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NON-UNIQUENESS FOR PLATEAU'S PROBLEM. A BIFURCATION PROCESS

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The discussion to follow will deal with aspects of Plateau's problem about which no exhaustive information is available today, and particular attention will be paid to the phenomenon of non-uniqueness. It is well known that the area of a minimal surface, its suggestive name notwithstanding, need not furnish a minimum (absolute or relative) among the areas of all surfaces having the same boundary. Let us consider a minimal surface $S = \{ \mathfrak{x} = \mathfrak{x}(u, v) ; u^2 + v^2 < R^2 \}$ which lies imbedded in Euclidean 3-space. We shall assume that u and v are isothermal parameters on S so that $\mathfrak{x}_u^2(u, v) = \mathfrak{x}_v^2(u, v) \equiv E(u, v) > 0$ and $\mathfrak{x}_u(u, v)\, \mathfrak{x}(u, v) \equiv F(u, v) = 0$. Let us further denote by $\mathfrak{X}(u, v)$ the unit normal vector of our surface and by $K(u, v)$ its Gaussian curvature. On S we consider the one-parameter family of Jordan curves

$$\Gamma_r = \{ \mathfrak{x} = \mathfrak{x}(r \cos \vartheta, r \sin \vartheta) ; 0 \leqq \vartheta \leqq 2\pi \}, \quad 0 < r < R,$$

which bound expanding portions

$$S(r) = \{ \mathfrak{x} = \mathfrak{x}(u, v) ; (u, v) \in \bar{P}_r \}$$

of S. Here $\bar{P}_r$ is used as an abbreviation for the closure of the disc $P_r = \{ u, v ; u^2 + v^2 < r^2 \}$. For sufficiently small values of r two assertions are valid:

i) $S(r)$ is the unique solution of Plateau's problem for Γ_r.

ii) The area $A[S(r)]$ of $S(r)$ furnishes the absolute minimum among the areas of all disc-type surfaces bounded by Γ_r.

As r increases, either one of these statements may become false. As for i), $S(r)$ will remain unique as long as Γ_r retains a simply covered convex curve as its parallel, or central, projection on a plane (see [10]); or as long as the total curvature of Γ_r does not exceed the value 4π (see [7]). On the other hand, the minimizing property of the surface $S(r)$ — more precisely, the question whether $A[S(r)]$ remains at least a relative

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minimum – depends on the second variation of its area which, in turn, is closely related to the eigenvalue problem

$$(1) \qquad \Delta\,\xi - 2\,\lambda\,E\,K\,\xi = 0 \quad \text{in} \quad P_r\,, \quad \xi = 0 \quad \text{on} \quad \partial P_r\,.$$

Because $K(u\,,v)$ is non-positive and vanishes at most in isolated points, this problem will have a sequence of eigenvalues $\{\lambda_n(r)\}$ satisfying the inequalities $0 < \lambda_1(r) < \lambda_2(r) \leqq \lambda_3(r) \leqq \dots$ and corresponding eigenfunctions $\xi_1(u\,,v\,;r)\,,\ \xi_2(u\,,v\,;r)\,,\ \dots$ subject to the ortho-normality relations

$$2 \iint\limits_{P_r} E\,|K|\,\xi_m\,\xi_n\,du\,dv = \delta_{mn}\,.$$

The smallest eigenvalue $\lambda_1(r)$ is a continuous and strictly decreasing function of r.

If $\eta(u\,,v) = \sum_{n=1}^{\infty} c_n\,\xi_n(u\,,v\,;r)$ denotes an arbitrary regular function, vanishing on ∂P_r, then the area of the comparison surface $S^{(\varepsilon)}(r) = \{\,\mathfrak{r} = \mathfrak{r}(u\,,v) + \varepsilon\,\eta(u\,,v)\,\mathfrak{X}(u\,,v)\,;\ (u\,,v) \in \bar{P}_r\,\}$ is found to be

$$\begin{aligned} A[S^{(\varepsilon)}(r)] &= A[S(r)] + \tfrac{1}{2}\,\varepsilon^2 \iint\limits_{P_r} (\eta_u^2 + \eta_v^2 + 2\,E\,K\,\eta^2)\,du\,dv + O(\varepsilon^3) \\ &= A[S(r)] + \tfrac{1}{2}\,\varepsilon^2 \sum_{n=1}^{\infty} (\lambda_n(r) - 1)\,c_n^2 + O(\varepsilon^3) \\ &\geqq A[S(r)] + \varepsilon^2\,(\lambda_1(r) - 1) \iint\limits_{P_r} E\,|K|\,\eta^2\,du\,dv + O(\varepsilon^3)\,. \end{aligned}$$

If $\lambda_1(r) > 1$, then $S(r)$ can be imbedded in a field of minimal surfaces (a detailed construction can be found in [7]), and by a classical argument going back to H. A. Schwarz ([11], pp. 224–240, 332–334) it is seen that the area of $S(r)$ furnishes a strong relative minimum. Moreover, since the boundary value problem $\Delta\,\eta - 2\,E\,K\,\eta = 0$ in P_r, $\eta = 0$ on ∂P_r under the assumption $\lambda_1(r) > 1$ has the unique solution $\eta(u\,,v) \equiv 0$, it follows (see e.g. [2], Lemma 3.4) that $S(r)$ is isolated or, as we shall also say, locally unique; see [8]. By [1] the inequality $\lambda_1(r)$ is guaranteed at least as long as the total curvature $\iint_{P_r} E\,|K|\,du\,dv$ of $S(r)$ remains smaller than 2π.

