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A THEOREM ON CLUSTER SETS OF AN ANALYTIC MAPPING INTO A RIEMANN SURFACE

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1. Introduction. The author [5] proved in 1952 that if an analytic mapping f of a punctured disk $0 < |z| < r$ into a Riemann surface R has $z = 0$ as an essential singularity in a certain sense, then the set of values in R taken by f in any neighborhood of $z = 0$ is conformally equivalent to a sphere possibly less two points or to a torus. Heins [1] and Marden, Richards and Rodin [2] gave different proofs and the latter applied the above result to the study of analytic self-mappings of Riemann surfaces.

In the present paper we shall treat the case where the singularity is not isolated. Our result gives a generalization of a theorem of Noshiro (see Theorem 6 at p. 26 of [4]). To explain his result, let f be a meromorphic function defined in a domain D, and K a compact set of logarithmic capacity zero on one component of the boundary ∂D. Let z_0 be a point of K not isolated on $(\partial D - K) \cup \{z_0\}$. Then it can be shown that the difference Ω between the cluster set at z_0 and the boundary cluster set defined at z_0 along $\partial D - K$ is an open set. Noshiro proved that f takes every value, with two possible exceptions, of each component of Ω in any neighborhood of z_0.

2. Preliminaries. Let R be a finite Riemann surface, and $\varrho_z |dz|$ be a conformal metric on R with strictly positive coefficient ϱ_z. Let S be a covering surface of R. One can regard $\varrho_z |dz|$ as a conformal metric on S. The coefficient may vanish on S; actually it does at each branch point of S. We set $L(c) = \int_c \varrho_z |dz|$ for a smooth arc c on R and $I(E) = \iint_E \varrho_z^2 \, dx \, dy$ for a measurable set E on R. We shall use the same notation for such quantities on S too.

Let S be a simply connected finite Riemann surface in particular. Ahlfors' main theorem asserts that there exists a constant h depending only on R such that

$$(1) \qquad 0 \geqq e_0 \, M(S) - h \, L(\partial S) \, ,$$

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where e_0 is the characteristic of R, $M(S) = I(S) / I(R)$ is the mean sheet number of S and ∂S is the boundary of S relative to R.

Let us call a simply connected domain on R with analytic boundary an open disk. Take open disks $\Delta_1, \dots, \Delta_q$ on R whose closures are mutually disjoint and contained in the interior of R. Denote the projection of S into R by f. A component of $f^{-1}(\Delta_j)$ is called an island lying above Δ_j if its boundary consists of inner points of S. Let $\{D_i\}$ be the islands lying above $\Delta_1, \dots, \Delta_q$. By the aid of (1) we derive

$$(2) \qquad -\sum e(D_i) \geqq (e_0+q)\, M(S) - h\, L(\partial S),$$

where $e(D_i)$ is the characteristic of D_i; if there is no island, then the left hand side is set to be zero. See (60) of [6] for our (2).

Let S now be a simply connected bordered covering surface of R. We call an increasing approximation $\{S_n\}$ of S a regular exhaustion of S if each S_n consists of finitely many finite surfaces and

$$\frac{L(\partial S_n)}{M(S_n)} \to 0 \qquad \text{as } n \to \infty,$$

where ∂S_n is the boundary of S_n relative to R. We note that our definition is general in the sense that S_n may not be connected. When there exists a regular exhaustion of S, we say that S is regularly exhaustible.

We have

Lemma. *Let S be a regularly exhaustible simply connected bordered Riemann surface which is a covering surface of a finite Riemann surface R of characteristic e_0. Then $e_0 \leqq 0$ and S covers all inner points of R except at most $-e_0$ points.*

Proof. Let $\{S_n\}$ be a regular exhaustion. From (1) it follows that $e_0 \leqq 0$. Next, assume that S does not cover $q = 1-e_0$ inner points of R, and take disks $\Delta_1, \dots, \Delta_q$ around them as above. Then there is no island above $\Delta_1, \dots, \Delta_q$. By (2) we have $0 \geqq M(S_n) - h\, L(\partial S_n)$ and meet a contradiction.

