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ON THE GREEN'S FUNCTION OF FUCHSIAN GROUPS

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1. Introduction

Let Γ be a Fuchsian group, that is a discontinuous group of Möbius transformations

$$\gamma(z) = e^{i\alpha}\frac{a-z}{1-\bar{a}z} \qquad (0 \leqq \alpha < 2\pi\,,\ |a| < 1) \tag{1.1}$$

of the unit disk $D = \{\,|z| < 1\,\}$ onto itself. For simplicity we assume throughout the paper that 0 is not an elliptic fixed point. Let ι denote the identity $\iota(z) \equiv z$ and let

$$F = \{\, z \in D:\ |\gamma'(z)| < 1 \text{ for all } \gamma \in \Gamma\,,\ \gamma \neq \iota\,\} \tag{1.2}$$

denote the normal fundamental domain with respect to 0.

A *character* of Γ is a complex-valued function $v(\gamma)$ satisfying

$$v(\varphi \circ \gamma) = v(\varphi)\, v(\gamma)\,, \qquad |v(\gamma)| = 1 \qquad (\varphi\,,\ \gamma \in \Gamma)\,.$$

An analytic function $f(z)$ $(z \in D)$ is called *character-automorphic* if

$$f(\gamma(z)) \equiv v(\gamma)\, f(z) \qquad (\gamma \in \Gamma) \tag{1.3}$$

for some character v of Γ. This is true if and only if $|f(\gamma(z))| \equiv |f(z)|$ for all $\gamma \in \Gamma$.

We assume now that Γ is of *convergence type*, that is

$$\sum_{\gamma \in \Gamma} (1 - |\gamma(z)|^2) \equiv (1 - |z|^2) \sum_{\gamma \in \Gamma} |\gamma'(z)| < \infty \qquad (z \in D)\,.$$

Then the *Green's function* of Γ with respect to 0 is defined as the Blaschke product (compare (1.1))

$$g(z) = \prod_{\gamma \in \Gamma} [e^{-i\vartheta(\gamma)}\, \gamma(z)] \qquad (\vartheta(\gamma) = \arg \gamma(0)\,,\ \vartheta(\iota) = 0)\,; \tag{1.4}$$

see Poincaré [15], Myrberg [13] and Nevanlinna [14, p. 214]. We have

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$$(1.5)\qquad |g(z)| = \prod_{\gamma\in\Gamma} |\gamma(z)|\,, \qquad \frac{g'(z)}{g(z)} = \sum_{\gamma\in\Gamma} \frac{\gamma'(z)}{\gamma(z)}\,.$$

The Green's function is character-automorphic and satisfies $g(0) = 0$, $|g(z)| < 1$ $(z \in D)$, and if $f(z)$ is any function with these properties then $|f(z)| \leqq |g(z)|$. Projecting $-\log |g(z)|$ to the Riemann surface D / Γ, we obtain the Green's function of D / Γ, the smallest positive harmonic function with a logarithmic pole at a certain point.

We say that an analytic function $f(z)$ $(z \in D)$ has the *angular limit* $f(\zeta)$ at $\zeta \in \partial D$ if $f(z) \to f(\zeta)$ as $z \to \zeta$ in every Stolz angle at ζ. The angular limit of the derivative is called the *angular derivative* and is denoted by $f'(\zeta)$ if it exists. The function is called of bounded characteristic (''beschränktartig'') if

$$(1.6)\qquad \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(r\,e^{i\vartheta})|\,d\vartheta \leqq K \qquad (0 < r < 1)\,.$$

This is true if and only if $f(z)$ is the quotient of two bounded analytic functions [14, p. 189].

The Green's function $g(z)$ of Γ has angular limits with $|g(\zeta)| = 1$ for almost all $\zeta \in \partial D$. Considering the angular derivative we define:

(a) Γ is of *accessible type* if $g'(\zeta)$ exists on a set of positive measure on ∂D;
(b) Γ is of *fully accessible type* if $g'(\zeta)$ exists almost everywhere on ∂D;
(c) Γ is of *Widom type* if the function $g'(z)$ is of bounded characteristic in D.

Since every function of bounded characteristic has finite angular limits almost everywhere [14, p. 208], it is clear that (c) $\Rightarrow$ (b) $\Rightarrow$ (a).