If $\lambda_1(r) < 1$, then (by choosing $\eta(u\,,v) = \xi_1(u\,,v\,;r)$) disc-type surfaces bounded by Γ_r and lying arbitrarily close to $S(r)$ can be constructed whose area is smaller than the area of $S(r)$. $S(r)$ is thus seen to be unstable. It is a matter of record, however, that Γ_r also bounds a solution surface of least area for Plateau's problem. Obviously, this surface must be distinct from $S(r)$. Whether it is close to $S(r)$ or not is, of course, another question. In any case, Γ_r bounds at least two solutions of Plateau's problem whenever $\lambda_1(r) < 1$.

The preceding remarks suggest the possible existence of two positive numbers r_1 and r_2 defined as follows: r_1 is the supremum of all those values r for which the surface $S(r)$ represents the unique solution of Plateau's problem for Γ_r. The number r_2 is the supremum of all those values r for which the area of $S(r)$ is a strong relative minimum among the areas of all disc-type surfaces bounded by Γ_r, i.e., $\lambda_1(r_2) = 1$. Obviously, the inequalities $0 < r_1 \leqq r_2$ hold. Since it is not clear whether the local uniqueness of $S(r)$ for $\lambda_1(r) > 1$ implies its global uniqueness, the strict inequality $r_1 < r_2$ cannot be excluded apriori. Our interest here is directed not so much to this question but rather to the study of the *bifurcation process* in which a second or more surfaces branch off from $S(r)$ as r passes through the critical value $r = r_2$. Considering that the treasure of general insights is still hidden today, we shall pursue this bifurcation process in detail for the case of an explicit example – Enneper's minimal surface whose position vector is given with the help of isothermal parameters u and v by

$$\mathfrak{x}(u, v) = \begin{Bmatrix} u + u v^2 - \frac{1}{3} u^3 \\ - v - u^2 v + \frac{1}{3} v^3 \\ u^2 - v^2 \end{Bmatrix}.$$

We have $E(u, v) = (1+u^2+v^2)^2$ and $K(u, v) = -4(1+u^2+v^2)^{-4}$. It is known (see [9], § 90) that the part of Enneper's surface which corresponds to the disc $u^2 + v^2 < 3$ is free of self-intersections. In particular, the image of every concentric circle $u^2 + v^2 = r^2$, $0 < r < \sqrt{3}$, is a Jordan curve $\Gamma_r = \{\mathfrak{x} = \mathfrak{z}(\vartheta; r);\ 0 \leqq \vartheta \leqq 2\pi\}$ with the position vector

$$\mathfrak{z}(\vartheta; r) = \mathfrak{x}(r \cos\vartheta, r \sin\vartheta) = \begin{Bmatrix} r\cos\vartheta - \frac{1}{3} r^3 \cos 3\vartheta \\ - r \sin\vartheta - \frac{1}{3} r^3 \sin 3\vartheta \\ r^2 \cos 2\vartheta \end{Bmatrix}.$$

Pictures of Γ_r can be found in [9], pp. 76, 79. The points ($u = \pm\sqrt{3}$, $v = 0$) on the circle $u^2 + v^2 = 3$ are mapped by $\mathfrak{x}(u, v)$ into the same point $\{0, 0, 3\}$ in space. Accordingly, we shall choose $R = \sqrt{3}$.

The curvature of the projection γ_r of Γ_r onto the (x, y)-plane is given by

$$\frac{(1+r^2)(1-3r^2) + 4r^2 \sin^2 2\vartheta}{r[1 + 2r^2 \cos 4\vartheta + r^4]^{3/2}}.$$

γ_r is a convex curve for $0 < r < 1/\sqrt{3}$ so that $S(r)$ is the unique solution of Plateau's problem for Γ_r if $0 < r \leqq 1/\sqrt{3}$. The second criterion for uniqueness mentioned above leads to a better result. A computation shows that the total curvature $\varkappa(\Gamma_r)$ of Γ_r is equal to

$$\varkappa(\Gamma_r) = \frac{8r}{1+r^2}\frac{1}{k(r)}\mathsf{E}(k(r)), \qquad k(r) = \frac{2r}{\sqrt{1+10r^2+9r^4}}$$

and that

$$\frac{d}{dr}\varkappa(\Gamma_r) \geqq 6\pi\frac{k(r)}{1+r^2}.$$

Here $\mathsf{E}(k) = \int_0^{\pi/2}\sqrt{1-k^2\sin^2\vartheta}\,d\vartheta$ denotes the complete elliptic integral of the second kind. We have $\varkappa(\Gamma_r) = 4\sqrt{3}\,\mathsf{E}(1/2) = 10.167... < 4\pi$ for $r = 1/\sqrt{3}$ and $\varkappa(\Gamma_r) = 4\sqrt{5}\,\mathsf{E}(1/\sqrt{5}) = 13.369... > 4\pi$ for $r = 1$. The equality $\varkappa(\Gamma_r) = 4\pi$ is achieved for $r = 0.882$ (see [9], § 828). It is therefore certain that $r_1 \geqq 0.882$, although we conjecture that $r_1 = 1$.

