) = a + \varepsilon\, g(t_1)\,\frac{1}{\lambda}\,(D-E)\,.$$

From (7.6) and (7.15) we obtain further ($w \in [E, D]$, $t \in [t_1, t_2]$)

$$(7.17)\qquad |dw| = \lambda\,\frac{f(t)}{g(t)}\,|dt|\,,\quad \text{where } w = \Phi(t)\,,$$

whence with $E_1 - E = i\,\vartheta\,\varepsilon = \varepsilon\,(D-C)$ and (7.8)

$$(7.18)\qquad \iint_{S'} \varrho_1^2(w)\,du\,dv = |E_1-E| \int_E^D \varrho_1^2(w)\,|dw|$$

$$= \varepsilon\,|D-C|\,\frac{1}{\lambda}\int_{t_1}^{t_2} g^2\,\frac{f}{g}\,dt\,.$$

Formulae (7.11) and (7.15) yield

$$(7.19)\qquad \iint_{S''} \varrho_1^2\,du\,dv = \frac{1}{\lambda^2}\,g^2(t_1)\,(C_1-C)\,|D_1-C_1|$$

$$= g^2(t_1)\,\frac{1}{\lambda^2}\,\varepsilon\,(D-E)\,(1+\varepsilon)\,|D-C|\,.$$

Combining (7.14) with (7.16), (7.18), (7.19) we obtain

(7.20)

$$\mu(\vartheta_1) \geqq \frac{L^2(\Gamma_1, \varrho_1)}{A_1(\varrho_1)}$$

$$= \frac{[a + \varepsilon\, g(t_1)\, \lambda^{-1}(D-E)]^2}{a\, b + \varepsilon\, \lambda^{-1}|D-C| \int_{t_1}^{t_2} f\, g\, dt + \varepsilon\,(1+\varepsilon)\, g^2(t_1)\, \lambda^{-2}(D-E)\, |D-C|}$$

$$= \frac{a}{b}\, \frac{1 + 2\,\varepsilon\, g(t_1)\, \lambda^{-1}(D-E)\,/\,a}{1 + \frac{\varepsilon}{a\, b}\lambda^{-1}|D-C| \int_{t_1}^{t_2} f\, g\, dt + \frac{\varepsilon}{a\, b} g^2(t_1)\, \lambda^{-2}(D-E)\, |D-C|} + O(\varepsilon^2)$$

$$= \frac{a}{b}\Big[1 + \frac{\varepsilon\, \lambda^{-1}}{a\, b}\Big\{b\, g(t_1)\,(D-E) - |D-C| \int_{t_1}^{t_2} f\, g\, dt\Big\}$$

$$+ \frac{\varepsilon\, \lambda^{-1}}{a\, b} g(t_1)\,(D-E)\,\{b - g(t_1)\, \lambda^{-1}|D-C|\}\Big] + O(\varepsilon^2)\,.$$

Using again (7.17) and the fact that g is nonvanishing and strictly decreasing in $t \in [1\,, t_2\rangle$ we obtain

$$(7.21)\quad b\, g(t_1)\,(D-E) = \int_1^{t_1} f\, dt \cdot \lambda \int_{t_1}^{t_2} f(t) \frac{g(t_1)}{g(t)}\, dt > \int_1^{t_1} f\, dt \cdot \lambda \int_{t_1}^{t_2} f\, 1\, dt$$

$$= \lambda \int_1^{t_1} \frac{f(t)}{g(t_1)}\, dt \cdot \int_{t_1}^{t_2} f(t)\, g(t_1)\, dt > \lambda \int_1^{t_1} \frac{f}{g}\, dt \cdot \int_{t_1}^{t_2} f\, g\, dt$$

$$= |D-C| \cdot \int_{t_1}^{t_2} f\, g\, dt\,,$$

and

$$(7.22)\qquad b = \int_1^{t_1} f\, dt > \int_1^{t_1} f(t) \frac{g(t_1)}{g(t)}\, dt = g(t_1) \cdot \int_1^{t} \frac{f}{g}\, dt$$

$$= g(t_1)\, \lambda^{-1}|D-C|\,.$$

Using now (7.21) and (7.22) in the last expression of (7.20) and comparing with (7.9) we finally conclude for any sufficiently small $\varepsilon > 0$

$$\mu((1+\varepsilon)\,\vartheta) = \mu(\vartheta_1) > \frac{a}{b} = \mu(\vartheta)$$

where $\vartheta > 1$, thus Lemma 3 is proved.

8. Recalling the notation

$$S = \langle 0\,, M\rangle \times \langle 0\,, M\rangle\,,$$
$$S^+ = \{\,(x\,,y)\,;\ 0 < y < x < M\,\}\,,$$
$$S^- = \{\,(x\,,y)\,;\ 0 < x < y < M\,\}\,,$$
$$\Delta = \{\,(\mu_1\,,\mu_2)\,;\ 0 < \mu_1 + \mu_2 < M\,,\ \mu_1 > 0\,,\ \mu_2 > 0\,\}\,,$$

we have

Theorem 2. *Let* $M > 0$ *and let* $h: \bar{S} \to \boldsymbol{R}^2$ *be given by (5.1). Then*

(i) *h maps $\bar{S}$ onto $\bar{\Delta}$;*

(ii) *the restriction of h to each of the two sets $\bar{S}^+$ and $\bar{S}^-$ is a homeomorphism onto $\bar{\Delta}$, this homeomorphism is moreover sense-preserving on $\bar{S}^+$ and sense-reversing on $\bar{S}^-$.*

Proof. We shall prove the part of statement (ii) which concerns S^+, since in view of (5.3) this suffices to prove the theorem. We shall proceed in several steps.

