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SOME COEFFICIENT PROBLEMS FOR STARLIKE FUNCTIONS

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0. Introduction

Consider the class S^* of normalized starlike functions f in the unit disc $D = \{ |z| < 1 \}$, whose basic results were established by *Rolf Nevanlinna* in his paper [6], and write

$$(0.1) \qquad f(z) = z + a_2 z^2 + \dots + a_n z^n + \dots .$$

The mapping

$$(0.2) \qquad A_n : f \mapsto (a_2 , \dots , a_n)$$

associates to each f a point in C^{n-1} and takes S^* onto some compact set S_n^* which is called the n-th coefficient body for the class S^*. This paper deals with some basic properties of S_n^*. It will be proved that S_n^* is homeomorphic to a ball in C^{n-1}, that for each boundary point a of S_n^* there is only one function f in S^* such that $A_n(f) = a$ and that $A_n(f)$ is on the boundary of S_n^* if and only if f takes the unit disc onto a domain which is bounded by at most $n-1$ rays

$$(0.3) \qquad R_j = \{ z = t\, e^{i\alpha_j} \mid t \geqq r_j \} , \quad \alpha_1 < \alpha_2 < \dots < \alpha_m < \alpha_1 + 2\pi ,$$

$1 \leqq m < n$. Contrary to the analogous problem for the class S (cf. [9] for example) the situation here is very explicit and elementary. The basic idea is to consider an analogous coefficient problem for the Carathéodory class C of functions g holomorphic in the unit disc which have positive real part and are normalized by the condition $g(0) = 1$. For each n the expansion

$$(0.4) \qquad g(z) = 1 + 2\, c_1 z + \dots + 2\, c_n z^n + \dots$$

defines a mapping

$$(0.5) \qquad \gamma_n : g \mapsto (c_1 , \dots , c_n)$$

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of C onto some compact set C_n in C^n, which is called the n-th coefficient body for the class C. The basic properties of C_n (Theorem A) are due to C. Carathéodory and O. Toeplitz; for completeness a full proof will be given in the third part of the paper.

C_n is a convex body in C^n, hence C_n admits for each boundary point a supporting hyperplane. This fact gives a set of inequalities for $a_2, \dots, a_{n+1}$ relative to a boundary point $(a_2^0, \dots, a_{n+1}^0)$ of S_{n+1}^*. Among them there are coefficient inequalities for the class S^* (Theorem 2) which are quite similar to those the extended general coefficient theorem of J. A. Jenkins gives for the class S. They imply that some sections of S_n^* are convex, i.e. if $a^0 = (a_2^0, \dots, a_\varrho^0)$ is a point of S_ϱ^* and if $W_n(a_2^0, \dots, a_\varrho^0)$ is the set of points $(a_{\varrho+1}, \dots, a_n)$ in $C^{n-\varrho}$ such that $(a_2^0, \dots, a_\varrho^0, a_{\varrho+1}, \dots, a_n)$ is in S_n^*, then $W_n(a_2^0, \dots, a_\varrho^0)$ is strictly convex for each point a^0 in S_ϱ^* provided that $n \leqq 2\varrho$, and this bound for n is sharp.

1. The n-th coefficient body

1.1. In this section we will prove the following

Theorem 1. *The n-th coefficient body S_n^* is homeomorphic to a ball in C^{n-1}. For each point a on the boundary of S_n^* there is only one function in S^* which is taken onto a by the mapping A_n while $A_n^{-1}(a)$ is an infinite set in S^* if a is in the interior of S_n^*. $A_n(f)$ is on the boundary of S_n^* if and only if there are distinct points $\varkappa_1, \dots, \varkappa_m$ on the unit circumference $\{|z| = 1\}$ and positive numbers $\mu_1, \dots, \mu_m$ where $\sum_{j=1}^m \mu_j = 1$ and $1 \leqq m < n$ such that*

$$f(z) = z \prod_1^m (1 - \varkappa_j z)^{-2\mu_j} . \tag{1.1}$$

Given the boundary point a, the numbers m, $\varkappa_j$ and μ_j are unique.

Remark. f takes the unit disc onto a domain which is bounded by m rays $(0, 3)$, where $\alpha_{j+1} - \alpha_j = 2\pi\mu_j$, $j = 1, \dots, m$. Conversely, any m such rays, $1 \leqq m < n$, up to a suitable homothety $z \to rz$, $r > 0$, determine via the mapping function a point on the boundary of S_n^*.

Since f belongs to S^* if and only if $zf'(z) / f(z)$ is in C, the differential equation

$$zf'(z) = g(z) f(z) \tag{1.2}$$

establishes a homeomorphism between C and S^* if C and S^* are provided with the topology of uniform convergence on compact subsets of D. Equation (1.2) implies the following relations between the coefficients a_j and c_j in (0.1) and (0.4) respectively:

$$(1.3)\qquad \begin{aligned} a_2 &= 2\,c_1\,, \\ 2\,a_3 &= 2\,(c_2 + a_2\,c_1)\,, \\ (n-1)\,a_n &= 2\,(c_{n-1} + a_2\,c_{n-2} + \dots + a_{n-1}\,c_1)\,. \end{aligned}$$

For each n, $n = 2, 3, \dots$, they define a homeomorphism of S_n^* onto C_{n-1}. Hence, for some basic properties of S_n^*, it suffices to study C_{n-1}.