The function $\zeta(u, v) = (1-u^2-v^2)/(1+u^2+v^2)$, which is positive for $u^2+v^2 < 1$ and zero for $u^2+v^2 = 1$, is a solution of the differential equation

$$\Delta\zeta - 2EK\zeta \equiv \Delta\zeta + \frac{8}{(1+u^2+v^2)^2}\zeta = 0.$$

From this fact it can be concluded that $\lambda_1(1) = 1$ so that $r_2 = 1$. Further properties of the curves Γ_r are discussed in [9], §§ 91, 390–396. It has been proved in [4] that Γ_r bounds three distinct solutions of Plateau's problem for r in a certain interval $r_0 < r < \sqrt{3}$. (A crude estimate for r_0 is $r_0 \leqq 1.682$.) From the following developments it will be seen that

as r passes increasingly through the critical value $r = 1$, the curves Γ_r acquire the capability of bounding, in addition to Enneper's surface, two further minimal surfaces which appear in a continuous bifurcation process.

It is now necessary to interject some basic definitions. Let $\Gamma = \{\mathfrak{x} = \mathfrak{z}(\tau)\,;\ 0 \leqq \tau \leqq 2\pi\}$ be a Jordan curve in Euclidean 3-space. We are concerned with vectors $\mathfrak{x}(u, v)$ defined in the closure $\bar{P}$ of the unit disc $P = \{u, v;\ u^2+v^2 < 1\}$ which map the boundary ∂P onto Γ. Setting $u + iv = w = \varrho e^{i\vartheta}$ we shall henceforth interchangeably use the notations $\mathfrak{x}(u, v)$, or $\mathfrak{x}(w)$, or $\mathfrak{x}(\varrho, \vartheta)$ (and later also $\mathfrak{x}(w; r)$ instead of $\mathfrak{x}(u, v; r)$, etc.) – whichever is most convenient. Denote by $\mathfrak{H} = \mathfrak{H}(\Gamma)$ the set of vectors $\mathfrak{x}(w) \in C^2(P) \cap C^0(\bar{P})$ which are harmonic in P and which map ∂P onto Γ monotonically so that three fixed distinct points w_1, w_2, w_3 on ∂P are transformed into three fixed distinct points $\mathfrak{z}_j = \mathfrak{z}(\tau_j)$, $(j = 1, 2, 3)$, on Γ, respectively. Once and for all we shall choose $w_1 = 1$,

$w_2 = i$, $w_3 = -1$. Then $\mathfrak{x}(e^{i\vartheta}) = \mathfrak{z}(\tau(\vartheta))$ where $\tau(\vartheta)$ is a continuous monotone function satisfying the conditions $\tau(0) = \tau_1$, $\tau(\pi/2) = \tau_2$, $\tau(\pi) = \tau_3$ and $\tau(\vartheta + 2\pi) = \tau(\vartheta) + 2\pi$. Endowed with the distance between two vectors,

$$\left|\mathfrak{x}_2 - \mathfrak{x}_1\right| = \max_{(u,v) \in \bar{P}} |\mathfrak{x}_2(u, v) - \mathfrak{x}_1(u, v)|,$$

the set $\mathfrak{H}$ becomes a metric space. Each element $\mathfrak{x}(u, v)$ of $\mathfrak{H}$ defines a harmonic surface $S = \{\mathfrak{x} = \mathfrak{x}(u, v)\,;\ (u, v) \in \bar{P}\}$. By definition, a solution of Plateau's problem for the curve Γ is a harmonic surface whose position vector satisfies in P the additional relations $\mathfrak{x}_u^2 = \mathfrak{x}_v^2$, $\mathfrak{x}_u \mathfrak{x}_v = 0$. We denote by $\mathfrak{M}$ the set of all such vectors. It is a matter of record that an element $\mathfrak{x}(u, v)$ of $\mathfrak{M}$ provides a topological mapping between ∂P and Γ.

Since we shall benefit from working with the fixed parameter domain P, a slight change of the introductory notation is advisable: The portion of Enneper's surface bounded by Γ_r will now be denoted by $S^{(0)}(r) = \{\mathfrak{x} = \mathfrak{x}^{(0)}(w\,; r)\,;\ w \in \bar{P}\}$ where

$$\mathfrak{x}^{(0)}(\varrho, \vartheta\,; r) = \begin{Bmatrix} r\,\varrho \cos\vartheta - \frac{1}{3} r^3 \varrho^3 \cos 3\vartheta \\ -r\,\varrho \sin\vartheta - \frac{1}{3} r^3 \varrho^3 \sin 3\vartheta \\ r^2 \varrho^2 \cos 2\vartheta \end{Bmatrix} = \mathrm{Re} \begin{Bmatrix} r\,w - \frac{1}{3} r^3 w^3 \\ i\,r\,w + i\,\frac{1}{3} r^3 w^3 \\ r^2 w^2 \end{Bmatrix}$$

and $\mathfrak{x}^{(0)}(1, \vartheta\,; r) = \mathfrak{z}(\vartheta\,; r)$.