I. *$h: \partial S^+ \to \partial\Delta$ is a homeomorphism which is sense-preserving.* On $\partial S^+ = [(0\,,0)\,,(M\,,0)] \cup [(M\,,0)\,,(M\,,M)] \cup [(M\,,M)\,,(0\,,0)]$ we have

(8.1) $$h(x\,,0) = (0\,,d(s_1 \cup [0\,,x]\,,s_2\,;Q)) \quad \text{for } x \in [0\,,M]\,,$$

(8.2) $$h(M\,,y) = (a_1(M\,,y)\,/\,b_1(M\,,y)\,,0) \quad \text{for } y \in [0\,,M]\,,$$

(8.3) $$h(x\,,x) = (x\,,M-x) \quad \text{for } x \in [0\,,M]\,.$$

(8.1) shows that h is injective on $[(0\,,0)\,,(M\,,0)]$ with $h(0\,,0) = (0\,,M)$ and $h(M\,,0) = [0\,,0]$, and the continuity of h implies that h maps $[(0\,,0)\,,(M\,,0)]$ homeomorphically onto $[(0\,,M)\,,(0\,,0)]$. The reasoning leading to (6.9) likewise shows $(\partial/\partial y)\,(a_1(M\,,y)\,/\,b_1(M\,,y)) > 0$ for $y \in \langle 0\,,M\rangle$ and one concludes from (8.2) that h maps $[M\,,0)\,,(M\,,M)]$ homeomorphically onto $[(0\,,0)\,,(M\,,0)]$. With (8.3) it is now immediate that $h: \partial S^+ \to \partial\Delta$ is a sense-preserving homeomorphism.

II. *h is injective on $\bar{S}^+$.* Since we know already that h is injective on ∂S^+ and since $h(x\,,y) \in \Delta$ for $(x\,,y) \in \bar{S}^+ - \partial S^+ = S^+$ we have just to show that h is injective on S^+.

Let therefore $(x_1\,,y_1)\,,\ (x_2\,,y_2) \in S^+$ be such that $h(x_1\,,y_1) = h(x_2\,,y_2)$. Putting $(\,j = 1, 2\,)$

$$(8.4)\qquad f_j(z) = \frac{1}{b_2(x_j, y_j)}\left[\int_i^z \left(\frac{1 - p(t)/p(y_j)}{1 - p(t)/p(x_j)}\right)^{1/2} dt + a_1(x_j, y_j)\right], \quad z \in Q,$$

and choosing that branch of the square root which has negative boundary values on $\langle 0, y_j \rangle$, we have in the notation (7.1)

$$(8.5)\qquad f_j(Q) = R(M_1^*(x_j, y_j) + M_2^*(x_j, y_j), M_2^*(x_j, y_j), \vartheta_j),$$

where

$$(8.6)\qquad \vartheta_j = b_1(x_j, y_j) / b_2(x_j, y_j) < 1,$$

the boundary points M, $M+i$, i, 0, y, x being mapped in this order to the points A, B, C, D, E, F of (7.4), replacing there (M, ξ, ϑ) by $(M_1^*(x_j, y_j) + M_2^*(x_j, y_j), M_2^*(x_j, y_j), \vartheta_j)$. From Lemma 3 we conclude $\vartheta_1 = \vartheta_2$, whence $f_2^{-1} \circ f_1$ is a conformal selfmap of Q keeping the boundary points 0, M, $M+i$, i individually fixed. Thus $f_2^{-1} \circ f_1$ is the identity I on Q, and we obtain

$$x_2 = f_2^{-1}(f_1(x_1)) = I(x_1) = x_1,$$
$$y_2 = f_2^{-1}(f_1(y_1)) = I(y_1) = y_1,$$

and $(x_1, y_1) = (x_2, y_2)$, showing that h is indeed injective on S^+.

III. $h: \bar{S}^+ \to \bar{\Delta}$ *is surjective.* Since $h: \partial S^+ \to \partial \Delta$ was already seen to be surjective, it remains to show that $h: S^+ \to \Delta$ is surjective. Let $w \in \Delta$ and assume that $w \notin h(S^+)$. Since $h(\partial S^+) \cap \Delta = \emptyset$ we have also $w \notin h(\bar{S}^+)$. If γ is the Jordan curve obtained by positive orientation of the boundary ∂S^+, the continuity of h on $\bar{S}^+$ yields by a standard homotopy argument that the winding number of $h(\gamma)$ with respect to w is zero, which by $h(\partial S^+) = \partial \Delta$ is obviously false. Thus indeed $h(\bar{S}^+) = \bar{\Delta}$.

IV. Since h is a continuous bijection of the compact set $\bar{S}^+$ onto $\bar{\Delta}$, h is a homeomorphism, and h is sense preserving since (by step I) it is sense preserving on ∂S^+, proving the Theorem.

9. Let $(x, y) \in \bar{S}$. $\sigma_{xy} > 0$ determines a directional field in Q prescribing at each point $z \in Q$ the direction

$$(9.1)\qquad \arg dz = -\frac{1}{2} \arg\left(-\frac{1 - p(z)/p(y)}{1 - p(z)/p(x)}\right) \mod \pi.$$

The direction (9.1) depends apparently continuously on all three variables $(x, y, z) \in [0, M] \times [0, M] \times Q$, and we showed that the pair $h = (M_1^*, M_2^*)$ not only is continuously depending on $(x, y) \in \bar{S}$, but that h is also a homeomorphism when y is restricted to $[0, x]$.