We shall give a number of characterizations of these concepts. In Theorem 1 we show, for instance, that

$$\Gamma \text{ is of accessible type} \Leftrightarrow \operatorname{mes}(\partial F \cap \partial D) > 0\,.$$

In Theorems 2 and 3 we characterize groups of accessible type in terms of their Riemann surface D / Γ, using results of J. E. McMillan [11] on the angular derivative of univalent functions. We construct a new example of a group of convergence type that is not of accessible type (compare [19, p. 515]).

Let $H^\infty(\Gamma, v)$ denote the Banach space of bounded analytic functions satisfying (1.3) for the character v of Γ. If Γ is of Widom type and $g^*(z)$ is the inner factor in the canonical representation of $g'(z)$ [6, p. 25], we show (Theorem 5) that

$$(1.7) \qquad f(z) = \frac{g(z)}{g'(z)} \sum_{\gamma \in \Gamma} \overline{v(\gamma)}\, h(\gamma(z))\, g^*(\gamma(z))\, \frac{\gamma'(z)}{\gamma(z)}$$

defines a bounded linear operator $h \in H^\infty \mapsto f \in H^\infty(\Gamma, v)$; this is a modification of a construction of Earle and Marden [7, p. 206]. We give an explicit formula for $g^*(z)$ in Theorem 8. It follows from Theorem 7 and from the remarkable results of Widom [21] (see also [20]) that

$$\Gamma \text{ is of Widom type} \Leftrightarrow H^\infty(\Gamma, v) \neq \{\text{const}\} \text{ for every } v$$

if Γ has no elliptic elements. Hardy classes of regular Riemann surfaces of Widom type were also considered by Hasumi [10].

Our definition (c) was suggested by a paper of Ahern and Clark [1] on the angular derivative of Blaschke products. In Theorem 6 we show that

$$\sum_{\gamma \in \Gamma} l(\gamma) = 2\pi, \qquad \sum_{\gamma \in \Gamma} l(\gamma) \log \frac{2\pi}{l(\gamma)} < \infty \Rightarrow \Gamma \text{ is of Widom type}$$

where $l(\gamma) = \operatorname{mes} \gamma(\partial F \cap \partial D)$, and in Theorem 7 that

$$\Gamma \text{ is of Widom type} \Rightarrow \sum_{\gamma \in \Gamma} |\gamma'(0)| \log \frac{1}{|\gamma'(0)|} < \infty .$$

2. Groups of accessible type

An *oricycle* at $\zeta \in \partial D$ is a disk in D touching ∂D at ζ. We call $\zeta \in \partial D$ an *oricyclic point* (with respect to Γ) if every oricycle at ζ contains only finitely many points $\gamma(0)$ $(\gamma \in \Gamma)$; it is easy to deduce that, for each $z \in D$, every oricycle contains only finitely many points $\gamma(z)$ $(\gamma \in \Gamma)$. This concept is motivated by the following lemma.

Lemma 1. *For every oricyclic point* ζ, *with at most countably many exceptions, there exists* $\gamma \in \Gamma$ *such that the normal fundamental domain* $\gamma(F)$ *with respect to* $\gamma(0)$ *is tangential to* ∂D *at* ζ.

The domain $H \subset D$ is called *tangential to* ∂D at ζ if H contains every Stolz angle

$$(2.1) \qquad S = \left\{ z \in D : |\arg(1 - \bar{\zeta} z)| < \frac{\pi}{2} - \delta ,\ |1 - \bar{\zeta} z| < \varrho \right\}$$

for $\delta > 0$ and some $\varrho = \varrho(\delta) > 0$.

Theorem 1. *Let* Γ *be a Fuchsian group and let* $0 < \beta \leqq 2\pi$. *Then the following four conditions are equivalent:*

(i) *The normal fundamental domains* $\gamma(F)$ *satisfy*

$$\sum_{\gamma \in \Gamma} \operatorname{mes}[\partial D \cap \partial \gamma(F)] \geqq \beta ;$$

(ii) *there exists a measurable set* $B \subset \partial D$ *containing no two* Γ-*equivalent points such that*

$$\sum_{\gamma \in \Gamma} \operatorname{mes} \gamma(B) \geqq \beta ;$$

(iii) Γ *is of convergence type and*

$$\operatorname{mes} \{ \zeta \in \partial D : g'(\zeta) \text{ exists} \} \geqq \beta ;$$

(iv) *the set of oricyclic points has measure* $\geqq \beta$.