1.2. The following result is due to C. Carathéodory [1] and O. Toeplitz [10] (cf. also [2]).

Theorem A. *C_n is a convex body in C^n containing the origin. To each point $\zeta = (c_1, \dots, c_n)$ in the interior of C_n there correspond infinitely many functions in C, i.e. $\gamma_n^{-1}(\zeta)$ is infinite; but for each point ζ on the boundary of C_n there is only one g in C which is taken onto ζ by the mapping γ_n. $\gamma_n(g)$ is on the boundary of C_n if and only if there are distinct points $\varkappa_1, \dots, \varkappa_m$ on the unit circumference $\{|z| = 1\}$ and positive numbers $\mu_1, \dots, \mu_m$ such that*

$$(1.4)\qquad g(z) = \sum_{j=1}^{m} \frac{1 + \varkappa_j z}{1 - \varkappa_j z}\,\mu_j\,,$$

where $1 \leqq m \leqq n$ and $\sum_{j=1}^{m} \mu_j = 1$. The numbers m, $\varkappa_j$, μ_j are determined uniquely by the boundary point ζ.

Proof of Theorem 1. Theorem A, together with (1.2) and (1.3), immediately implies Theorem 1. Since (1.3) establishes a homeomorphism between the boundaries of C_{n-1} and S_n^*, by integration of (1.2), the functions (1.4) give exactly those functions in S^* which are taken onto the boundary of S_n^* under the mapping A_n.

1.3. The implication of Theorem 1 for extremal problems within the class S^* is immediate. Let $F(a_2, \dots, a_n)$ be a real valued function of the complex variables $a_2, \dots, a_n$, which is defined and continuously differentiable with respect to the real variables $x_j = \operatorname{Re} a_j$, $y_j = \operatorname{Im} a_j$, $j = 2, \dots, n$, in some neighborhood N of S_n^*, such that $|\operatorname{grad} F|$ is positive there. Then the function F attains its maximum on S_n^* only on the boundary. Hence *each function f which maximizes F* (considered as a functional on S^*) *on S^* is necessarily of type* (1.1). This result was proved by J. A. Hummel by variational methods of Schiffer's type within the class S^* (cf. [4]).

Furthermore, let F attain its maximum at a point $(a_2^0, \dots, a_n^0)$ of the boundary ∂S_n^*. Then, from

$$F(a_2, \dots, a_n) = F(a_2^0, \dots, a_n^0) + 2 \operatorname{Re} \sum_{j=2}^{n} \frac{\partial F}{\partial a_j} \Delta a_j$$
$$+ o\left(\max_j \{ |\Delta a_j| \}\right), \qquad \Delta a_j = a_j - a_j^0 ,$$

it follows that

$$\operatorname{Re} \left\{ \sum_2^n \frac{\partial F}{\partial a_j} (a_2^0, \dots, a_n^0) \Delta a_j \right\} + o\left(\max_j \{ |\Delta a_j| \}\right) \leqq 0$$

and this shows that $(\partial F / \partial a_2, \dots, \partial F / \partial a_n)$ is an outer normal vector to S_n^* at $(a_2^0, \dots, a_n^0)$. (Cf. also Paragraph 2.1.)

1.4. If $\zeta = (c_1, \dots, c_n)$ is an interior point of C_n, then $\gamma_n^{-1}(\zeta)$ is an infinite set in C. It is possible, however, to define in a natural way a subset of C which is homeomorphic to C_n under the mapping γ_n. Assume first that $\zeta \neq (0, \dots, 0)$. Since C_n is a convex body containing the origin, there is a unique number $t > 1$ such that $t\zeta$ is on the boundary of C_n. By Theorem A there is a unique set of numbers $\varkappa_j$ and μ_j, $j = 1, \dots, m$, $1 \leqq m \leqq n$, such that

$$g(z) = \sum_{j=1}^{m} \frac{1 + \varkappa_j z}{1 - \varkappa_j z} \mu_j$$

corresponds to $t\zeta$, i.e. $\gamma_n(g) = t\zeta$. The function $g^* = g/t + 1 - 1/t$, which can be written in the form

$$g^*(z) = 1 + 2 \sum_{j=1}^{m} \frac{\varkappa_j z}{1 - \varkappa_j z} \mu_j^* , \qquad \sum_1^m \mu_j^* = \frac{1}{t} ,$$

is in C and $\gamma_n(g^*) = \zeta$. If $\zeta = (0, \dots, 0)$ we choose $1/t = 0$, i.e. μ vanishes and g^* is just the constant 1, which case may be characterized also by setting $m = 0$. Thus we proved

Theorem A′. *To each point $\zeta = (c_1, \dots, c_n)$ of C_n there corresponds a unique set of distinct points $\varkappa_1, \dots, \varkappa_m$ on the unit circumference and a set of positive numbers $\mu_1, \dots, \mu_m$, where $\sum_{j=1}^m \leqq 1$ and $0 \leqq m \leqq n$, such that*

$$g(z; \zeta) = 1 + 2 \sum_{j=1}^{m} \frac{\varkappa_j z}{1 - \varkappa_j z} \mu_j$$

is in C and $\gamma_n(g(.\,; \zeta)) = \zeta$. The set of the numbers $\varkappa_j$ and μ_j might be empty in which case we set $m = 0$. ζ is on the boundary of C_n if and only if $\sum_1^m \mu_j = 1$. The correspondence $\zeta \mapsto g(.\,; \zeta)$ defines a homeomorphic mapping of C_n onto the subset $\{ g(.\,; \zeta) \}$ of C.