3. Main theorem. Let G be an open set in the z-plane, $z_0 \in \partial G$ and K a compact subset of ∂G containing z_0. Let f be a mapping of G into a Riemann surface R. We define the cluster set $f(z_0; G)$ at z_0 to be

$$\bigcap_{U \in \mathscr{U}(z_0)} \overline{f(U \cap G)},$$

where $\mathscr{U}(z_0)$ is the system of neighborhoods of z_0 and $\overline{f(U \cap G)}$ denotes the closure of $f(U \cap G)$, and define a boundary cluster set $f(z_0; \partial G - K)$ by

$$\bigcap_{U \in \mathscr{U}(z_0)} \overline{\bigcup_{z \in U \cap (\partial G - K)} f(z; G)}.$$

We shall prove

Theorem. *Let f be an analytic mapping of an open set G in the z-plane into a Riemann surface R, $z_0 \in \partial G$ and K a compact set of logarithmic capacity zero which contains z_0 and which is contained in one component of ∂G. If $f(z_0\,;G)$ contains more than one point, then $f(z_0\,;G) - f(z_0\,;\partial G-K)$ is an open set and the genus of every component D of $f(z_0\,;G) - f(z_0\,;\partial G-K)$ is at most one. If the genus is 0 (1 resp.), then every point of D is taken by f in any neighborhood of z_0 except for at most two (with no exception resp.).*

Proof. Suppose $f(z_0\,;G) - f(z_0\,;\partial G - K)$ is not open. Then the boundary $\partial f(z_0\,;G)$ is not contained in $f(z_0\,;\partial G - K)$. Let P_0 be a point of $\partial f(z_0\,;G) - f(z_0\,;\partial G-K)$, and N be a closed neighborhood of P_0 which does not include the whole $f(z_0\,;G)$ and which is disjoint from $f(z_0\,;\partial G-K)$. Let w be a local parameter such that N contains the local disk which corresponds to $|w| \leqq 1$ and P_0 corresponds to $w=0$. Denote the composed mapping $w(f(z))$ by $g(z)$, and let Ω be the inverse image of $|w|<1$ in G. Since $P_0 \in f(z_0\,;G)$, there exists a sequence $\{z_n\}$ in G tending to z_0 such that $f(z_n) \to P_0$. Hence z_0 is a boundary point of Ω. Suppose z_0 is isolated on $(\partial\Omega - K) \cup \{z_0\}$. Then $\Omega \cap U = U - K = G \cap U$ for some neighborhood U of z_0, and hence $f(z_0\,;G) \subset N$. This is against our choice of N. Thus z_0 is not isolated on $(\partial\Omega-K) \cup \{z_0\}$. We see that $g(z_0\,;\partial\Omega-K)$ is contained in $|w|=1$ and that $w=0$ belongs to $\partial g(z_0\,;\Omega)$. Thus $\partial g(z_0\,;\Omega) \not\subset g(z_0\,;\partial G-K)$. However, if we use Theorem 4 at p. 17 of [4] in the theory of cluster sets for functions, then we conclude that $\partial g(z_0\,;\Omega) \subset g(z_0\,;\partial\Omega-K)$. This contradiction shows that $f(z_0\,;G) - f(z_0\,;\partial G-K)$ is open.

We shall prove next that, in case the genus of a component D of $f(z_0\,;G) - f(z_0\,;\partial G-K)$ is zero, f takes on every value of D except for at most two in any neighborhood of z_0. The proof is exactly the same as for Noshiro's theorem referred to in the introduction. For the sake of completeness, however, we shall prove it. Suppose P_1, P_2, $P_3 \in D$ are not taken by f on $G \cap \{\,|z-z_0| \leqq r\,\}$. Draw an analytic simple closed curve c which passes through P_3, whose interior Δ contains P_1 and P_2 and above which lies no branch point of G as a covering surface of R. We take c in D so that Δ is included in D. Since $D \subset f(z_0\,;G)$, there is a sequence $\{z_k\}$ tending to z_0 whose image $\{f(z_k)\}$ is contained in Δ and tends to P_1. We may assume that no z_k is a branch point. Let $(f(z_k)\,P_1)^{\sim}$ be a curve in D which passes through no image of any branch point and converges to P_1 as $k \to \infty$, and let γ_k be the inverse image of $(f(z_k)\,P_1)^{\sim}$ starting from z_k. If there are infinitely many γ_k which intersect $|z-z_0|=r$, these γ_k cluster to a continuum F connecting z_0 and $|z-z_0|=r$. Since f is not constant, F has no point in G. Accordingly

$F \subset \partial G$. We see that every $U \in \mathscr{U}(z_0)$ contains some point z of $F - K$ and $f(z\,;G)$ contains P_1 at such z . Accordingly $f(z_0\,; \partial G - K)$ contains P_1 . This is impossible. It follows that γ_k must terminate at a point of $K \cap \{ |z-z_0| < r \}$ if k is large. Let S be a component of

$$f^{-1}(\bar{\Delta}) \cap \{ |z-z_0| \leqq r \}$$

which contains such a γ_k . Since f does not assume P_3 on $G \cap \{ |z-z_0| \leqq r \}$, S is simply connected.