We want to extend this homeomorphism when the zero y of the quadratic differential σ_{xy} is no longer restricted to $[0, x]$ but varies on $\partial Q - \{x\}$. To this end we shall define a directional field in Q also for these values of y in a natural manner. The form (9.1), however, is not suitable since $p(y_0) = 0$ for some $y_0 \in [i, M+i] \cup [M, M+i]$ making (9.1) undefined for $y = y_0$ [(9.1) has a limit direction at each point $z \in Q$ for fixed x, when $y \to y_0$ on ∂Q from *one* side; the two limit directions thus obtained differ by $\pi/2 \bmod \pi$, though].

We put

$$(9.2) \qquad T = \{ (x, y) ; \ x \in \langle 0, M \rangle , \ y \in \partial Q - \{x\} \} ,$$

and for $(x, y) \in T$, $z \in \overline{Q}$ we define

$$(9.3) \quad \sigma^*_{xy} = -\frac{1 + p(x)}{1 + |p(y)|}\frac{p(z) - p(y)}{p(z) - p(x)} dz^2 , \quad \text{when } y \in \langle 0, x \rangle ,$$

$$= \frac{1 + p(x)}{p(z) - p(x)} dz^2 , \quad \text{when } y = 0 ,$$

$$= \frac{1 + p(x)}{1 + |p(y)|}\frac{p(z) - p(y)}{p(z) - p(x)} dz^2 , \quad \text{when } y \in \partial Q - [0, x] .$$

For fixed $x \in \langle 0, M \rangle$ the discontinuity of (9.3) in y is now placed at the point $y = x$ since

$$\lim_{y \to x-} \sigma^*_{xy} = - dz^2 = \sigma_{xx} , \quad \lim_{y \to x+} \sigma^*_{xy} = + dz^2 = - \sigma_{xx} ,$$

and comparing with (3.1), we have for $(x, y) \in \overline{S}^+ \cap T$

$$(9.4) \qquad \sigma^*_{xy} = \frac{1 + 1/p(x)}{1 + 1/p(y)} \sigma_{xy} .$$

Thus for $(x, y) \in \overline{S}^+ \cap T$, $\sigma^*_{xy} > 0$ and $\sigma_{xy} > 0$ have the same trajectories; using σ^*_{xy} instead of σ_{xy} in (4.3) and (4.4), the quantities (4.3) pick up the factor $(1 + 1/p(x)) / (1 + 1/p(y)) > 0$ while the quotient (4.4) remains unchanged. Moreover, the directional field defined in Q by $\sigma^*_{xy} > 0$ apparently depends continuously on $((x, y), z) \in T \times Q$. [$T$ is topologized in the natural manner, carrying over the product topology of $\langle 0, M \rangle \times \partial Q$.]

For $(x, y) \in T$ we denote by K^*_{xy} the closure of the union of the trajectories of $\sigma^*_{xy} > 0$ which have the limit point y. Thus first

$$(9a) \qquad K^*_{xy} = K_{xy} \quad \text{for} \quad (x, y) \in \overline{S}^+ \cap T = T_0 .$$

Letting

$$(9.5)\qquad \begin{aligned} T_1 &= \langle 0\,, M\rangle \times (\langle 0\,, i] \cup [i\,, M+i\rangle)\,, \\ T_2 &= (\langle 0\,, M\rangle \times [M\,, M+i\rangle) \cup S^-\,, \end{aligned}$$

and observing $T = T_0 \cup T_1 \cup (\langle 0\,, M\rangle \times \{M+i\}) \cup T_2$ we obtain in the other cases

$$(9b)\qquad K^*_{xy} = [y\,, i] \cup K'_{xy} \quad \text{for } (x\,, y) \in T_1\,,$$

$$(9c)\qquad K^*_{x,\,M+i} = [i\,, M+i] \cup [M\,, M+i] \quad \text{for } x \in \langle 0\,, M\rangle\,,$$

$$(9d)\qquad K^*_{xy} = [y\,, M] \cup K'_{xy} \quad \text{for } (x\,, y) \in T_2\,.$$

Here K'_{xy} is the carrier of an analytic arc with initial point y and terminal point (in case (9b)) on $\langle x\,, M\rangle$ or (in case (9d)) on $\langle 0\,, i\rangle$, the arc lying up to initial and terminal point in Q. $[M\,, M+i]$ is a trajectory of $\sigma^*_{xy} > 0$ for $(x\,, y) \in T_1$, while $[i\,, i+M]$ is a trajectory when $(x\,, y) \in T_2$; $[0\,, x]$ is the closure of a trajectory in all three cases (9b), (9c), (9d). Considering Q as a pentagon as in section 1, we have

$$(9.6)\qquad K^*_{xy} \in \mathfrak{K}([0\,, x]\,, [M\,, M+i]\,; i) \quad \text{for } (x\,, y) \in T_1\,,$$

$$(9.7)\qquad K^*_{xy} \in \mathfrak{K}([0\,, x]\,, [i\,, M+i]\,; M) \quad \text{for } (x\,, y) \in T_2\,,$$

leading to the cases II and III of section 2. We put therefore

$$(9.8)\qquad s_1(x\,, y) = [0\,, x] \quad \text{for } (x\,, y) \in T_1 \cup T_2\,,$$

$$(9.9)\qquad \begin{aligned} s_2(x\,, y) &= [M\,, M+i] \quad \text{for } (x\,, y) \in T_1\,, \\ &= [i\,, M+i] \quad \text{for } (x\,, y) \in T_2\,. \end{aligned}$$

10. We define now on T a function h^*, coinciding with h of (5.1) on S^+, in a natural way, obtaining results which are analogous to the Lemmata 1, 2, 3 and the Theorems 1, 2.