Let $S(r) = \{\mathfrak{x} = \mathfrak{x}(w\,; r)\,;\ w \in \bar{P}\}$ be another solution of Plateau's problem for the curve Γ_r. By a theorem of H. Lewy [3] the position vector of $S(r)$ is analytic in $\bar{P}$. Moreover, owing to the results of [6], the derivatives of $\mathfrak{x}(w\,; r)$ can be estimated uniformly for all possible solutions. We shall refer here to the validity of the inequality

$$\max_{0 \leqq \vartheta \leqq 2\pi} \{ |\mathfrak{x}(1, \vartheta\,; r)|,\ |\mathfrak{x}_\vartheta(1, \vartheta\,; r)|,\ |\mathfrak{x}_{\vartheta\vartheta}(1, \vartheta\,; r)| \} \leqq K(r)$$

with a universal constant $K = K(r)$ depending on r alone, $0 < r < \sqrt{3}$. From the proofs in [6] it would not be difficult to extract an explicit expression for $K(r)$ in terms of the geometric properties of Γ_r. For the purposes at hand knowledge of the precise form of $K(r)$ would only be of minor advantage, however.

For the time being a value r, $1 \leqq r < \sqrt{3}$ is chosen and then kept fixed. A computation shows that $|\mathfrak{x}_\vartheta^{(0)}(1, \vartheta\,; r)| = r(1 + r^2)$ and

$$|\mathfrak{x}_{\vartheta\vartheta}^{(0)}(1, \vartheta\,; r)| = r[1 + r^2(6 + 4\cos^2 2\vartheta) + 9\,r^4]^{1/2} \leqq r\sqrt{1 + r^2}\sqrt{1 + 9\,r^2}\,.$$

From [8], p. 406, we know that $|\mathfrak{r}_\vartheta(1, \vartheta)| \geqq r(1+r^2)/2$ as long as $|\mathfrak{r}-\mathfrak{r}^{(0)}| < \varepsilon_1 \equiv \min(\pi/50, 1/175 K)$. Since Γ_r is the topological image of ∂P under the mapping of the position vectors of both surfaces $S^{(0)}$ and S, the boundary values of $\mathfrak{r}$ are related to those of $\mathfrak{r}^{(0)}$ through an expression of the form

$$\mathfrak{r}(1, \vartheta; r) = \mathfrak{r}^{(0)}(1, \vartheta + \lambda(\vartheta); r). \tag{2}$$

Here $\lambda(\vartheta)$ is a periodic analytic function satisfying the inequality $\lambda'(\vartheta) \geqq -1$. We write $\lambda(\vartheta)$ as a Fourier series,

$$\lambda(\vartheta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\vartheta + b_n \sin n\vartheta)$$

with the norm $||\lambda||$ defined by

$$||\lambda||^2 = \frac{1}{\pi} \int_0^{2\pi} \lambda^2(\vartheta)\, d\vartheta = \frac{1}{2} a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2).$$

In view of the "three point condition" above, $\lambda(\vartheta)$ is subject to the conditions $\lambda(0) = \lambda(\pi/2) = \lambda(\pi) = 0$. The relation $\mathfrak{r}_\vartheta(1, \vartheta; r) = (1+\lambda'(\vartheta))\, \mathfrak{r}_\vartheta^{(0)}(1, \vartheta + \lambda(\vartheta); r)$ which follows from (2) implies the inequalities

$$\tfrac{1}{2} \leqq 1 + \lambda'(\vartheta) \leqq \tfrac{1}{2} K \qquad \text{whenever } |\mathfrak{r}-\mathfrak{r}^{(0)}| < \varepsilon_1.$$

From [8], p. 406, we also conclude that

$$\frac{1}{8} |\mathfrak{r}-\mathfrak{r}^{(0)}| \leqq |\lambda(\vartheta)| \leqq |\mathfrak{r}-\mathfrak{r}^{(0)}| \qquad \text{for } |\lambda(\vartheta)| \leqq \frac{\pi}{3}.$$

It follows that the size of the neighborhood of $\mathfrak{r}^{(0)}$, as measured according to the metric of $\mathfrak{H}$, is also governed by the magnitude of $L = \max_{0 \leqq \vartheta \leqq 2\pi} |\lambda(\vartheta)|$. A series expansion of (2) leads to

$$\mathfrak{r}(1, \vartheta; r) = \mathfrak{r}^{(0)}(1, \vartheta; r) + \lambda(\vartheta)\, \mathfrak{r}_\vartheta^{(0)}(1, \vartheta; r) + \tfrac{1}{2} \lambda^2(\vartheta)\, \mathfrak{r}_{\vartheta\vartheta}^{(0)}(1, \vartheta; r) + \ldots. \tag{3}$$

We denote by $\mathfrak{r}^{(k)}(\varrho, \vartheta; r)$ the harmonic vector in $\bar{P}$ with boundary values

$$\mathfrak{r}^{(k)}(1, \vartheta; r) = \frac{1}{k!} \lambda^k(\vartheta) \frac{\partial^k}{\partial \vartheta^k} \mathfrak{r}^{(0)}(1, \vartheta; r). \tag{4}$$