The Fuchsian group Γ is called of *accessible type* if it satisfies the above (equivalent) conditions for some $\beta > 0$. We can replace (i) and (ii) by the more concise conditions

(i′) $\operatorname{mes} (\partial F \cap \partial D) > 0$;

(ii′) there exists a set of positive measure on ∂D that contains no two Γ-equivalent points.

The group Γ is called of *fully accessible type* if it satisfies the above conditions with $\beta = 2\pi$. Every group of the second kind is of accessible kind as (i′) shows, but need not be of fully accessible type as Example 2 will show.

Remark. One might attempt to "prove" (ii) for all groups as follows: We choose a representative in each Γ-equivalence class. Their union B contains no two Γ-equivalent points and satisfies

$$\bigcup_{\gamma \in \Gamma} \gamma(B) = \partial D ,$$

and this would seem to imply (ii) with $\beta = 2\pi$. Unfortunately, the set B need not be measurable as the existence of groups not of accessible type shows.

We need the following result of Frostman [9] on Blaschke products; see also Ahern and Clark [1].

Lemma 2. *Let* $\zeta \in \partial D$. *If* $|g(\zeta)| = 1$ *and* $g'(\zeta) \neq \infty$ *exist then*

$$|g'(\zeta)| = \sum_{\gamma \in \Gamma} |\gamma'(\zeta)| . \tag{2.2}$$

Conversely, if this sum converges then $|g(\zeta)| = 1$ *and* $g'(\zeta) \neq \infty$ *exist.*

Cargo [3] has shown that, in the above case,

$$g(z) \to g(\zeta) \text{ as } z \to \zeta \text{ in every oricycle at } \zeta . \tag{2.3}$$

Proof of Theorem 1. (i) $\Rightarrow$ (ii): It is sufficient to show that only countably many points on $\partial F \cap \partial D$ can be Γ-equivalent to some other point on $\partial F \cap \partial D$. Let ζ , $\zeta' \in \partial F \cap \partial D$ and $\zeta = \gamma(\zeta')$ for some $\gamma \in \Gamma$, $\gamma \neq \iota$. Since F is n.e. (= non-euclidean) convex and contains a disk around 0 , it is easy to see that the radial segments $[0 , \zeta]$ and

$[0, \zeta']$ lie in F. Hence the n.e. segment from $\gamma(0)$ to $\gamma(\zeta') = \zeta$ lies in $\gamma(F)$, and we deduce that the n.e. bisector of $[0, \gamma(0)]$ also ends at ζ. There exist only countably many such bisectors and thus only countably many such points ζ.

(ii) $\Rightarrow$ (iii): It is easy to deduce from (ii) [19, p. 514] that Γ is of convergence type. Furthermore, (ii) implies

$$\beta \leqq \sum_{\gamma \in \Gamma} \operatorname{mes} \gamma(B) = \sum_{\gamma \in \Gamma} \int_B |\gamma'(\zeta)|\, |d\zeta| = \int_B \Big(\sum_{\gamma \in \Gamma} |\gamma'(\zeta)| \Big) |d\zeta| . \tag{2.4}$$

Hence the sum (2.2) converges almost everywhere on B, and Lemma 2 shows that $g'(\zeta) \neq \infty$ exists almost everywhere on B and therefore almost everywhere on $\bigcup_{\gamma \in \Gamma} \gamma(B)$. It follows from (ii) that this is a disjoint union and that it has measure $\geqq \beta$. We remark that, by (2.4) and (2.2),

$$\int_B |g'(\zeta)|\, |d\zeta| = \sum_{\gamma \in \Gamma} \operatorname{mes} \gamma(B) . \tag{2.5}$$

(iii) $\Rightarrow$ (iv): It is sufficient to show that $\zeta \in \partial D$ is an oricyclic point if $|g(\zeta)| = 1$ and $g'(\zeta) \neq \infty$ exist. Lemma 2 shows that, under these conditions,

$$\sum_{\gamma \in \Gamma} |\gamma'(\zeta)| = |g'(\zeta)| < \infty .$$

If $\varepsilon > 0$ it follows that, for some finite subset $\Gamma_0 = \Gamma_0(\varepsilon)$ of Γ,

$$\frac{1 - |a|^2}{|\zeta - a|^2} = |\gamma'(\zeta)| < \varepsilon \qquad \text{for } \gamma \in \Gamma \setminus \Gamma_0 \tag{2.6}$$

where we use the notation (1.1). This is our assertion because $a = \gamma^{-1}(0)$ and because all oricycles at ζ have the form

$$\left\{ \frac{1 - |z|^2}{|\zeta - z|^2} \equiv \operatorname{Re} \frac{\zeta + z}{\zeta - z} \geqq \varepsilon \right\} \qquad (0 < \varepsilon < \infty) . \tag{2.7}$$

(iv) $\Rightarrow$ (i): This assertion follows at once from Lemma 1.