Observing (9.6) and (9.7) we put in the notation of section 1, using (9.8) and (9.9),

$$(10.1)\qquad \begin{aligned} D_1^*(x\,, y) &= D_1(s_1(x\,, y)\,, s_2(x\,, y)\,; K^*_{xy})\,, \quad (x\,, y) \in T_1 \cup T_2\,, \\ D_2^*(x\,, y) &= D_2(s_1(x\,, y)\,, s_2(x\,, y)\,; K^*_{xy})\,, \quad (x\,, y) \in T_1 \cup T_2\,. \end{aligned}$$

Again, $(j = 1, 2)$, $K^*_{xy} \cap \partial D_j^*$ is connected for $(x\,, y) \in T_1 \cup T_2$, so we put

$$(10.2)\qquad \begin{aligned} M_1^*(x\,, y) &= -\,d(s_1(x\,, y)\,, K^*_{xy} \cap \partial D_1^*(x\,, y)\,;\ D_1^*(x\,, y)) \\ &\qquad \text{for } (x\,, y) \in T_1 \cup T_2\,, \\ M_1^*(x\,, M+i) &= -\,d([0\,, x]\,, [i\,, M+i] \cup [M\,, M+i]\,; Q) \\ &\qquad \text{for } x \in \langle 0\,, M\rangle\,, \end{aligned}$$

$$(10.3)\quad \begin{aligned} M_2^*(x\,,y) &= d(s_2(x\,,y)\,,K_{xy}^* \cap \partial D_2^*(x\,,y)\,;\; D_2^*(x\,,y)) \\ &\qquad\qquad\qquad \text{for } (x\,,y) \in T_1\,, \\ M_2^*(x\,,M+i) &= 0 \qquad \text{for } x \in \langle 0\,,M\rangle\,, \\ &= -\,d(s_2(x\,,y)\,,K_{xy}^* \cap \partial D_2^*(x\,,y)\,;\; D_2^*(x\,,y)) \\ &\qquad\qquad\qquad \text{for } (x\,,y) \in T_2\,, \end{aligned}$$

and define h^* on T using (10.2), (10.3) and the notation of (9a):

$$(10.4)\quad h^*(x\,,y) = \begin{cases} h(x\,,y) & \text{for } (x\,,y) \in T_0\,, \\ (M_1^*(x\,,y)\,,M_2^*(x\,,y)) & \text{for } (x\,,y) \in T - T_0\,. \end{cases}$$

The quantities (10.2), (10.3) have expressions similar to (4.4): We have

$$(10.5)\quad \begin{aligned} M_1^*(x\,,y) &= \begin{cases} \dfrac{a_1^*(x\,,y)}{b_1^*(x\,,y)}\,, & (x\,,y) \in T - (\langle 0\,,M\rangle \times \{0\})\,, \\ 0\,, & (x\,,y) \in \langle 0\,,M\rangle \times \{0\}\,, \end{cases} \\ M_2^*(x\,,y) &= \begin{cases} \dfrac{a_2^*(x\,,y)}{b_2^*(x\,,y)}\,, & (x\,,y) \in T - (\langle 0\,,M\rangle \times \{M+i\})\,, \\ 0\,, & (x\,,y) \in \langle 0\,,M\rangle \times \{M+i\}\,, \end{cases} \end{aligned}$$

where $a_1^*\,,\ b_1^*\,,\ a_2^*\,,\ b_2^*$ are obtained from the table below in the

Table 1

y	a_1^*	b_1^*	a_2^*	b_2^*
$[x\,,0]$	$[0\,,y]$	$[0\,,i]$	$[x\,,M]$	$[M\,,M+i]$
0	0	—	$[x\,,M]$	$[M\,,M+i]$
$[0\,,i]$	$-[0\,,y]$	$[0\,,x]$	$[i\,,M+i]$	$[M\,,M+i]$
$[i\,,M+i]$	$-[0\,,i]$	$[0\,,x]$	$[y\,,M+i]$	$[M\,,M+i]$
$M+i$	$-[0\,,i]$	$[0\,,x]$	0	—
$[M+i\,,M]$	$-[x\,,M]$	$[0\,,x]$	$-[y\,,M+i]$	$[i\,,M+i]$
$[M\,,x]$	$-[x\,,y]$	$[0\,,x]$	$-[M\,,M+i]$	$[i\,,M+i]$

following way: if $x \in \langle 0\,,M\rangle$ and $y \in \langle \alpha\,,\beta\rangle$ when $[\alpha\,,\beta]$ is an entry in the first column, if $c \in \{a_1^*\,,b_1^*\,,a_2^*\,,b_2^*\}$, and if $[\gamma\,,\delta]$ or $-[\gamma\,,\delta]$ is the entry determined by $[\alpha\,,\beta]$ and c, then