The complex-valued expression

$$\begin{aligned} \Phi(\varrho, \vartheta) + i\, \Psi(\varrho, \vartheta) &= \tfrac{1}{2} (\mathfrak{r}_\vartheta^2 - \varrho^2 \mathfrak{r}_\varrho^2) + i\, \varrho\, \mathfrak{r}_\varrho \mathfrak{r}_\vartheta \\ &= \tfrac{1}{2} \varrho^2 e^{2i\vartheta} [(\mathfrak{r}_v^2 - \mathfrak{r}_u^2) + i\, 2\, \mathfrak{r}_u \mathfrak{r}_v] \end{aligned} \tag{5}$$

is an analytic function in P for *every* element of $\mathfrak{H}$, i.e., for *every* surface whose position vector is harmonic in P. It is therefore easily seen that the surface S is a solution of Plateau's problem if, and only if, $\Psi(1, \vartheta) = 0$. From [8] we know that this equation is equivalent with the condition

$$\sum_{m=1}^{\infty} \Lambda^{(m)}(1, \vartheta; r, \lambda) = 0. \tag{6}$$

Here the $\Lambda^{(m)}$ are defined by

$$\Lambda^{(m)}(1, \vartheta; r, \lambda) = \sum_{k=0}^{m} \frac{1}{k!} \lambda^k(\vartheta)\, \mathfrak{r}_\varrho^{(m-k)}(1, \vartheta; r) \frac{\partial^{k+1}}{\partial\vartheta^{k+1}} \mathfrak{r}^{(0)}(1, \vartheta; r). \tag{7}$$

Applying the estimating procedures used in [5], [8] we find that there are positive constants $\mathscr{C}$ and M such that for $\max_{0 \leqq \vartheta \leqq 2\pi} |\lambda(\vartheta)| \leqq L$, $\max_{0 \leqq \vartheta \leqq 2\pi} |\bar{\lambda}(\vartheta)| \leqq L$

$$\|\Lambda^{(m)}(1, \vartheta; r, \lambda)\| \leqq \mathscr{C} \frac{(M\, L)^{m-1}}{(m-1)!} \|\lambda'\|, \tag{8}$$

$$\|\Lambda^{(m)}(1, \vartheta; r, \bar{\lambda}) - \Lambda^{(m)}(1, \vartheta; r, \lambda)\| \leqq \mathscr{C} \frac{(M\, L)^{m-2}}{(m-2)!} (\|\lambda'\| + \|\bar{\lambda}'\|)\, \|\bar{\lambda}' - \lambda'\|.$$

From [8], p. 408 we take over the computation of $\sigma(\vartheta; r, \lambda) \equiv \Lambda^{(1)}(1, \vartheta; r, \lambda)$:

$$\sigma(\vartheta; r, \lambda) = r^2(r^2+1) \left\{ a_2\, c_2 - \frac{r^2-1}{r^2+1} b_2\, s_2 + \sum_{n=3}^{\infty} [(r^2+1)\, n - (3\, r^2+1)]\, (a_n\, c_n + b_n\, s_n) \right\}. \tag{9}$$

In this formula the abbreviations $c_n = \cos n\vartheta$ and $s_n = \sin n\vartheta$ have been used. Condition (6) can be written as

$$\sigma(\vartheta; r, \lambda) = -\sum_{m=2}^{\infty} \Lambda^{(m)}(1, \vartheta; r, \lambda) \equiv \Lambda(1, \vartheta; r, \lambda). \tag{10}$$

We turn now to the attempt to find a solution $\lambda = \lambda(\vartheta; r)$ of (10) for values $r > 1$ close to $r = 1$. For this purpose we set $\varepsilon = \sqrt{r^2 - 1}$ and assume λ to be of the form

$$\lambda(\vartheta; r) = \varepsilon\, \lambda^{(1)}(\vartheta) + \varepsilon^2\, \lambda^{(2)}(\vartheta) + \ldots$$

where

$$\lambda^{(k)}(\vartheta) = \tfrac{1}{2} a_0^{(k)} + \sum_{n=1}^{\infty} (a_n^{(k)} \cos n\vartheta + b_n^{(k)} \sin n\vartheta).$$

Some preparatory beuristic remarks are in order. A lengthy computation [1] shows that

$$(11) \qquad \Lambda^{(2)}(1\,,\vartheta\,;r\,,p\sin 2\vartheta) = -\frac{r^2}{4}(1+14\,r^2+3\,r^4)\,p^2\sin 4\vartheta$$
$$= -\frac{9}{2}\,r^2\,p^2\sin 4\vartheta + p^2\,O(\varepsilon^2)\,.$$

Accordingly, we shall choose the first two terms in the expansion of $\lambda(\vartheta\,;r)$ as follows:

$$\lambda^{(1)}(\vartheta) = b_2^{(1)}\sin 2\vartheta\,,$$
$$\lambda^{(2)}(\vartheta) = b_2^{(2)}\sin 2\vartheta + b_4^{(2)}\sin 4\vartheta\,.$$