As before, we obtain from this Theorem A′, together with (1.3) and (1.2),

Theorem 1′. *Let M denote the set of measures defined by m distinct points $\varkappa_1, \dots, \varkappa_m$ on the unit circumference with assigned positive numbers $\mu_1, \dots, \mu_m$ such that $\sum_j^m \mu_j \leqq 1$ and $0 \leqq m < n$. Then by (1.1) there corresponds to M a subset of functions in S^* which is homeomorphic to S_n^*.*

2. Coefficient inequalities

2.1. Here we derive coefficient inequalities relative to a given boundary point $a^0 = (a_2^0, \dots, a_n^0)$ of S_n^*. By (1.3) there corresponds to a^0 a point $\zeta^0 = (c_1^0, \dots, c_{n-1}^0)$ on the boundary of C_{n-1}. Choose a supporting hyperplane at ζ^0, with normal direction $\alpha = (\alpha_1, \dots, \alpha_{n-1})$. Then, for any point $\zeta = (c_1, \dots, c_{n-1})$ of C_{n-1} we have the inequality

$$\text{Re} < \zeta - \zeta^0, \alpha > \leqq 0. \tag{2.1}$$

Since by (1.3) $c_j - c_j^0$ depends on the coefficients $a_2, a_2^0, \dots, a_{j+1}, a_{j+1}^0$ only, (2.1) represents an inequality involving the boundary point a^0 and an arbitrary point a of S_n^*. We transform now (2.1) considering that $\zeta - \zeta^0$ could be small. Let g and g_0 be functions in C such that $\gamma_{n-1}(g) = \zeta$ and $\gamma_{n-1}(g_0) = \zeta^0$:

$$g(z) = \sum_0^\infty 2\, c_j z^j, \qquad g_0(z) = \sum_0^\infty 2\, c_j^0 z^j, \qquad 2\, c_0 = 1.$$

Define

$$\Phi(z) = \int_0^z (g(t) - g_0(t)) \frac{dt}{t} = \varphi_1 z + \dots + \varphi_n z^n + \dots \tag{2.2}$$

where $2\,(c_j - c_j^0) = j\, \varphi_j$, $j = 1, 2, \dots$. Let the functions f and f_0 in S^* correspond to g and g_0. Equation (1.2) implies then

$$f(z) = z \exp \int_0^z (g(t) - 1) \frac{dt}{t} = f_0(z) \exp \Phi(z). \tag{2.3}$$

Write now

$$\alpha(z) = \frac{\alpha_{n-1}}{z^{n-1}} + \dots + \frac{\alpha_1}{z}, \tag{2.4}$$

$$k(z) = -\frac{\alpha'(z)}{f_0(z)} = \frac{k_n}{z^{n+1}} + \dots + \frac{k_2}{z^3} + \dots \tag{2.5}$$

and correspondingly $\bar{\alpha}(z) = \overline{\alpha(\bar{z})}$, $\bar{k}(z) = \overline{k(\bar{z})}$. It follows then

$$(2.6)\qquad 2 < \zeta - \zeta^0 , \alpha > = \frac{1}{2\pi i} \oint \bar{\alpha}(z)\,(g(z) - g_0(z)) \frac{dz}{z}$$

$$= -\frac{1}{2\pi i} \oint \bar{\alpha}'(z)\, \Phi(z)\, dz = \frac{1}{2\pi i} \oint \bar{k}(z)\, f_0(z)\, \Phi(z)\, dz$$

and by (2.1)

$$(2.7)\qquad \operatorname{Re} \frac{1}{2\pi i} \oint \bar{k}(z)\, f_0(z)\, \Phi(z)\, dz \leqq 0 ,$$

where the integration is taken along a positively oriented circuit around the origin.

2.2. Choose a variation of f_0 in S^*, i.e. a mapping $\varepsilon \mapsto f_\varepsilon$ of some interval $(0 , \varepsilon_0)$ into S^* such that

$$(2.8)\qquad f_\varepsilon = f_0 + \varepsilon f_1 + o(\varepsilon) ,$$

where f_1 is holomorphic in D with $f_1(z) = O(z^2)$, and $o(\varepsilon) / \varepsilon$ converges to zero uniformly on compact subsets of D. If

$$f_\varepsilon(z) = \sum_{j=1}^{\infty} a_j(\varepsilon)\, z^j \quad \text{and} \quad f_1(z) = \sum_{j=2}^{\infty} a'_j\, z^j ,$$

then $a' = (a'_2 , \ldots , a'_n)$ is a tangent vector to the curve $\varepsilon \mapsto a(\varepsilon) = (a_2(\varepsilon) , \ldots , a_n(\varepsilon))$, $0 \leqq \varepsilon \leqq \varepsilon_0$ at $a(0)$: $a(\varepsilon) = a(0) + \varepsilon a' + o(\varepsilon)$ for $\varepsilon \to 0$. Corresponding to (2.8) we have

$$(2.9)\qquad g_\varepsilon = g_0 + \varepsilon g_1 + o(\varepsilon) \quad \text{and} \quad \Phi_\varepsilon = \varepsilon \Phi_1 + o(\varepsilon) ,$$

where $\Phi_1(z) = \int_0^z g_1(\zeta)\, d\zeta / \zeta$. From (2.2) and (2.3) it then follows that

$$(2.10)\qquad f_1 = f_0\, \Phi_1 .$$

With $g_\varepsilon(z) = \sum_{j=0}^{\infty} c_j(\varepsilon)\, z^j$ and $g_1(z) = \sum_{j=1}^{\infty} c'_j\, z^j$ we get $\zeta(\varepsilon) = \zeta(0) + \varepsilon \zeta' + o(\varepsilon)$ for $\varepsilon \to 0$, where $\zeta(\varepsilon) = (c_1(\varepsilon) , \ldots , c_{n-1}(\varepsilon))$ and $\zeta' = (c'_1 , \ldots , c'_{n-1})$. If $\alpha = (\alpha_1 , \ldots , \alpha_{n-1})$ is an outer normal vector to C_{n-1} at the boundary point $\zeta(0)$, it follows from (2.1), (2.6), (2.9) and (2.2) that

$$\operatorname{Re} \frac{1}{2\pi i} \oint \bar{k}(z)\, f_1(z)\, dz = \operatorname{Re} \{ \bar{k}_2 a'_2 + \ldots + \bar{k}_n a'_n \} \leqq 0$$

for all tangent vectors a', i.e. $(k_2 , \ldots , k_n)$ is an outer normal vector to S_n^* at a^0.

