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AREA METHOD AND UNIVALENT FUNCTIONS WITH QUASICONFORMAL EXTENSIONS

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1. Introduction

For a function $z \mapsto z + \sum b_n z^{-n}$, univalent for $|z| > 1$, the area of the set of its omitted values can be easily expressed in terms of the coefficients b_n. The obvious fact that this area is non-negative leads to the area theorem

$$\sum_{n=1}^{\infty} n\,|b_n|^2 \leqq 1 .$$

Let S be the class of functions f, holomorphic and univalent in the unit disc D, and so normalized that

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n . \tag{1.1}$$

Rolf Nevanlinna [4] was the first to observe that some basic properties of functions $f \in S$ can be derived from the area theorem in a very simple manner: not only does the inequality $|a_2| \leqq 2$ follow immediately, but straightforward integration yields sharp bounds for $|f(z)|$ and $|f'(z)|$.

Our aim in this paper is to show that the area method and its modifications lead quite easily to general inequalities for univalent functions with quasiconformal extensions. In Section 2 we first give a (known) generalization of the area theorem for class S. A sharp version of it is derived in Section 3, by means of Schwarz's lemma, for the subclasses of S whose functions admit quasiconformal extensions with uniformly bounded maximal dilatations. Some of its consequences are discussed in Sections 4 and 5.

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2. Power inequality in S

For a function $f \in S$ we write

$$[f(z)]^{-\nu} = \sum_{n=-\nu}^{\infty} b_{n\nu} z^n , \qquad \nu = 1, 2, \dots . \tag{2.1}$$

Owing to normalization, the first coefficients $b_{-\nu\nu}$ are equal to 1. Choose complex parameters $x_1, x_2, \dots, x_N$ and denote

$$F(z) = \sum_{\nu=1}^{N} x_\nu [f(z)]^{-\nu} .$$

Then

$$F(z) = \sum_{n=-N}^{\infty} y_n z^n \tag{2.2}$$

with

$$y_n = \sum_{\nu=1}^{N} b_{n\nu} x_\nu , \tag{2.3}$$

if we set $b_{n\nu} = 0$ for $\nu < -N$.

For a positive $\varrho < 1$, direct computation gives

$$\int_{|z|=\varrho} \bar{F}(z)\, dF(z) = 2\pi i \sum_{n=-N}^{\infty} n\, |y_n|^2 \varrho^{2n} . \tag{2.4}$$

Application of Green's formula shows that

$$i \int_{|z|=\varrho} \bar{F}(z)\, dF(z) \geqq 0 .$$

Hence, letting $\varrho \to 1$ we obtain from (2.4) the "Power inequality"

$$\sum_{n=1}^{\infty} n\, |y_n|^2 \leqq \sum_{n=1}^{N} n\, |y_{-n}|^2 \tag{2.5}$$

(Schiffer – Tammi [6], Ahlfors [1]).

This formula contains some well-known inequalities as special cases. For $N = 1$, (2.5) reduces to the classical area theorem

$$\sum_{n=1}^{\infty} n\, |b_{n1}|^2 \leqq 1 . \tag{2.6}$$

The Grunsky inequalities also follow from (2.5) with a suitable choice of the parameters x_ν.

3. Power inequality in S_k

Let S_k, $0 < k < 1$, be the class of quasiconformal homeomorphisms f of the plane for which $f \mid D \in S$ and whose complex dilatation μ satisfies the condition $||\mu||_\infty \leqq k$. The subclass of S_k whose functions map the point ζ to infinity is denoted by $S_k(\zeta)$. The class $S_0(\zeta)$ contains the single element

$$z \mapsto z(1 - z / \zeta)^{-1} . \tag{3.1}$$

Suppose μ is a measurable function which satisfies $||\mu||_\infty < k$ and vanishes in D. If ζ and w are given complex numbers, $|\zeta| \geqq 1$, $|w| < 1$, then there is a unique mapping $f(\ , w)$ of class $S_{|w|}(\zeta)$ with complex dilatation $w \mu / k$. We denote by $a_n(w)$, $b_{n\nu}(w)$ and $y_n(w)$ the coefficients of $f(\ , w) \mid D$, defined by (1.1), (2.1) and (2.2).

T h e o r e m 1. *If* $f \in S_k$, *then*

$$\sum_{n=1}^{\infty} n \, |y_n|^2 \leqq k^2 \sum_{n=1}^{N} n \, |y_{-n}|^2 . \tag{3.2}$$

Proof. Let μ be the complex dilatation of f, and $f(\ , w)$ the mapping with complex dilatation $w \mu / k$ which belongs to class $S_{|w|}(f^{-1}(\infty))$. Then $w \mapsto f(z, w)$ is defined in the unit disc, and $f = f(\ , k)$.