Let a conformal metric $\varrho_w|dw|$ be given on Δ . Let us see that S is regularly exhaustible as a covering surface of Δ . It is well known that there exists a logarithmic potential $U(z)$ of a unit measure supported by K such that $U = \infty$ on K . Let V be a conjugate of U and let $\zeta = F(z)$ be a single-valued branch in S of $\exp(U(z) + i\,V(z))$. If λ_0 is large, then the level set $\{ z\,; e^{U(z)} = \lambda \}$ intersects γ_k for every $\lambda \geqq \lambda_0$. Take ζ as a local parameter at every point z of S at which $F'(z) \neq 0$, i.e., grad $U \neq 0$. Denote the part of S on which $e^U < \lambda$ by S_λ , and the level set $\{ z \in S\,; |F(z)| = e^{U(z)} = \lambda \}$ by Θ_λ . Denoting by $(\lambda\,, \vartheta)$ the polar coordinates in the ζ-plane, we have

$$I(S_\lambda) - I(S_{\lambda_0}) = \int_{\lambda_0}^{\lambda} \int_{\Theta_\lambda} \varrho_\zeta^2\, \lambda\, d\vartheta\, d\lambda\,,$$

where $\varrho_\zeta|d\zeta|$ is the conformal metric on S expressed in terms of ζ and equals $\varrho_w|dw|$ at every point z at which $F'(z) \neq 0$ and which is not a branch point of f. Set

$$L(\lambda) = \int_{\Theta_\lambda} \varrho_\zeta\, \lambda\, d\vartheta\,.$$

Denote by $\alpha > 0$ the distance between c and $(f(z_k)\,P_1)^{\sim}$, measured with respect to $\varrho_w|dw|$. Then $L(\lambda) \geqq 2\,\alpha$ for $\lambda \geqq \lambda_0$. Applying Schwarz's inequality we derive

$$(L(\lambda))^2 \leqq \int_{\Theta_\lambda} \lambda\, d\vartheta \int_{\Theta_\lambda} \varrho_\zeta^2\, \lambda\, d\vartheta = \lambda\, \vartheta(\lambda) \int_{\Theta_\lambda} \varrho_\zeta^2\, \lambda\, d\vartheta\,,$$

where $\vartheta(\lambda) = \int_{\Theta_\lambda} d\vartheta$, and

$$\frac{(L(\lambda))^2}{\lambda\, \vartheta(\lambda)} \leqq \int_{\Theta_\lambda} \varrho_\zeta^2\, \lambda\, d\vartheta = \frac{dI(S_\lambda)}{d\lambda}\,.$$

Using the relation

$$\vartheta(\lambda) = \int_{\Theta_\lambda} dV \leqq \int_{U=\log\lambda} \frac{\partial U}{\partial n} ds = 2\pi ,$$

we have

$$\frac{2\alpha^2}{\pi} \int_{\lambda_0}^{\lambda} \frac{d\lambda}{\lambda} \leqq \int_{\lambda_0}^{\lambda} dI(S_\lambda) = I(S_\lambda) - I(S_{\lambda_0}) .$$

It follows that $I(S_\lambda) \to \infty$ as $\lambda \to \infty$. If there existed $\beta > 0$ and $\lambda_1 > \lambda_0$ such that $I(S_\lambda) \leqq \beta L(\lambda)$ for all $\lambda \geqq \lambda_1$, then

$$\frac{1}{2\pi} \int_{\lambda_1}^{\lambda} \frac{d\lambda}{\lambda} \leqq \beta^2 \int_{\lambda_1}^{\lambda} \frac{dI(S_\lambda)}{(I(S_\lambda))^2} \leqq \frac{\beta^2}{I(S_{\lambda_1})} < \infty .$$

This is absurd. Therefore there exists $\{\lambda_n\}$ tending to ∞ such that

$$\frac{L(\partial S_{\lambda_n})}{I(S_{\lambda_n})} = \frac{L(\lambda_n) + L(\partial S \cap \{ |z-z_0| = r \})}{I(S_{\lambda_n})} \to 0 \quad \text{as } n \to \infty .$$

Thus S is a regularly exhaustible simply connected bordered covering surface of $\bar{\Delta}$. Our lemma implies that S covers all points of Δ except at most one point. This is not true. Consequently, every component of $f(z_0 ; G) - f(z_0 ; \partial G - K)$ of genus zero is covered by the image of any neighborhood of z_0 except for at most two points.