Conversely, let k be an outer normal to S_n^* at $a(0)$. For any $g \in C$ define $g_\varepsilon = g_0 + \varepsilon (g - g_0)$, $0 \leqq \varepsilon \leqq 1$. This is a variation of g_0 in C. It infers a variation $f_\varepsilon = f_0 + \varepsilon f_1 + o(\varepsilon)$ of f_0 in S^*, and a curve $\varepsilon \mapsto a(\varepsilon) = a(0) + \varepsilon a' + o(\varepsilon)$ in S_n^*. Let now the vector $\alpha = (\alpha_1, \ldots, \alpha_{n-1})$ be given by (2.5) and (2.4). From (2.6) and (2.10) it follows that $\operatorname{Re} \sum_{j=1}^{n-1} \bar{\alpha}_j (c_j - c_j^0) \leqq 0$ for all points $\zeta = (c_1, \ldots, c_{n-1})$ of C_{n-1} and this shows that $\alpha = (\alpha_1, \ldots, \alpha_{n-1})$ is an outer normal vector to C_{n-1}. Thus, by (2.5), we proved

P r o p o s i t i o n 2.2. There is a one to one correspondence between the outer normal vectors to C_{n-1} and S_n^* at associated boundary points, which is given by the equations

$$
\begin{aligned}
\alpha_1 &= k_n a_{n-1} + k_{n-1} a_{n-2} + \ldots + k_3 a_2 + k_2 \\
2\alpha_2 &= k_n a_{n-2} + k_{n-1} a_{n-3} + \ldots + k_3 \\
&\cdots \\
(n-2)\alpha_{n-2} &= k_n a_2 + k_{n-1} \\
(n-1)\alpha_{n-1} &= k_n .
\end{aligned}
\tag{2.11}
$$

2.3. The preceding considerations suggest to develop (2.3) into powers of Φ and to write (2.7) in the form

$$
\operatorname{Re} \left\{ \frac{1}{2\pi i} \oint \bar{k}(z) \left(f(z) - f_0(z) - \frac{1}{2!} f_0(z)\, \Phi^2(z) - \ldots \right) dz \right\} \leqq 0 . \tag{2.12}
$$

Only the powers $\Phi^2, \ldots, \Phi^{n-1}$ are relevant for the evaluation of the integral, since Φ has a zero at the origin. However, the higher is the order of this zero the less powers of Φ are needed. Observe that by (2.2) we have

$$
\Phi(z) = \varphi_\varrho z^\varrho + \varphi_{\varrho+1} z^{\varrho+1} + \ldots , \qquad 1 \leqq \varrho < n , \tag{2.13}
$$

or $c_j = c_j^0$ for $j = 1, \ldots, \varrho - 1$ if and only if $a_j = a_j^0$, $j = 2, \ldots, \varrho$, and that in case of (2.13) we have

$$
\begin{aligned}
\bar{k}(z) f_0(z)\, \Phi^2(z) &= \left(\frac{\bar{k}_n}{z^{n+1}} + \ldots \right) (z + a_2^0 z^2 + \ldots)(\varphi_\varrho^2 z^{2\varrho} + \ldots) \\
&= \bar{k}_n \varphi_\varrho^2 z^{2\varrho - n} + \ldots .
\end{aligned}
\tag{2.14}
$$

We consider two cases.

1° $2\varrho \geqq n$. In this case inequality (2.12) reduces to

$$
\operatorname{Re} \left\{ \sum_{j=\varrho+1}^{n} \bar{k}_j (a_j - a_j^0) \right\} \leqq 0 . \tag{2.15}
$$

Equality occurs if and only if it holds in (2.7) also. $k = (k_2, \dots, k_n)$ is an outer normal vector to S_n^* at the point $a^0 = (a_2^0, \dots, a_n^0)$. To a^0 and k there corresponds the outer normal vector $\alpha = (\alpha_1, \dots, \alpha_{n-1})$ to C_{n-1} at $\zeta^0 = (c_1^0, \dots, c_{n-1}^0)$. From (2.6) it follows that equality holds in (2.7) if and only if

$$\operatorname{Re} <\zeta - \zeta^0, \alpha> = \operatorname{Re} \sum_{j=1}^{n-1} (c_j - c_j^0) \bar{\alpha}_j = 0,$$

i.e. the point $\zeta = (c_1, \dots, c_{n-1})$ lies on the supporting hyperplane through ζ^0 with normal direction α. Since, by assumption, $c_j = c_j^0$ for $j = 1, \dots, \varrho - 1$ and $2\varrho \geqq n$ it follows from Lemma 3.5 (in Paragraph 3.5, with $\varrho - 1$ and $n-1$ instead of ϱ and n respectively) that this occurs only if $\zeta = \zeta^0$, hence only if $a = a^0$ or $f = f_0$.