The $N \times N$ determinant whose rows and columns consist of the coefficients $b_{-n\nu}$ of f, $n, \nu = 1, 2, \dots, N$, is equal to 1. In view of (2.3), we can thus associate with $f(\ , w)$ the parameters x_ν so that $y_{-n}(w) = y_{-n}$, $n = 1, 2, \dots, N$, for each $w \in D$.

The functions $w \to a_n(w)$ are known to be holomorphic in D ([3]). Thus every $w \mapsto b_{n\nu}(w)$, being a polynomial of the coefficients $a_j(w)$, is holomorphic. By formula (2.3), the same is true of $w \mapsto y_n(w)$.

From (3.1) we see that $b_{n\nu}(0) = 0$ for all positive values of n. Hence, by (2.3),

$$y_n(0) = 0 , \quad n = 1, 2, \dots . \tag{3.3}$$

Set $\lambda_n = |y_n|^2 / y_n^2$, $n = 1, 2, \dots$, if $y_n \neq 0$; otherwise $\lambda_n = 1$. Having fixed a natural number M, we consider the function ψ defined by

$$\psi(w) = \sum_{n=1}^{M} n \, \lambda_n (y_n(w))^2 .$$

It is holomorphic in the unit disc, and by (3.3) has a double zero at the origin. From (2.5) it follows that

$$|\psi(w)| \leqq \sum_{n=1}^{N} n \, |y_{-n}|^2 .$$

Hence, applying Schwarz's lemma to the function $w \mapsto \psi(w) / w$ we obtain

$$|\psi(w)| \leqq |w|^2 \sum_{n=1}^{N} n \, |y_{-n}|^2 .$$

For $w = k$ and $M \to \infty$ this yields (3.2).

Setting $u_n = n \, y_{-n}$ and applying Schwarz's inequality to (3.2) we conclude that

$$\left| \sum_{n=1}^{N} u_n \, y_n \right| \leqq k \sum_{n=1}^{N} |u_n|^2 / n . \tag{3.4}$$

4. Functional $a_2^2 - a_3$

An immediate consequence of Theorem 1 is the (known) area theorem for S_k:

$$\sum_{n=1}^{\infty} n \, |b_{n1}|^2 \leqq k^2 .$$

Since $b_{11} = a_2^2 - a_3$, it follows that in S_k

$$|a_2^2 - a_3| \leqq k . \tag{4.1}$$

Equality can hold only if $b_{n1} = 0$ for $n > 1$. Then

$$f(z) = z(1 - a_2 \, z + k \, e^{i\vartheta} \, z^2)^{-1} , \quad z \in D . \tag{4.2}$$

This function is univalent. It is holomorphic if and only if $k \, e^{i\vartheta} \, z^2 - a_2 \, z + 1 \neq 0$ in D. This is equivalent to the condition

$$a_2 \in E_\vartheta , \tag{4.3}$$

where E_ϑ is the closed ellipse onto whose exterior $z \mapsto 1 / z + k \, e^{i\vartheta} \, z$ maps D. Condition (4.3) implies

$$|a_2| \leqq 1 + k . \tag{4.4}$$

If this inequality holds, there is at least one ϑ for which (4.3) is true.

Suppose (4.3) is fulfilled. Then (4.2), together with

$$f(z) = z \, \bar{z}(\bar{z} - a_2 \, z \, \bar{z} + k \, e^{i\vartheta} \, z)^{-1} , \quad |z| \geqq 1 , \tag{4.5}$$

defines an element of S_k. By a result of Strebel [7], (4.5) is the only extension of (4.2) with this property. Hence, equality holds in (4.1) if and only if f is defined by (4.2) and (4.5), and condition (4.3) is satisfied.

The restriction (4.3) (or (4.4)) is not void if $0 < k < 1$: Schiffer and Schober [5] have proved that

$$\max_{S_k} |a_2| = 2 - 4 \, \varkappa^2 , \quad \varkappa = (\operatorname{arc} \cos k) / \pi .$$

Since this is greater than $1+k$ for $0 < k < 1$, we have $\max |a_2^2 - a_3| = k$ for $0 \leqq |a_2| \leqq 1+k$, while $|a_2^2 - a_3| < k$ for $1+k < |a_2| \leqq 2 - 4\varkappa^2$. Because the functions maximizing $|a_2|$ are unique up to trivial rotations, direct computation gives $|a_2^2 - a_3| = 1 - 16\varkappa^2/3 + 16\varkappa^4/3$ for $|a_2| = 2 - 4\varkappa^2$.