Proof of Lemma 1. Let ζ be oricyclic and not one of the countably many points where $|\gamma_1'(\zeta)| = |\gamma_2'(\zeta)|$ for some $\gamma_1 \neq \gamma_2$. Then, by definition, every oricycle (2.7) contains only finitely many points $a = \gamma^{-1}(0)$ $(\gamma \in \Gamma)$. Hence (2.6) holds for some finite set $\Gamma_0 = \Gamma_0(\varepsilon)$. It follows that, for $\gamma \in \Gamma \setminus \Gamma_0$,

$$|\gamma'(z)| = \frac{1 - |a|^2}{|1 - \bar{a}z|^2} \leqq \varepsilon \left| \frac{\zeta - 1/\bar{a}}{z - 1/\bar{a}} \right|^2 \qquad (z \in D) . \tag{2.8}$$

Let $\delta > 0$ and let S be the Stolz angle (2.1). Since $|1/\bar{a}| > 1$ it is easy to deduce geometrically from (2.8) that $|\gamma'(z)| < 1$ for $z \in S$ and $\gamma \in \Gamma \setminus$

Γ_0 if $\varepsilon > 0$ is sufficiently small. By the above property of ζ we can choose $\varrho(\delta)$ in (2.1) so small that $|\gamma_1'(z)| \neq |\gamma_2'(z)|$ for $z \in S$ and distinct $\gamma_1, \gamma_2 \in \Gamma_0$. Then there exists a unique $\varphi \in \Gamma_0$ such that

$$|\gamma'(z)| < |\varphi'(z)| \qquad (z \in S)$$

for $\gamma \in \Gamma_0$, $\gamma \neq \varphi$. Since $|\varphi'(z)| \geqq \iota'(z) = 1$ this relation holds for all $\gamma \in \Gamma$, $\gamma \neq \varphi$, and it follows that $S \subset \varphi^{-1}(F)$.

3. Groups of accessible type and Riemann surfaces

We give first a characterization in terms of simply connected domains.

Theorem 2. *The Fuchsian group Γ is of accessible type if and only if there exists a simply connected domain $G \subset D$ containing no two Γ-equivalent points, such that $\partial G \cap \partial D$ has positive harmonic measure relative to G.*

If $z = \psi(s)$ maps $\{ |s| < 1 \}$ conformally onto G and if A is the set of points $e^{i\vartheta}$ where the angular limit $\psi(e^{i\vartheta})$ exists and satisfies $|\psi(e^{i\vartheta})| = 1$, then the last condition of Theorem 2 means that $\operatorname{mes} A > 0$.

Proof. (a) Let Γ be of accessible type. Then we choose the normal fundamental domain F as G. Since ∂F is a rectifiable Jordan curve and since $\operatorname{mes}(\partial F \cap \partial D) > 0$ by Theorem 1 (i), it follows from Riesz' theorem [5, p. 50] that $\operatorname{mes} A > 0$.

(b) Conversely, let the condition of the theorem be satisfied and let $e^{i\vartheta} \in A$. Since $|\psi(e^{i\vartheta})| = 1 > |\psi(s)|$ it is clear that $\arg(\psi(s) - \psi(e^{i\vartheta}))$ is bounded in $s \in D$. Hence it follows from McMillan's twist point theorem [11, Th. 1] [16, p. 326] that the angular derivative $\psi'(e^{i\vartheta})$ exists and is $\neq 0, \infty$ on a set $A_0 \subset A$ with $\operatorname{mes} A_0 = \operatorname{mes} A > 0$. Another result of McMillan [11, Th. 2 (iii)] [16, p. 328] then shows that $B_0 = \psi(A_0) \subset \partial D$ has positive measure. By a simple property of the angular derivative [16, p. 303], the domain G is tangential to ∂D at every $\zeta \in B_0$. This implies that the sets $\gamma(B_0)$ $(\gamma \in \Gamma)$ are disjoint because the domains $\gamma(G)$ are disjoint by the hypothesis of the theorem. Hence B_0 satisfies condition (ii) of our Theorem 1 and Γ is therefore of accessible type.