2° $n = 2\varrho + 1$. In this case the residue of $\bar{k}(z) f_0(z) \Phi^2(z)$ at the origin is $\bar{k}_n \varphi_\varrho^2$ and from (2.3) it follows $\varphi_\varrho = a_{\varrho+1} - a_{\varrho+1}^0$. Hence (2.12) implies

$$\operatorname{Re} \left\{ \sum_{j=\varrho+1}^{n} \bar{k}_j (a_j - a_j^0) - \frac{\bar{k}_n}{2} (a_{\varrho+1} - a_{\varrho+1}^0)^2 \right\} \leqq 0 .$$

If $a_{\varrho+1} = a_{\varrho+1}^0$ equality occurs only if $a = a^0$, because then we are in the preceding case, i.e. $n < 2(\varrho + 1)$. Thus we proved

T h e o r e m 2. *Let $a^0 = (a_2^0, \dots, a_n^0)$ be a boundary point of the coefficient body S_n^*, let $k = (k_2, \dots, k_n)$ be an outer normal vector to S_n^* at a^0 and let the integer ϱ satisfy the condition $\varrho < n \leqq 2\varrho + 1$. If $\varepsilon = 0$ for $n \leqq 2\varrho$ and $\varepsilon = 1$ for $n = 2\varrho + 1$, then the inequality*

$$\operatorname{Re} \left\{ \sum_{j=\varrho+1}^{n} \bar{k}_j (a_j - a_j^0) - \varepsilon \frac{\bar{k}_n}{2} (a_{\varrho+1} - a_{\varrho+1}^0)^2 \right\} \leqq 0 \tag{2.16}$$

holds for all points a of S_n^* such that $a_j = a_j^0$, $j = 2, \dots, \varrho$. In the case that $n \leqq 2\varrho$ or that $n = 2\varrho + 1$ and $a_{\varrho+1} = a_{\varrho+1}^0$ equality occurs in (2.16) if and only if $f = f_0$.

R e m a r k. Theorem 2 gives a coefficient inequality which is quite similar to the one J. A. Jenkins has given in his general coefficient Theorem [5] for the particular case of the normalized schlicht functions in the unit disc.

2.4. In the case $n \leqq 2\varrho$ Theorem 2 has an interesting corollary. Choose in S_ϱ^* a fixed point $(a_2^0, \dots, a_\varrho^0)$, $1 < \varrho < n$. Denote by $W_n(a_2^0, \dots, a_\varrho^0)$ the set of all points $(a_{\varrho+1}, \dots, a_n)$ in $C^{n-\varrho}$ such that $(a_2^0, \dots, a_\varrho^0, a_{\varrho+1}, \dots, a_n)$ is in S_n^*. Let $(a_{\varrho+1}^0, \dots, a_n^0) = A^0$ be on the boundary of W_n. Then $(a_2^0, \dots, a_\varrho^0, a_{\varrho+1}^0, \dots, a_n^0) = a^0$ is a boundary point of S_n^*. Choose there an outer normal vector $k = (k_2, \dots, k_n)$. By Theorem 2 we have (2.15) for all points $(a_{\varrho+1}, \dots, a_n)$ in $W_n(a_2^0, \dots, a_\varrho^0)$.

This shows that at each boundary point W_n has a supporting hyperplane, hence W_n is convex. It is even strictly convex, i.e. each supporting hyperplane to W_n contains only one point of W_n because equality occurs in (2.15) only if $a_j = a_j^0$, $j = \varrho+1, \dots, n$.

Now we show that $W_n(a_2, \dots, a_\varrho)$ is no more convex for arbitrary points $(a_2, \dots, a_\varrho)$ in S_ϱ^*, if $n > 2\varrho$. More precisely, we show that $W_{2\varrho+1}(0, \dots, 0)$ is not convex. For this purpose we consider the two functions

$$f_1(z) = (k(z^\varrho))^{1/\varrho} = z + \frac{2}{\varrho} z^{\varrho+1} + \frac{\varrho+2}{\varrho^2} z^{2\varrho+1} + \dots \text{ and}$$

$$f_2(z) = \varepsilon^{-1} f_1(\varepsilon z) = z - \frac{2}{\varrho} z^{\varrho+1} + \frac{\varrho+2}{\varrho^2} z^{2\varrho+1} + \dots$$

in S^*, where $\varepsilon^\varrho = -1$ and k is the Koebe function $k(z) = z / (1-z)^2$. They show that $W_{2\varrho+1}(0, \dots, 0)$ contains the points

$$(2/\varrho, 0, \dots, 0, (\varrho+2)/\varrho^2) \quad \text{and} \quad (-2/\varrho, 0, \dots, 0, (\varrho+2)/\varrho^2).$$

But the midpoint $(0, \dots, 0, (\varrho+2)/\varrho^2)$ of them does not belong to $W_{2\varrho+1}(0, \dots, 0)$, because for any schlicht function, hence for any starlike function $f(z) = z + a_{2\varrho+1} z^{2\varrho+1} + \dots$ we have $|a_{2\varrho+1}| \leqq 1/\varrho$ by a result of Prawitz ([8]) and because $1/\varrho < (\varrho+2)/\varrho^2$. Thus we proved

T h e o r e m 3. *Associate to a point* $(a_2^0, \dots, a_\varrho^0)$ *in* S_ϱ^* *and an integer* $n > \varrho$ *the set* $W_n(a_2^0, \dots, a_\varrho^0)$ *of those points* $(a_{\varrho+1}, \dots, a_n)$ *in* $C^{n-\varrho}$ *for which* $(a_2^0, \dots, a_\varrho^0, a_{\varrho+1}, \dots, a_n)$ *is in* S_n^*. *Then* $W_n(a_2^0, \dots, a_\varrho^0)$ *is a strictly convex body if* $n \leqq 2\varrho$. *However,* $W_{2\varrho+1}(0, \dots, 0)$ *is no longer convex.*

A similar theorem holds for the coefficient bodies of the class S (cf. [7]).