$$(10.6)\quad c(x\,,y) = \int_\alpha^\beta |\sigma_{xy}^*|^{1/2} \quad \text{or} \quad -\int_\alpha^\beta |\sigma_{xy}^*|^{1/2}\,, \text{ integration on } [\alpha\,,\beta]\,;$$

the expression (10.6) is used also for $y \in \{\alpha, \beta\} - \{x\}$ excepting $c = b_1^*$, $y = 0$ and $c = b_2^*$, $y = M+i$, leaving (as indicated by dashes in Table 1) b_1^* undefined at $(x, 0)$ and b_2^* undefined at $(x, M+i)$; in each case, however, $\lim_{y\to\alpha} c(x, y)$ and $\lim_{y\to\beta} c(x, y)$ do exist and are finite, and we put in particular

$$(10.7)\quad \lim_{\substack{y\to 0\\ y>0}} b_1^*(x, y) = b_1^*(x, 0+) > 0, \qquad \lim_{\substack{y\to 0\\ iy<0}} b_1^*(x, y) = b_1^*(x, 0-) > 0,$$

$$(10.8)\quad \begin{aligned} \lim_{\substack{y\to M\\ y<M}} b_2^*(x, i+y) &= b_2^*(x, (M+i)+) > 0, \\ \lim_{\substack{y\to 1\\ y<1}} b_2^*(x, M+iy) &= b_2^*(x, (M+i)-) > 0. \end{aligned}$$

Though the intervals in the first column of Table 1 overlap, an application of Cauchy's Integral Theorem shows at once: a_1^* and a_2^* are well defined on T, b_1^* is well defined on $T - (\langle 0, M\rangle \times \{0\})$, b_2^* is well defined on $T - (\langle 0, M\rangle \times \{M+i\})$. We have e.g.

$$a_1^*(x, y) = -\int_0^y |\sigma_{xy}^*|^{1/2} \quad \text{for } (x, y) \in \langle 0, M\rangle \times [0, i].$$

Now we may write instead of (10.4)

$$(10.9)\quad h^*(x, y) = (M_1^*(x, y), M_2^*(x, y)) \quad \text{for } (x, y) \in T.$$

L e m m a 4. a) *The functions* a_1^*, b_1^*, a_2^*, b_2^* *are continuous on their respective domains of definition,* b_1^* *and* b_2^* *are positive, and we have for each* $x \in \langle 0, M\rangle$

$$(10.10)\quad \begin{aligned} b_1^*(x, 0+) + b_1^*(x, 0-) &= b_2^*(x, 0), \\ b_2^*(x, (M+i)+) + b_2^*(x, (M+i)-) &= b_1^*(x, M+i); \end{aligned}$$

b) *the functions* M_1^*, M_2^* *are continuous on* T.

The easy proof, similar to parts of the proof of Lemma 1, is omitted.

L e m m a 5. *Let* $x \in \langle 0, M\rangle$, *let* $s_1 = [0, x]$, $s_2 = [M, M+i]$ *and* $K \in \mathfrak{K}(s_1, s_2; i)$. *Then for any* $y \in \langle 0, i] \cup [i, M+i\rangle$ *we have the inequality*

$$(10.11)\quad \begin{aligned} &b_1^{*2}(x, y)\, M_1(s_1, s_2; K) + b_2^{*2}(x, y)\, M_2(s_1, s_2; K) \\ &\leqq b_1^{*2}(x, y)\,(-M_1^*(x, y)) + b_2^{*2}(x, y)\, M_2^*(x, y), \end{aligned}$$

with equality if and only if $K = K_{xy}^*$; *for* $y \in \{0, M+i\}$ *formula (10.11) holds with strict inequality when* $b_1^*(x, y)$ *and* $b_2^*(x, y)$ *are replaced by the corresponding limits.*

L e m m a 6. *Let* $x \in \langle 0 , M \rangle$, *let* $s_1 = [0 , x]$, $s_2 = [i , M+i]$ *and* $K \in \mathfrak{K}(s_1 , s_2 ; M)$. *Then for any* $y \in \langle M+i , M] \cup [M , x\rangle$ *we have the inequality*

$$(10.12) \qquad b_1^{*2}(x , y)\, M_1(s_1 , s_2 ; K) + b_2^{*2}(x , y)\, M_2(s_1 , s_2 ; K) \leqq b_1^{*2}(x , y)\,(- M_1^*(x , y)) + b_2^{*2}(x , y)\,(- M_2^*(x , y)) ,$$

with equality if and only if $K = K_{xy}$; *for* $y = M+i$, *formula (10.12) holds with strict inequality when* $b_2^*(x , y)$ *is replaced by* $b_2^*(x , (M+i)-)$; *we have also*

$$(10.13) \qquad x^2\, M_1(s_1 , s_2 ; K) + M^2\, M_2(s_1 , s_2 ; K) < M .$$

The proofs of Lemma 5 and Lemma 6 are analogous to the proof of Lemma 2; (10.13) is obtained from (10.12) using

$$(10.14) \qquad \lim_{\substack{y\to x\\ y>x}} b_1^*(x , y) = x , \qquad \lim_{\substack{y\to x\\ y>x}} b_2^*(x , y) = M ,$$

$$\lim_{\substack{y\to x\\ y>x}} M_1^*(x , y) = 0 , \qquad \lim_{\substack{y\to x\\ y>x}} M_2^*(x , y) = -\frac{1}{M} ,$$

where these latter relations follow easily from (9.3), (10.5), Table 1.