We turn now to necessary and sufficient (conformally invariant) conditions in terms of the Riemann surface D / Γ obtained by identifying Γ-equivalent points. We assume that Γ has no elliptic elements, so that D is (conformally equivalent to) the universal covering surface of D / Γ. Then Theorem 2 states that Γ is of accessible type if and only if D / Γ contains a simply connected domain H such that $\partial H \cap \partial(D / \Gamma)$ has positive harmonic measure.

In the next criterion, we allow multiply connected domains. The sufficiency proof is based on a modification of the Lusin–Privalov construction due to McMillan [11]. We denote by $\Lambda(E)$ the *linear measure* (one-dimensional Hausdorff measure) of $E \subset C$.

Theorem 3. *The Fuchsian group Γ without elliptic elements is of accessible type if and only if there exists a domain $\Delta \subset D / \Gamma$ with the following properties:*

(1) *there is a conformal map h of some plane domain H onto Δ;*

(2) *there is a set $E \subset \partial H$ with $\Lambda(E) > 0$ such that, for every $w \in E$, the interior $T(w)$ of some equilateral triangle of apex w lies in G;*

(3) *if $w_n \in H$, $w_n \to w \in E$ $(n \to \infty)$ then the points $h(w_n) \in \Delta$ have no limit point in D / Γ.*

Proof. (a) Let Γ be of accessible type and let $\psi(s)$ map D conformally onto F. Since ∂F is a rectifiable Jordan curve and since $\operatorname{mes}(\partial F \cap \partial D) > 0$, we can find a set $A_0 \subset \psi^{-1}(\partial F \cap \partial D)$ with $\operatorname{mes} A_0 > 0$ such that the angular derivative $\psi'(s) \neq \infty$ exists for all $s \in A_0$ [5, p. 51], [16, p. 320]. Hence F is tangential to ∂D at each point $w \in E = \psi(A_0)$ so that F contains a triangle, and $\operatorname{mes} A_0 > 0$ implies $\Lambda(E) = \operatorname{mes} E > 0$.

Since F contains no two Γ-equivalent points the projection h of D onto D / Γ maps F (one-to-one) conformally onto some domain $\Delta \subset D / \Gamma$. Finally let $w_n \in F$, $w_n \to w \in E$. We have $|w_n| \leqq |\gamma(w_n)|$ $(\gamma \in \Gamma)$ by the definition of the normal fundamental domain. Since $|w| = 1$ it follows that $(h(w_n))$ has no limit point in D / Γ.

(b) Conversely, let the condition of the theorem be satisfied. We may assume that the triangle $T(w)$ has the rational angle $\alpha(w)$ at w and that its base lies on the (oriented) line $L(w)$ of rational inclination and rational distance from 0. Since $\{(\alpha(w), L(w)) : w \in E\}$ is countable and since $\Lambda(E) > 0$, there exists $E_0 \subset E$ with $\Lambda(E_0) > 0$ such that

$$\alpha(w) \equiv \alpha_0, \qquad L(w) \equiv L_0 \qquad \text{for } w \in E_0. \tag{3.1}$$

The union of the domains $T(w)$ $(w \in E_0)$ has a connected component H_0 such that $\Lambda(E_0 \cap \partial H_0) > 0$. It follows from (3.1) that H_0 is simply connected and that $\Lambda(\partial H_0) > 0$. Let $\varphi(s)$ map D conformally onto H_0. Then $\varphi(s)$ is continuous in $\bar{D}$, and

$$A_0 = \{ e^{i\vartheta} : \varphi(e^{i\vartheta}) \in E_0 \cap \partial H_0 \} \subset \partial D$$

satisfies $\operatorname{mes} A_0 > 0$ [16, p. 322].