Then

$$\sigma(\vartheta\,;r\,,\lambda) = 8\,\varepsilon^2\,b_4^{(2)}\sin 4\vartheta + O(\varepsilon^3)$$

and

$$\Lambda(1\,,\vartheta\,;r\,,\lambda) = \frac{9}{2}\,\varepsilon^2\,[b_2^{(1)}]^2\sin 4\vartheta + O(\varepsilon^3)$$

so that

$$(12) \qquad b_4^{(2)} = \frac{9}{16}\,[b_2^{(1)}]^2\,.$$

Now the necessity arises to compare certain terms in ε^3. For this purpose we note the following relations which can be obtained only by extensive computations (remember the abbreviations $s_n = \sin n\vartheta$):

$$\Lambda^{(2)}(1\,,\vartheta\,;r\,,p\,s_2+q\,s_4)$$
$$= -r^2\left\{\frac{9}{2}\,p^2\,s_4 + 9\,p\,q\,(s_2+s_6) + \frac{1}{16}\,q^2\,(22\,s_4+117\,s_8+2\,s_{12})\right\}$$
$$+\,p^2\,O(\varepsilon^2) + p\,q\,O(\varepsilon^2) + q^2\,O(\varepsilon^2)\,,$$

$$\Lambda^{(3)}(1\,,\vartheta\,;r\,,p\,s_2) = r^2\,\frac{11}{6}\,p^3\,(3\,s_2-s_6) + p^3\,O(\varepsilon^2)\,.$$

It is now seen that the coefficients of $\sin 2\vartheta$ are

in $\frac{1}{r^2}\,\sigma(1\,,\vartheta\,;r\,,\lambda)$: $\quad -\varepsilon^3\,b_2^{(1)} + O(\varepsilon^4)\,,$

in $\frac{1}{r^2}\,\Lambda(1\,,\vartheta\,;r\,,\lambda)$: $\quad 9\,\varepsilon^3\,b_2^{(1)}\,b_4^{(2)} - \frac{11}{2}\,\varepsilon^3\,[b_2^{(1)}]^3 + O(\varepsilon^4)\,.$

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[1] Unfortunately, neither this computation nor the even more extensive explicit computations, which will become necessary later on, can be reproduced here.

Consequently, we shall have

$$-b_2^{(1)} = 9\, b_2^{(1)}\, b_4^{(2)} - \frac{11}{2}\, [b_2^{(1)}]^3$$

or, combined with (12),

$$b_2^{(1)} + \left(\frac{81}{16} - \frac{11}{2}\right) [b_2^{(1)}]^3 = \left(1 - \frac{7}{16}\, [b_2^{(1)}]^2\right) b_2^{(1)} = 0\,.$$

This condition has three distinct solutions

$$b_2^{(1)} = 0\,, \quad b_2^{(1)} = \frac{4}{\sqrt{7}}\,, \quad b_2^{(1)} = -\frac{4}{\sqrt{7}}$$

and, corresponding to these solutions, $b_4^{(2)} = 0$ in the first case and $b_4^{(2)} = 9/7$ in the other two cases.

The preceding beuristic remarks indicate that a bifurcation appears as ε increases from zero to small positive values. i.e., as r increases from one to values larger than one. The choice $b_2^{(1)} = 0$ leads, of course, to $\lambda(\vartheta\,;\,r) \equiv 0$, that is to say, back to our original surface $S^{(0)}(r)$. Of the other two cases we shall now discuss rigorously the second for which $b_2^{(1)} = 4/\sqrt{7}$. The third case can be treated similarly.

We shall try to find a solution $\lambda(\vartheta\,;\,r)$ of (10) for small positive values of ε in the form

$$\text{(13)} \qquad \lambda(\vartheta\,;\,r) = \left[\frac{4}{\sqrt{7}}\,\varepsilon + \varepsilon^2\, b_2(\varepsilon)\right] s_2 + \left[\frac{9}{7}\,\varepsilon^2 + \varepsilon^3\, b_4(\varepsilon)\right] s_4 + \varepsilon^3 \left\{\frac{1}{2}\, a_0(\varepsilon) + a_1(\varepsilon)\, c_1 + b_1(\varepsilon)\, s_1 + \mu(\vartheta\,;\,\varepsilon)\right\}$$

where

$$\mu(\vartheta\,;\,\varepsilon) = a_2(\varepsilon)\, c_2 + a_3(\varepsilon)\, c_3 + b_3(\varepsilon)\, s_3 + a_4(\varepsilon)\, c_4 + \sum_{n=5}^{\infty} (a_n(\varepsilon)\, c_n + b_n(\varepsilon)\, s_n)$$

and the $a_n(\varepsilon)$, $b_n(\varepsilon)$ are themselves power series in ε. In view of the three point condition we must have

$$\begin{aligned}
\tfrac{1}{2}\, a_0(\varepsilon) + a_1(\varepsilon) \qquad\qquad &+ \mu(0\,;\,\varepsilon) = 0\,,\\
\tfrac{1}{2}\, a_0(\varepsilon) \qquad\qquad + b_1(\varepsilon) &+ \mu(\pi/2\,;\,\varepsilon) = 0\,,\\
\tfrac{1}{2}\, a_0(\varepsilon) - a_1(\varepsilon) \qquad\qquad &+ \mu(\pi\,;\,\varepsilon) = 0\,.
\end{aligned}$$