11. For $x \in \langle M , 0\rangle$ we use now the two-point compactification $\partial\hat{Q}_x$ of $Q - \{x\}$ by adding the two points

$$x- = \lim_{\substack{y\to x\\ y<x}} y \quad \text{and} \quad x+ = \lim_{\substack{y\to x\\ y>x}} y .$$

We put further

$$(11.1) \qquad \Delta_1 = \{ (\mu_1 , \mu_2) ; \ \mu_1 < 0 , \ 0 < \mu_2 < M \} ,$$

$$(11.2) \qquad \Delta_2 = \{ (\mu_1 , \mu_2) ; \ \mu_1 < 0 , \ - 1/M < \mu_2 < 0 \} ,$$

and with $\bar{\Delta}$ of (5.2),

$$(11.3) \qquad \bar{E} = \bar{\Delta} \cup \bar{\Delta}_1 \cup \bar{\Delta}_2 ,$$

$$(11.4) \qquad E = \text{interior of } \bar{E} .$$

Then we have in analogy to Theorem 2

T h e o r e m 3. *Let* $x \in \langle 0 , M \rangle$ *and let*

$$\partial\hat{Q}_x = (\partial Q - \{x\}) \cup (\{x-\} \cup \{x+\})$$

be the two-point compactification of $\partial Q - \{x\}$; *let* $h_x^* : \partial\hat{Q}_x \to \boldsymbol{R}^2$ *be the continuous function, satisfying*

$$h_x^*(y) \;=\; (M_1^*(x\,,y)\,,\,M_2^*(x\,,y)) \qquad \text{for } y\in\partial Q-\{x\}\,.$$

Then

(i) h_x^* *is injective* ;

(ii) $\Gamma_x^*=h_x^*(\partial\hat{Q}_x)$ *is a piecewise smooth curve in* $\bar{E}$, *the points* $h_x^*(0)$ *and* $h_x^*(M+i)$ *being the points at which the slope of* Γ_x^* *is discontinuous* ;

(iii) $h_x^*(\partial Q-\{x\})\subset E$, *in particular*

$$h_x^*(\langle 0\,,x\rangle)\subset\Delta\,,$$

$$h_x^*(\langle 0\,,i]\cup[i\,,M+i\rangle)\subset\Delta_1\,,$$

$$h_x^*(\langle M+i\,,M]\cup[M\,,x\rangle)\subset\Delta_2\,,$$

and $h_x(\partial\hat{Q}_x)$ *is a cross-cut of* E ;

(iv) *for* $y\in\langle x\,,0\rangle\cup\langle 0\,,i]\cup[i\,,M+i\rangle\cup\langle M+i\,,M]\cup[M\,,x\rangle$, Γ_x^* *has at* $h_x^*(y)$ *the slope*

(11.5) $\lambda_x^*(y)$

$$= -\,b_1^{*2}(x\,,y)\,/\,b_2^{*2}(x\,,y)\;<\;0\,,\quad y\in\langle x\,,0\rangle\cup\langle M+i\,,M]\cup[M\,,x\rangle\,,$$

$$= \;b_1^{*2}(x\,,y)\,/\,b_2^{*2}(x\,,y)\;>\;0\,,\quad y\in\langle 0\,,i]\cup[i\,,M+i\rangle\,;$$

(v) *the angle* $\varphi_x^*(y)\in\langle 0\,,2\pi\rangle$ *which the tangent-vector to* Γ_x^* , *oriented in the sense corresponding to the negative orientation of* ∂Q , *forms at* $h_x^*(y)$ *with the direction of the positive* μ_1 *-axis in the* $(\mu_1\,,\mu_2)$ *-plane is strictly increasing from*

$$\varphi_x^*(x-)\;=\;3\pi/4 \qquad \textit{at } h_x^*(x-)=(x\,,M-x)$$

to

$$\varphi_x^*(x+)\;=\;\arctan\lambda_x^*(x+)\in\langle 7\pi/4\,,2\pi\rangle \qquad \textit{at } h_x^*(x+)=(0\,,-1/M)\,,$$

over

$$\varphi_x^*(0+)\;=\;\arctan\lambda_x^*(0+)\in\langle 3\pi/4\,,\pi\rangle\,,$$

$$\varphi_x^*(0-)\;=\;\arctan\lambda_x^*(0-)\in\langle \pi\,,5\pi/4\rangle\,,$$

$$\varphi_x^*(i)\;=\;5\pi/4\,,$$

$$\varphi_x^*((M+i)+)\;=\;\arctan\lambda_x^*((M+i)+)\in\langle 5\pi/4\,,\;3\pi/2\rangle\,,$$

$$\varphi_x^*((M+i)-)\;=\;\arctan\lambda_x^*((M+i)-)\in\langle 3\pi/2\,,\;7\pi/4\rangle\,,$$

$$\varphi_x^*(M)\;=\;7\pi/4\,;$$

(vi) *the map* $x\to\varphi_x^*(x+)$ *is a decreasing homeomorphism of* $\langle 0\,,M\rangle$ *onto* $\langle 7\pi/4\,,\;2\pi\rangle$.