Secondly, let D be a component of $f(z_0 ; G) - f(z_0 ; \partial G - K)$ of genus at least one, and assume that $P_0 \in D$ is not taken by f in $G \cap \{ |z-z_0| \leqq r \}$. Take a subdomain Δ of genus one of D bounded by an analytic simple closed curve c in D which passes through P_0 and above which lies no branch point of G. If there is $P_1 \in \Delta$ which is not taken by f in a neighborhood of z_0, then we observe that there is a curve γ in any neighborhood of z_0 which terminates at a point of K and along which f tends to P_1. Let S be a component of $f^{-1}(\bar{\Delta}) \cap \{ |z-z_0| \leqq r \}$ containing γ. It is simply connected. We can show as above that it is a regularly exhaustible bordered covering surface of $\bar{\Delta}$. Since the characteristic of Δ is one, our lemma gives a contradiction. Hence all points of Δ are taken in any neighborhood of z_0. Since Δ contains a topological handle, there are analytic simple closed curves c_1 and c_2 in Δ such that they intersect mutually only at a point P_2 and no branch point of G lies above $c_1 \cup c_2$. Let $\{z_k\}$ be a sequence of points tending to z_0 such that $f(z_k) = P_2$ for each k. Consider the component l_k of $f^{-1}(c_1)$ which passes through z_k. Suppose there is no l_k which terminates at K for large k. Then for some k_0 l_{k_0} must be a closed curve in $G \cap \{ |z-z_0| < r \}$. Consider the component l'_{k_0} of $f^{-1}(c_2)$ which starts from z_{k_0} and runs in the interior

D_{k_0} of l_{k_0}. Since the part of D_{k_0} near l_{k_0} corresponds to one shore of c_1, l'_{k_0} can not intersect l_{k_0} again. Hence it must terminate at some point of K. It is now concluded that, in any neighborhood of z_0, there is a curve which terminates at a point of K and whose image by f is contained in c_1 or c_2. Let S be a component of $f^{-1}(\bar{\varDelta}) \cap \{ |z-z_0| \leqq r\}$ containing such a curve. It is a simply connected regularly exhaustible bordered covering surface of $\bar{\varDelta}$. This is again impossible. Thus every point of D is taken in any neighborhood of z_0.

Finally, suppose the genus of a component D is at least two. Take a subdomain $\varDelta$ of genus one of D bounded by an analytic simple closed curve c in D. Suppose there is a component S_0 of

$$f^{-1}(\bar{\varDelta}) \cap \{ |z-z_0| \leqq r\}$$

which is not simply connected. Then there exists a closed curve γ' on ∂S_0 corresponding to c. Take a non-branch point $z' \in S_0$ close to γ' and let P be its image. Since $D - \bar{\varDelta}$ is not planar, it contains two analytic simple closed curves c_1' and c_2' which meet only at P and above which no branch point of G lies. The inverse image of c_1' passing through z' must be a closed curve. If we start from z' in one direction along the inverse image of c_2', we have no place to go after all. Consequently, every component of $f^{-1}(\bar{\varDelta}) \cap \{ |z-z_0| \leqq r\}$ is simply connected. The rest of the proof is the same as above. The proof of our theorem is now completed.

Remark 1. In our theorem the case when $f(z_0\,; G)$ consists of a single point or is empty is not treated. Therefore it does not include author's result in [5]. See [5] in this aspect.

Remark 2. In [5] the author stated that he could not apply Ahlfors' theory of covering surfaces to prove Theorem 1. The present paper surmounts that difficulty.

Remark 3. If the condition that K is contained in one component of ∂G is removed in our theorem, then the conclusion is not true in general. Actually, Matsumoto [3] proved that, given any K_σ-set E of logarithmic capacity zero in the w-plane, there exist a compact set K of logarithmic capacity zero in the z-plane and a meromorphic function $w = f(z)$ defined outside K such that every point of K is a singularity for f and E is the set of exceptional values at each point of K.

In connexion with this remark we mention the following open question due to M. Suzuki:

Let D be a domain in a plane and K be a compact subset of D of logarithmic capacity zero. Let f be an analytic mapping of $D - K$ into a Riemann surface R such that the cluster set of f at a point of K coincides with R. Then, is the genus of R at most one?

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