From these equations the three coefficients a_0, a_1, b_1 can be determined:

$$a_0(\varepsilon) = -2\,[a_2(\varepsilon) + a_4(\varepsilon) + a_6(\varepsilon) + \dots]$$

(14) $$a_1(\varepsilon) = -[a_3(\varepsilon) + a_5(\varepsilon) + a_7(\varepsilon) + \dots]$$

$$b_1(\varepsilon) = 2\,[a_2(\varepsilon) + a_6(\varepsilon) + a_{10}(\varepsilon) + \dots] + [b_3(\varepsilon) - b_5(\varepsilon) + b_7(\varepsilon) - \dots]$$

so that

(15) $$\frac{1}{2}\,a_0^2(\varepsilon) + a_1^2(\varepsilon) + b_1^2(\varepsilon) \leqq \frac{5}{2}\,\|\mu'(\vartheta\,;\varepsilon)\|^2 .$$

After substitution of $\lambda(\vartheta\,;r)$ from (13), $(1/r^2)\,\Lambda(1\,,\vartheta\,;r\,,\lambda)$ attains an expansion

$$\frac{1}{r^2}\,\Lambda(1\,,\vartheta\,;r\,,\lambda) = \sum_{n=2}^{\infty}[A_n(\varepsilon\,,\lambda)\,c_n + B_n(\varepsilon\,,\lambda)\,s_n] .$$

From (5) it is clear, that neither a constant term nor terms with $\cos\vartheta$ or $\sin\vartheta$ can appear on the right hand side. In determining the coefficients A_n and B_n it is necessary to carry the computations to the point that explicit expressions become available for all terms up to those of order ε^4 in B_2 and all terms up to those of order ε^3 in the other coefficients. The result of these computations is as follows:

$$B_2(\varepsilon\,,\lambda) = -\frac{4}{\sqrt{7}}\,\varepsilon^3 + \left[\frac{180}{49} - \frac{183}{7}\,b_2(\varepsilon) + \frac{36}{\sqrt{7}}\,b_4\,(\varepsilon)\right]\varepsilon^4 + \varepsilon^5\,\tilde{B}_2(\varepsilon\,,\lambda) ,$$

$$B_4(\varepsilon\,,\lambda) = \frac{72}{7}\,\varepsilon^2 + \frac{36}{\sqrt{7}}\,\varepsilon^3\,b_2(\varepsilon) + \varepsilon^4\,\tilde{B}_4(\varepsilon\,,\lambda) ,$$

$$B_6(\varepsilon\,,\lambda) = \frac{1324}{21\sqrt{7}}\,\varepsilon^3 + \varepsilon^4\,\tilde{B}_6(\varepsilon\,,\lambda) .$$

All the other coefficients have the form

$$A_n(\varepsilon\,,\lambda) = \varepsilon^4\,\tilde{A}_n(\varepsilon\,,\lambda) , \qquad n \geqq 2 ,$$

$$B_n(\varepsilon\,,\lambda) = \varepsilon^4\,\tilde{B}_n(\varepsilon\,,\lambda) , \qquad n \geqq 2\,,\; n \neq 2, 4, 6 .$$

On the other hand,

$$\frac{1}{r^2}\,\sigma(\vartheta\,;r\,,\lambda) = (2+\varepsilon^2)\,\varepsilon^3\,a_2(\varepsilon)\,c_2 - \left(\frac{4}{7}\,\varepsilon^3 + \varepsilon^4\,b_2(\varepsilon)\right)s_2$$

$$+\; 2\,\varepsilon^3\,(2+\varepsilon^2)\,[a_3(\varepsilon)\,c_3 + b_3(\varepsilon)\,s_3]$$

$$+\; \varepsilon^3\,(2+\varepsilon^2)\,(4+\varepsilon^2)\,a_4(\varepsilon)\,c_4$$

$$+ (2+\varepsilon^2)(4+\varepsilon^2)\left(\frac{9}{7}\varepsilon^2 + \varepsilon^3 b_4(\varepsilon)\right) s_4$$

$$+ \varepsilon^3 (2+\varepsilon^2) \sum_{n=5}^{\infty} [(2+\varepsilon^2)\, n - (4+3\,\varepsilon^2)]\, (a_n(\varepsilon)\, c_n + b_n(\varepsilon)\, s_n) .$$

The conditions for the coefficients $a_0(\varepsilon)$, $a_1(\varepsilon)$, $b_1(\varepsilon)$ are contained in (14). A comparison leads to a system of equations for the Fourier coefficients of $\lambda(\vartheta\,;r)$. The three "special" equations are

$$-b_2(\varepsilon) = \frac{180}{49} - \frac{183}{7} b_2(\varepsilon) + \frac{36}{\sqrt{7}} b_4(\varepsilon) + \varepsilon\, \tilde{B}_2(\varepsilon\,,\lambda)\,,$$

$$(8+6\,\varepsilon^2+\varepsilon^4)\, b_4(\varepsilon) = \frac{36}{\sqrt{7}} b_2(\varepsilon) - \varepsilon\,(6+\varepsilon^2) + \varepsilon\, \tilde{B}_4(\varepsilon\,,\lambda)\,,$$

$$(16+14\,\varepsilon^2+3\,\varepsilon^4)\, b_6(\varepsilon) = \frac{1324}{21\sqrt{7}} + \varepsilon\, \tilde{B}_6(\varepsilon\,,\lambda)\,.$$