Proof. That h_x^* is continuous on $\partial Q-\{x\}$ with continuous extension to $\partial\hat{Q}_x$ follows from (10.5) with table 1, and one obtains in particular

$h_x^*(x+) = (0\,,\,-1/M)$. The restriction of the statements of Theorem 3 to $\langle 0\,,\,x-\rangle \in \partial\hat{Q}_x$ is contained in Theorem 1. The proof of Theorem 1 was based on Lemma 2 and the expression (4.4). Using now Lemma 5 for $y \in \langle 0\,,\,i] \cup [i\,,\,M+i\rangle$ and Lemma 6 for $y \in \langle M+i\,,\,M] \cup [M\,,\,x\rangle$ together with the expressions (10.5), a similar reasoning gives those statements of Theorem 3 which concern $y \in \langle 0\,,\,i] \cup [i\,,\,M+i\rangle \cup \langle M+i\,,\,M] \cup [M\,,\,x\rangle$. With $h_x^*(0) \in (\bar{\varDelta} \cup \bar{\varDelta}_1) \cap E$ and $h_x^*(M+i) \in (\bar{\varDelta}_1 \cup \bar{\varDelta}_2) \cap E$ the statements (i) through (v) of Theorem 3 follow. Finally, (10.14) and (11.5) give immediately statement (vi), completing the proof.

12. We recall the definition (9.2) of T and the topology of T . Denoting for $(x\,,\,y) \in T$ by $d_{xy} \in \langle 0\,,\,2+2\,M\rangle$ the distance of y from $x-$ along ∂Q , the map

$$(x\,,\,y) \rightarrow (x\,,\,x - d_{xy})\,, \quad (x\,,\,y) \in T\,,$$

is a homeomorphism of T onto the parallelogram

$$\tilde{T} = \{\,(x\,,\,y)\,;\; 0 < x < M\,,\; -2 - 2\,M + x < y < x\,\}\,;$$

the restriction of this homeomorphism to $S^+ \subset T$ is the identity, and we orient T by carrying over the usual positive orientation of the plane set $\tilde{T}$. Likewise we compactify T by carrying over the compactification of $\tilde{T}$; we call $\hat{T}$ this compactification of T (which corresponds to the compactification $\partial\hat{Q}_x$ of $Q - \{x\}$) . In the notation (10.9), (11.4) we have as an extension of Theorem 2:

T h e o r e m 4. *The map* $h^*: T \rightarrow E$ *is a sense-preserving homeomorphism.*

To prove Theorem 4, we adapt the proof of Theorem 2, minor changes are due to the fact that h^* cannot be extended homeomorphically to $\hat{T}$ (already $h^*(x\,,\,x+) = (0\,,\,-1/M)$ for each $x \in \langle 0\,,\,M\rangle$) . We proceed again in several steps.

I. $h^*(T) \subset E$ follows from Theorem 3 (iii).

II. h^* *is injective on* T . We shall indicate later how this can be seen.

III. $h^*: T \rightarrow E$ *is surjective.* We use the fact (Theorem 3) that for $x \in \langle 0\,,\,M\rangle$, Γ_x^* is a cross-cut of E joining the boundary points $(x\,,\,M-x)$ and $(0\,,\,-1/M)$. Γ_x^* decomposes E into two simple connected components, we denote by E_x that component which has the point $(0\,,\,0)$ on the boundary. Using II we conclude further: if $\xi \in \langle x\,,\,M\rangle$ then Γ_ξ^* is a cross-cut in E_x . So in order to prove that h^* takes the value $\mu = (\mu_1\,,\,\mu_2) \in E$, it suffices, by a standard homotopy-argument, to show that there exist $x \in \langle 0\,,\,M\rangle$ and $\xi \in \langle x\,,\,M\rangle$ such that $\mu \in E_x$ and that Γ_ξ^* separates

μ from $(0, 0)$ in E_x, since, as is easily seen, h^* has a continuous extension to the closure $\overline{T}_{x\xi}$ of

$$(12.1) \qquad T_{x\xi} = \{ (t, y) ; \ t \in \langle x, \xi \rangle, \ y \in \partial Q - \{t\} \}$$

in $\hat{T}$.

Assume first $\mu = (\mu_1, \mu_2) \in E$ satisfies $\mu_1 < 0$, $\mu_2 \geqq 0$. From Theorem 3 it follows that $\mu \in E_x$ once there exists $y \in \langle 0, i] \cup [i, M+i]$ such that $M_1^*(x, y) < \mu_1$, $M_2^*(x, y) > \mu_2$. To find such a pair (x, y) we choose first $y \in \langle 0, i]$ such that

$$(12.2) \quad \mu_2 < \int_i^{M+i} \left| \frac{1 - p(z)/p(y)}{1 + 1/|p(y)|} \right|^{1/2} |dz| \Big/ \int_M^{M+i} \left| \frac{1 - p(z)/p(y)}{1 + 1/|p(y)|} \right|^{1/2} |dz| .$$