Let consider the particular case that $\varrho = n-1$. If $(a_2^0, \dots, a_{n-1}^0)$ is on the boundary of S_{n-1}^*, then obviously $W_n(a_2^0, \dots, a_{n-1}^0)$, the range of a_n, is a point. Thus we may assume that $(a_2^0, \dots, a_{n-1}^0)$ is in the interior of S_{n-1}^*. The corresponding point $(c_1^0, \dots, c_{n-2}^0)$ is in the interior of C_{n-2}. As was remarked by Carathéodory ([1]), the range of C_n is a disc. Hence, by (1.3), it follows: *For a given point* $(a_2^0, \dots, a_{n-1}^0)$ in S_{n-1}^* *the range of* a_n *is either a disc or a point.* Based on a different method this result was given by J. A. Hummel in [3].

3. Proof of Theorem A

3.1. C_n is a compact and convex set in C^n, since C is convex and compact (in the topology of uniform convergence on compact subsets of D), and γ_n is continuous and linear.

3.2. Let ζ be an interior point of C_n. There is a λ, $\lambda > 1$, such that $\lambda\,\zeta = (\lambda\, c_1, \dots, \lambda\, c_n)$ is still in C_n. Choose in C a function g such that $\gamma_n(g) = \lambda\,\zeta$. Then $g_1 = \lambda^{-1} g + (1 - 1/\lambda)$ is in C and satisfies $\operatorname{Re} g_1(z) > 1 - 1/\lambda$. Hence, $g_1 + h$ is in C and $\gamma_n(g_0 + h)$ equals to ζ for each function $h(z) = b_{n+1} z^{n+1} + \dots$ which is holomorphic in D, such that $\sup_{z \in D} |h(z)| \leq 1 - 1/\lambda$. This proves that $\gamma_n^{-1}(\zeta)$ is an infinite set in C.

3.3. Let P denote the set of probability measures supported by the unit circumference $\{ |z| = 1 \}$. According to a result of Herglotz g belongs to the class C if and only if

$$g(z) = \int_0^{2\pi} \frac{1 + e^{i\theta} z}{1 - e^{i\theta} z}\, d\mu_\theta\,, \qquad \mu \in P\,,$$

or equivalently if and only if the coefficients c_n of g (in $(0, 4)$) are the trigonometric moments of a probability measure, i.e.

$$c_n = \int_0^{2\pi} e^{in\theta}\, d\mu_\theta\,, \qquad \mu \in P\,, \qquad n = 0, 1, 2, \dots\,. \tag{3.1}$$

In the sequel we represent points in $\boldsymbol{R}^{2n}$ in the form $\zeta = (\zeta_1, \dots, \zeta_n)$, $\zeta_j \in \boldsymbol{C}$, as points in $\boldsymbol{C}^n$, and consequently, we write the standard scalar-product in $\boldsymbol{R}^{2n}$ as $\operatorname{Re} <\zeta, \zeta^1> = \operatorname{Re} \sum_{j=1}^n \bar{\zeta}_j\, \zeta_j'$.

Hence, the norm of ζ is given by $|\zeta| = (<\zeta, \zeta>)^{1/2}$.

Defining

$$\zeta(\mu) = \int_0^{2\pi} (e^{i\theta}, \dots, e^{in\theta})\, d\mu_\theta \tag{3.2}$$

for any real measure (supported by the unit circumference) we have

$$C_n = \{ \zeta(\mu) \mid \mu \in P \}\,,$$

i.e. C_n is the convex hull of the curve

$$\Gamma:\ \theta \mapsto (e^{i\theta}, \dots, e^{in\theta})\,, \qquad 0 \leq \theta \leq 2\pi\,.$$

Let α be a unit vector:

$$\alpha = (\alpha_1, \dots, \alpha_n)\,, \qquad |\alpha| = 1\,.$$

Define

$$h(\alpha) = \max_\theta \operatorname{Re} \{ \bar{\alpha}_1 e^{i\theta} + \dots + \bar{\alpha}_n e^{in\theta} \} \tag{3.3}$$

and

$$T(e^{i\theta}, \alpha) = h(\alpha) - \operatorname{Re}\left\{\sum_{j=1}^{n} \bar{\alpha}_j e^{ij\theta}\right\}. \tag{3.4}$$

Obviously $T(e^{i\theta}, \alpha) \geqq 0$ for all θ and α. Hence, by a lemma of Fejer and Riesz, there is a polynomial $p(z) = \xi_0 + \xi_1 z + \dots + \xi_n z^n$ such that

$$T(e^{i\theta}, \alpha) = |p(e^{i\theta})|^2 . \tag{3.5}$$

Let now $\zeta_0 = \zeta(\mu_0)$, $\mu_0 \in P$, be a point on the boundary of C_n. Since C_n is convex there is a supporting hyperplane

$$\operatorname{Re} <\zeta - \zeta_0, \alpha> = 0, \quad |\alpha| = 1, \text{ i.e. } \max_{\zeta \in C_n} \operatorname{Re} <\zeta, \alpha> = \operatorname{Re} <\zeta_0, \alpha> .$$

Since C_n is the convex hull of Γ we have also

$$\max_{\theta} \operatorname{Re} \sum_{j=1}^{n} \bar{\alpha}_j e^{ij\theta} = \operatorname{Re} <\zeta_0, \alpha>$$

or by (3.3) $h(\alpha) - \operatorname{Re} <\zeta_0, \alpha> = 0$. With the notations (3.4) and (3.5) and with (3.1) it follows

$$\int_0^{2\pi} T(e^{i\theta}, \alpha)\, d\mu_0 = \int_0^{2\pi} |p(e^{i\theta})|^2\, d\mu_0 = 0$$

and this shows that a measure μ_0, $\mu_0 \in P$, such that $\zeta(\mu_0)$ is on the boundary of C_n, is a measure supported by at most n points on the unit circumference.