In the subclass $S_k(\zeta)$, the equations (4.2) and (4.5) define an extremal function, provided that $|a_2| \leqq (1+k)/|\zeta|$. In particular, for $\zeta = \infty$ we have $\max |a_2^2 - a_3| = k$ if and only if $a_2 = 0$. The maximum value of $|a_2|$ is $2k$, the corresponding functions f being defined by $f(z) = z(1 + k e^{i\vartheta} z)^{-2}$ in D. Consequently, $|a_2^2 - a_3| = k^2$ for $|a_2| = 2k$.

5. Coefficient a_4

Let f belong to $S_k(\infty)$ and have the power series coefficients a_n. The function φ, defined by $\varphi(z) = (f(z^2))^{1/2}$, is then also in $S_k(\infty)$. The standard way to estimate a_4 is to apply (3.4) to φ, with the choice $N = 3$, $u_1 = u$, $u_2 = 0$, $u_3 = 1$. It follows that

$$(5.1) \qquad \begin{aligned} &| a_4/2 - a_2 a_3 + 13 a_2^3/24 + a_2 u^2/2 + (a_3 - 3 a_2^2/4) u | \\ &\leqq k(1/3 + |u|^2). \end{aligned}$$

In estimating $|a_4|$ we can suppose, without loss of generality, that a_4 is positive. Choose $u = a_2$. Since $|a_2| \leqq 2k$, the inequality (5.1) then yields (Kühnau [2])

$$|a_4| \leqq 2k/3 + 38k^3/3.$$

This estimate is asymptotically correct as $k \to 0$, but becomes quite inaccurate as $k \to 1$. Kühnau [2] has proved that

$$(5.2) \qquad |a_4| \leqq 2k/3 + 10k^3/3$$

if $k \geqq (7/15)^{1/2}$. For $k = 1$, this gives the sharp estimate $|a_4| \leqq 4$.

Using (5.1) we shall show that if the coefficients a_2, a_3, a_4 are real, then (5.2) remains valid for $k \geqq 0.41$. Again we can assume that $a_4 > 0$. If (5.1) is written in the form $a u^2 + 2 b u + c \leqq 0$, one sees that $u = -b/a$ is an optimal choice. This gives the inequality

$$(5.3) \qquad a_4 \leqq 2k/3 + d,$$

with

$$d = 2 a_2 a_3 - 13 a_2^3/12 - (a_3 - 3 a_2^2/4)^2 (2k - a_2)^{-1}.$$

Wanting to establish (5.2) we can exclude the case $a_2 = 2k$ (then $a_4 = 4k^3$). Hence $2k - a_2 > 0$.

Rearranging the terms we obtain

$$d = 2\,k\,a_2^2 - 7\,a_2^3\,/\,12 - (a_3 + a_2^2\,/\,4 - 2\,k\,a_2)^2\,(2\,k - a_2)^{-1}\,.$$

With a_2 fixed and a_3 variable, d attains its maximum $M(a_2) = 2\,k\,a_2^2 - 7\,a_2^3\,/\,12$ for

$$a_3 = -a_2^2\,/\,4 + 2\,k\,a_2\,. \tag{5.4}$$

We observe that

$$M(2\,k) = 10\,k^3\,/\,3\,. \tag{5.5}$$

Condition $|a_2^2 - a_3| \leqq k$, coupled with (5.4), yields

$$a_2 \geqq h = 4\,k\,/\,5 - 2(4\,k^2 + 5\,k)^{1/2}\,/\,5\,.$$

For a fixed a_2 satisfying

$$-2\,k \leqq a_2 \leqq h\,, \tag{5.6}$$

we have $a_3 + a_2^2/4 - 2\,k\,a_2 \geqq 0$ if $a_3 \geqq -k + a_2^2$. Hence, on the interval (5.6) the choice $a_3 = -k + a_2^2$ gives an upper bound for d, i.e.

$$d \leqq 2\,k\,a_2^2 - 7\,a_2^3/12 - (5\,a_2^2/4 - 2\,k\,a_2 - k)^2\,(2\,k - a_2)^{-1}\,. \tag{5.7}$$

Thus far, the computations have been easily carried out. It remains to determine the maximum value of the majorant in (5.7) for $-2\,k \leqq a_2 \leqq h$. We were glad to leave it to the computer to show that the maximum does not exceed $10\,k^3\,/\,3$ if $k \geqq 0.41$. In view of (5.3) and (5.5), we thus obtain the desired estimate (5.2).

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