This is possible since the quotient in (12.2) tends to M for $y \to 0$, $y \in \langle 0, i]$, and since $\mu_2 < M$ for $\mu \in E$. From (9.3) we have then for $x \in \langle 0, M \rangle$

$$\sigma_{xy}^* = \frac{1 + 1/p(x)}{1 + 1/|p(y)|} \frac{1 - p(z)/p(y)}{p(z)/p(x) - 1} dz^2 ,$$

and comparison with (12.2) yields: there exists $x_0 \in \langle 0, M \rangle$ such that $M_2^*(x, y) > \mu_2$ once $x \in \langle 0, x_0 \rangle$, since $|\sigma_{xy}^*|^{1/2}$ tends uniformly to the integrands in (12.2) for $x \to 0$. Using the Laurent-series of $p(z)$ near $z = 0$ it is easily seen that

$$b_1^*(x, y) = \int_0^x |\sigma_{xy}|^{1/2}$$

is bounded for fixed y and $x \in \langle 0, x_0 \rangle$, while

$$a_1^*(x, y) = - \int_0^y |\sigma_{xy}^*|^{1/2}$$

tends to $-\infty$ when $x \to 0$, $x > 0$. Thus for suitable $x \in \langle 0, x_0 \rangle$ we have $M_1^*(x, y) < \mu_1$ and $M_2^*(x, y) > \mu_2$ as desired, and $\mu \in E_x$. If now $\xi \in \langle x, M \rangle$ is such that $\mu \notin \overline{E}_\xi$ then Γ_ξ^* separates μ from $(0, 0)$ in E_x. By Theorem 3 again $\mu \notin \overline{E}_\xi$ certainly holds when $M_1^*(\xi, M+i) > \mu_1$ since $M_2^*(\xi, M+i) = 0$, and (observing $\mu_1 < 0$) this is by (10.2) obviously the case when ξ is sufficiently close to M. Similar considerations in the other cases ($\mu_1 \geqq 0$ or equivalently $\mu \in \overline{\Delta} \cap E$ already covered in the proof of Theorem 2, and $\mu_2 < 0$ or equivalently $\mu \in \Delta_2$) show again the existence of $x \in \langle 0, M \rangle$ and $\xi \in \langle x, M \rangle$ so that $\mu \in E_x$

and that Γ_ξ^* separates μ from $(0, 0)$ in E_x. So any value $\mu \in E$ is indeed taken by h^* in T.

IV. $h^* : T \to E$ *is a homeomorphism.* To see that it suffices now to show that h^* is an open mapping. We show therefore: if (x_0, y_0) is any point of T and if G is a sufficiently small open set containing (x_0, y_0) then $h^*(G)$ is open. We choose x, ξ so that $0 < x < x_0 < \xi < M$ and that Γ_ξ^* separates $\mu = h^*(x_0, y_0)$ from $(0, 0)$ in E_x. The cross-cut Γ_ξ^* decomposes E_x into two components, we call $E_{x\xi}$ that component which has Γ_x^* on the boundary, and we note

$$\partial E_{x\xi} = [(\xi, M-\xi), (x, M-x)] \cup \Gamma_x^* \cup \Gamma_\xi^* .$$

We have $\mu \in E_{x\xi}$, and the continuous extension of h^* to $\bar{T}_{x\xi}$ satisfies $h^*(t, t-) = (t, M-t)$ and $h^*(t, t+) = (0, -1/M)$, for $t \in [x, \xi]$. The homotopy-argument of III then implies

$$E_{x\xi} = h^*(T_{x\xi}) . \tag{12.3}$$

Since $h^*(G) \subset E_{x\xi}$ once G is sufficiently small, by (12.3), we have

$$h^*(G) = E_{x\xi} \cap [R^2 - h^*(\bar{T}_{x\xi} - G)] \tag{12.4}$$

and since $h^*(\bar{T}_{x\xi} - G)$ is compact, $h^*(G)$ is by (12.4) clearly open. So Theorem 4 is proved once it is shown that $h^* : T \to E$ is injective, which fact will be dealt with now.

13. In section 8, the proof that the function h is injective on $\bar{S}^+$, did use Lemma 3. By similar reasoning it is shown that $h^* : T_1 \to \Delta_1$ is injective by the use of

L e m m a 7. *Let* M, ξ, ϑ, $R(\vartheta)$ *be as in Lemma 3; let*

$$\begin{aligned} \mu_1(\vartheta) &= d([-\xi+i, i], [\vartheta(M-\xi), \vartheta(M-\xi+i)]; R(\vartheta)) \quad \textit{for } 0 < \vartheta \leqq 1, \\ &= d([-\xi+i, i] \cup [i, i\vartheta], [\vartheta(M-\xi), \vartheta(M-\xi+i)]; R(\vartheta)) \\ &\qquad \textit{for } 1 < \vartheta < \infty . \end{aligned}$$

Then μ_1 *is strictly decreasing for* $\vartheta \in \langle 0, \infty \rangle$.

Concerning the proof of Lemma 7, the following remarks may suffice. For $\vartheta \in \langle 0, 1]$, the Lemma can be proved by the method used in the proof of Lemma 3; the same method can be used to show that the conjugate extremal distance

$$\frac{1}{\mu_1(\vartheta)} = d([-\xi, -\xi+i] \cup [-\xi, \vartheta(M-\xi)], [\vartheta i, \vartheta(M-\xi+i)]; R(\vartheta))$$

is strictly increasing for $\vartheta \in [1, \infty\rangle$.

In a similar way it is proved that $h^*: T_2 \rightarrow \Delta_2$ is injective. Since in the notation of section 9, $T = T_0 \cup T_1 \cup (\langle 0, M\rangle \times \{M+i\}) \cup T_2$, since h^* was already seen to be injective on T_0 (Theorem 2), since the sets $h^*(T_0)$, $h^*(T_1)$, $h^*(\langle 0,M\rangle \times \{M+i\})$, $h^*(T_2)$ all are disjoint, and since by (10.2) h^* is clearly injective on $\langle 0, M\rangle \times \{M+i\}$, $h^*: T \rightarrow E$ is indeed injective, completing the proof of Theorem 4.

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