Conversely, for an integer m, $1 \leq m \leq n$, choose m distinct points $\varkappa_j$ on the unit circumference and positives numbers μ_j, $j = 1, \dots, m$ such that $\sum \mu_j = 1$. Let the pairs $\{ (\varkappa_j, \mu_j) \}$ define the measure μ_0. Then $\zeta(\mu_0)$ is on the boundary of C_n. In fact, setting $p(z) = \Pi_1^m (z - \varkappa_j)$ we have

$$\int_0^{2\pi} |p(e^{i\theta})|^2\, d\mu_0 = \sum_1^m p(\varkappa_j)\, \mu_j = 0 . \tag{3.6}$$

But

$$|p(e^{i\theta})|^2 = \alpha_0 - \operatorname{Re} \sum_{j=1}^{n} \bar{\alpha}_j e^{ij\theta}$$

for suitably chosen numbers α_j. As a positive factor is not relevant we can assume that $\alpha = (\alpha_1, \dots, \alpha_n)$ is a unit vector and this implies $\alpha_0 = h(\alpha)$. Equivalently to (3.6) we have then

$$\int_0^{2\pi} T(e^{i\theta}, \alpha)\, d\mu_0 = 0\,, \quad \text{and} \quad \int_0^{2\pi} T(e^{i\theta}, \alpha)\, d\mu \geqq 0 \quad \text{for all} \quad \mu \in P\,,$$

i.e.

$$\max_{\zeta \in C_n} \operatorname{Re} <\zeta\,, \alpha> \;=\; \operatorname{Re} <\zeta(\mu_0)\,, \alpha>\,.$$

This shows that $\zeta(\mu_0)$ is on the boundary of C_n ending the proof that $\zeta = \gamma_n(g)$ belongs to the boundary of C_n if and only if g is given by (1.4).

3.4. It will be shown now that for a point $\zeta^0 = (c_1\,, \dots\,, c_n)$ on the boundary of C_n there is only one measure μ in P such that $\zeta(\mu) = \zeta^0$, and then this implies that there is a unique g in C satisfying $\gamma_n(g) = \zeta_0$.

Let μ be a measure in P such that $\zeta(\mu) = \zeta^0$. μ is carried by some points $\varkappa_1\,, \dots\,, \varkappa_m\,,\ 1 \leqq m \leqq n$; hence,

$$\zeta^0 = \mu_1\,\zeta^1 + \dots + \mu_m\,\zeta^m\,, \qquad \mu_j > 0\,, \qquad \sum_1^m \mu_j = 1\,,$$

where

$$\zeta^j = (\varkappa_j\,, \varkappa_j^2\,, \dots\,, \varkappa_j^n)\,, \qquad j = 1, \dots, m\,. \tag{3.7}$$

These vectors ζ^j are linearly independent (because the points $\varkappa_j$ are distinct). Their convex hull coh $\{\, \zeta^1\,, \dots\,, \zeta^m \,\}$ lies on the $m-1$ dimensional hyperplane

$$\zeta = \sum_{j=1}^m \lambda_j\,\zeta^j\,, \qquad \sum_1^m \lambda_j = 1\,, \tag{3.8}$$

and contains ζ^0 in its interior, since all the weights μ_j are positive.

Let now μ' be another measure in P such that $\zeta(\mu') = \zeta(\mu) = \zeta^0$. μ' is carried by some points $\varkappa_{1j}\,,\ j = 1, \dots, m_1\,,\ 1 \leqq m_1 \leqq n$ such that

$$\zeta^0 = \sum_{j=1}^{m_1} \zeta_1^j\,\mu_j'\,, \qquad \mu_j' > 0\,, \qquad \sum_1^{m_1} \mu_j' = 1\,,$$

where

$$\zeta_1^j = (\varkappa_{1j}\,, \varkappa_{1j}^2\,, \dots\,, \varkappa_{1j}^n)\,, \qquad j = 1, \dots, m_1\,.$$

Since ζ^0 is in the interior of either convex hull, say of coh $\{\, \zeta'\,, \dots\,, \zeta^m \,\}$ and of coh $\{\, \zeta_1^1\,, \dots\,, \zeta_1^{m_1} \,\}$, these convex hulls lie on the same hyperplane of dimension $m-1 = m_1-1$. Choose a $\varkappa_{1\varrho}\,,\ \varrho = 1\,, \dots, m_1\,,$ and write $\varkappa_{1\varrho} = \varkappa$. From $\zeta_1^\varrho = \sum_{j=1}^m \lambda_j\,\zeta^j\,,\ \sum_1^m \lambda_j = 1$, according to (3.8) it follows that the vectors $\zeta_1^\varrho - \zeta^1\,,\ \zeta^2 - \zeta^1\,, \dots\,,\ \zeta^m - \zeta^1$ are linearly dependent, hence

$$\begin{vmatrix} \varkappa - \varkappa_1 , & \varkappa_2 - \varkappa_1 , & \dots , & \varkappa_m - \varkappa_1 \\ \varkappa^2 - \varkappa_1^2 , & \varkappa_2^2 - \varkappa_1^2 , & \dots , & \varkappa_m^2 - \varkappa_1^2 \\ & \dots & & \\ \varkappa^m - \varkappa_1^m , & \varkappa_2^m - \varkappa_1^m , & \dots , & \varkappa_m^m - \varkappa_1^m \end{vmatrix} = 0 .$$