The remaining equations are

$$(2+\varepsilon^2)\, a_2(\varepsilon) = \varepsilon\, \tilde{A}_2(\varepsilon\,,\lambda)\,,$$

$$(2+\varepsilon^2)\,[(2+\varepsilon^2)\, n - (4+3\,\varepsilon^2)]\, a_n(\varepsilon) = \varepsilon\, \tilde{A}_n(\varepsilon\,,\lambda)\,, \quad n \geqq 3\,,$$

$$(2+\varepsilon^2)\,[(2+\varepsilon^2)\, n - (4+3\,\varepsilon^2)]\, b_n(\varepsilon) = \varepsilon\, \tilde{B}_n(\varepsilon\,,\lambda)\,, \quad n \geqq 3\,, \ n \neq 4, 6\,.$$

For $\varepsilon = 0$ we obtain

$$a_2(\varepsilon) = a_3(\varepsilon) = \ldots = 0\,,$$

$$b_3(\varepsilon) = b_5(\varepsilon) = b_7(\varepsilon) = b_8(\varepsilon) = \ldots = 0\,,$$

and

$$-b_2(0) = \frac{180}{49} - \frac{183}{7} b_2(0) + \frac{36}{\sqrt{7}} b_4(0)\,,$$

$$8\, b_4(0) = \frac{36}{\sqrt{7}} b_2(0)\,,$$

$$16\, b_6(0) = \frac{1324}{21\sqrt{7}}\,,$$

or

$$\frac{176}{7} b_2(0) - \frac{36}{\sqrt{7}} b_4(0) = \frac{180}{49},$$

$$-\frac{36}{\sqrt{7}} b_2(0) + 8\, b_4(0) = 0$$

and $b_6(0) = 331 / (84\sqrt{7})$, so that

$$b_2(0) = \frac{90}{49}, \quad b_4(0) = \frac{405}{49\sqrt{7}}, \quad b_6(0) = \frac{331}{84\sqrt{7}}.$$

In view of the estimates (8), (15) and the solvability for $\varepsilon = 0$ it can be shown that our system of equations has, for all small non-negative values of ε, in fact a solution $\lambda^{(+)}(\vartheta\,;r)$ of the form (13) whose derivative has a uniformly bounded norm:

$$\lambda^{(+)}(\vartheta\,;r) = \varepsilon \frac{4}{\sqrt{7}} s_2 + \varepsilon^2 \left[\frac{90}{49} s_2 + \frac{9}{7} s_4\right] + O(\varepsilon^3).$$

Let us summarize: We have seen that equations (10) have three distinct solutions for values of $r \geqq 1$ close to $r = 1$:

$$\lambda^{(0)}(\vartheta\,;r) = 0,$$

$$\lambda^{(+)}(\vartheta\,;r) = \frac{4}{\sqrt{7}} \sqrt{r^2-1}\, \sin 2\vartheta + O(r^2-1),$$

$$\lambda^{(-)}(\vartheta\,;r) = -\frac{4}{\sqrt{7}} \sqrt{r^2-1}\, \sin 2\vartheta + O(r^2-1).$$

Each generates a solution of Plateau's problem for the Jordan curve Γ_r. $\lambda^{(0)} = 0$ leads us back to Enneper's surface $S^{(0)}(r)$ with the position vector $\mathfrak{r}^{(0)}(\varrho, \vartheta\,; r)$. But in addition to Enneper's surface two new surfaces $S^{(+)}(r)$ and $S^{(-)}(r)$ appear. In view of (3) the position vectors of these surfaces are, respectively,

$$\mathfrak{r}^{(\pm)}(\varrho, \vartheta\,; r) = \mathfrak{r}^{(0)}(\varrho, \vartheta\,; r) + \frac{2}{\sqrt{7}} \sqrt{r^2-1} \begin{Bmatrix} \varrho^3 c_3 - \varrho^5 c_5 \\ -\varrho^3 s_3 - \varrho^5 s_5 \\ -2 + 2\varrho^4 c_4 \end{Bmatrix} + O(r^2-1).$$

We know that Enneper's surface is not stable for $r > 1$. The surfaces $S^{(+)}(r)$ and $S^{(-)}(r)$ will be the two surfaces of least area bounded by Γ_r. (It is the special symmetry of Γ_r which gives rise to *two* distinct, but congruent surfaces of least area.) While the point $u = v = 0$ is mapped by $\mathfrak{r}^{(0)}$ into the origin of 3-space for all values of r, its image under the mapping by the vectors $\mathfrak{r}^{(+)}$ and $\mathfrak{r}^{(-)}$, roughly the point

$$\left(0\,,\,0\,,\,-\frac{4}{\sqrt{7}}\sqrt{r^2-1}\right) \text{ or } \left(0\,,\,0\,,\frac{4}{\sqrt{7}}\sqrt{r^2-1}\right),$$

moves along the z-axis, down or up, as r increases. In this way the surfaces $S^{(+)}$ and $S^{(-)}$ arrange to decrease the surface area of $S^{(0)}$.

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