Consider this determinant as a polynomial in $\varkappa$. It is of degree m and does not vanish identically; its roots are $\varkappa_1 , \dots , \varkappa_m$; hence, $\varkappa_{1\varrho} = \varkappa$ equals to one of these roots and this implies that the two sets $\{ \varkappa_1 , \dots , \varkappa_m \}$ and $\{ \varkappa_{11} , \dots , \varkappa_{1m_1} \}$ are identical, i.e. the two measures μ and μ' have the same support $\{ \varkappa_1 , \dots , \varkappa_m \}$. Their values μ_j carried by the points $\varkappa_j$ have to satisfy the linear system

$$\sum_{j=1}^{m} \varkappa_j^\lambda \mu_j = c_\lambda , \qquad \lambda = 1, \dots, n .$$

Since the matrix $(\varkappa_j^\lambda)_{j=1,\dots,m}^{\lambda=1,\dots,n}$ has rank m, the point $\zeta^0 = (c_1 , \dots , c_n)$ uniquely determines the weights μ_j and this implies $\mu^1 = \mu$. We conclude that for a boundary point ζ^0 of C_n there is a unique function g in C such that $\gamma_n(g) = \zeta^0$, and this completes the proof of Theorem A.

3.5. The lemma we used in Paragraph 2.3 easily follows by a similar argument as used just ahead. Let a supporting hyperplane to C_n, with normal direction α, be given. If the polynomial $h(\alpha) - \operatorname{Re} \sum_{j=1}^{n} \overline{\alpha}_j z^j$ has the zeros $\varkappa_1 , \dots , \varkappa_m$ on the unit circumference, then the intersection of C_n with the given supporting hyperplane is the convex hull of the points ζ^j, $j = 1, \dots, m$, where the ζ^j are given by (3.7). Furthermore, let the coefficients $c_1 , \dots , c_\varrho$ be given. With $c_{-k} = \overline{c}_k$, $k = 1, \dots, \varrho$ and $c_0 = 1$ they have to satisfy the equations

$$\sum_{j=1}^{m} \varkappa_j^k \mu_j = c_k , \qquad k = 0 , \pm 1 , \dots , \pm \varrho ,$$

because the μ_j are real. The matrix $(\varkappa_j^k)_{k=-\varrho,\dots,\varrho}^{j=1,\dots,m}$ has rank m if $2\varrho + 1 \geqq m$. This shows that the coefficients $c_1 , \dots , c_\varrho$ uniquely determine the point $(c_1 , \dots , c_n)$ on the supporting hyperplane with the given normal direction α. Thus we proved

L e m m a 3.5. *Let a given supporting hyperplane to C_n touch the curve*

$$\Gamma : \theta \mapsto (e^{i\theta} , e^{2i\theta} , \dots , e^{in\theta}) , \qquad 0 \leqq \theta \leqq 2\pi ,$$

in m distinct points, $1 \leqq m \leqq n$. A point $(c_1 , \dots , c_n)$ of C_n on this hyperplane is then uniquely determined by its coordinates $c_1 , \dots , c_\varrho$, if $2\varrho + 1 \geqq m$.

References

[1] CARATHÉODORY, C.: Über den Variabilitätsbereich der Fourier'schen Konstanten von positiven harmonischen Funktionen. - Rend. Circ. Mat. Palermo 32, 1911, 193–217.

[2] GRENANDER, U., and G. SZEGÖ: Toeplitz forms and their applications. - California Monographs in Mathematical Sciences, University of California Press, Berkeley–Los Angeles, 1958.

[3] HUMMEL, J. A.: The coefficient regions of starlike functions. - Pacific J. Math. 7, 1957, 1381–1389.

[4] —»— A variational method for starlike functions. - Proc. Amer. Math. Soc. 9, 1958, 82–87.

[5] JENKINS, J. A.: An extension of the general coefficient theorem. - Trans. Amer. Math. Soc. 95, 1960, 387–407.

[6] NEVANLINNA, R.: Über die konforme Abbildung von Sterngebieten. - Översikt av Finska vetenskaps-societetens förhandlingar 63. A. 6, Helsingfors, 1920–1921, 1–21.

[7] PFLUGER, A.: О выпуклости некоторых сечений n-тел козффициентов однолистных функций. - Некоторые проблемы математики и механики, К семидесятилетию академика М. А. Лаврентъева, Издателъство "Наука", Ленинград, 1970, 233—241. Translation: On the convexity of some sections of the nth coefficient body for schlicht functions. - Some problems of mathematics and mechanics, On the occasion of the seventieth birthday of Academician M. A. Lavrent'ev, Amer. Math. Soc. Transl. (2) 104, 1976, 215–222.

[8] PRAWITZ, H.: Über Mittelwerte analytischer Funktionen. - Ark. Mat. Astronom. Fys. 20. A. 6, 1927–1928, 1–12.

[9] SCHAEFFER, A. C., and D. C. SPENCER: Coefficient regions for schlicht functions. - American Mathematical Society Colloquium Publications 35, New York, 1950.

[10] TOEPLITZ, O.: Über die Fourier'sche Entwickelung positiver Funktionen. - Rend. Circ. Mat. Palermo 32, 1911, 191–192.

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