Since $H_0 \subset H$ by property (2), we see from (1) that $h(H_0)$ is a simply connected subdomain of $\Delta \subset D / \Gamma$. Since Γ contains no elliptic elements the inverse p^{-1} of the projection maps $h(H_0)$ conformally onto some simply connected domain G_0 containing no two Γ-equivalent points, and $\psi = p^{-1} \circ h \circ \varphi$ maps D conformally onto G_0. It follows from property

(3) that $\psi(A_0) \subset \partial D$. Hence $\partial G_0 \cap \partial D$ has harmonic measure $\geqq \operatorname{mes} A_0 > 0$, and we conclude from Theorem 2 that Γ is of accessible type.

We construct now an *example*. Let L_1 be an open arc on ∂D; we allow $L_1 = \emptyset$. We choose a countable set $P \subset D$ with

$$\bar{P} \cap \partial D = L_0 = \partial D \setminus L_1$$

such that, at each $w \in L_0$, the symmetric Stolz angle of opening $\pi / 2$ contains infinitely many points of P. Let Γ be the Fuchsian group associated with the domain $G = D \setminus P$. Thus G is conformally equivalent to D / Γ. Hence the projection map is an automorphic function $f(z)$ ($z \in D$) with $f(D) = G$ which is thus non-constant and bounded. In particular, it follows that Γ is of convergence type. We may assume that $f(0) = 0$.

By Fatou's theorem the angular limit $f(\zeta)$ exists for almost all $\zeta \in \partial D$. We set

$$E_j = \{ \zeta \in \partial D : f(\zeta) \in L_j \} \qquad (j = 1, 2). \tag{3.2}$$

Since all angular limits $f(\zeta)$ lie on $\partial G = L_0 \cup L_1 \cup P$ and since P has zero capacity, it follows [14, p. 209] that $\operatorname{mes} E_0 + \operatorname{mes} E_1 = 2\pi$.

We show now that

$$\operatorname{mes} \left[E_0 \cap \bigcup_{\gamma \in \Gamma} \partial\gamma(F) \right] = 0. \tag{3.3}$$

Otherwise there would exist $\gamma \in \Gamma$ such that $\operatorname{mes}(E_0 \cap \partial\gamma(F)) > 0$. Let $\varphi(s)$ map D conformally onto $\gamma(F)$. Since $\partial\gamma(F)$ is rectifiable it follows from Riesz' theorem that $A_0 = \varphi^{-1}(E_0 \cap \partial\gamma(F))$ has positive measure. Now $\psi(s) = f(\varphi(s))$ maps D conformally onto $f(\gamma(F)) = f(F) \subset G$. It follows from (3.2) for every $e^{i\vartheta} \in A_0$ that $\psi(s)$ tends to a limit on L_0 as $s \to e^{i\vartheta}$ along a suitable arc. Since ψ is a bounded function it follows that the angular limit exists and satisfies $\psi(e^{i\vartheta}) \in f(E_0) = L_0$, in particular $|\psi(e^{i\vartheta})| = 1$. As in the proof of Theorem 2 we therefore deduce from McMillan's twist point theorem that $f(F)$, and thus G, is tangential to ∂D at some point of L_0. But this is false by our choice of $P = D \setminus G$. Thus (3.3) has been proved.

If we choose $L_1 = \emptyset$ then $\operatorname{mes} E_0 = 2\pi$. It follows from (3.3) that $\operatorname{mes}(\partial F \cap \partial D) = 0$. Hence we have obtained (compare Tsuji [19, p. 515]:

E x a m p l e 1. *There is a Fuchsian group not of accessible type for which there exists a non-constant bounded automorphic function and which is therefore of convergence type.*

Let now L_1 be an arc of length ε and let $\omega(w)$ be the harmonic measure of L_1 at w relative to D. Then $\omega(f(z))$ is bounded and harmonic in D and has, by (3.2), the angular limit 0 on E_0 and 1 on E_1. Therefore

$$\varepsilon = 2\pi\,\omega(0) = 2\pi\,\omega(f(0)) = \int_{E_1} |d\zeta| = \operatorname{mes} E_1 .$$

Using (3.3) and the fact that $\operatorname{mes} E_0 + \operatorname{mes} E_1 = 2\pi$ we deduce that

$$\operatorname{mes}\Big[\partial D \cap \bigcup_{\gamma \in \Gamma} \partial\gamma(F)\Big] \leqq \operatorname{mes} E_1 = \varepsilon .$$

Hence we have shown:

E x a m p l e 2. *For every* $\varepsilon > 0$, *there is a Fuchsian group of the second kind (thus of accessible type) such that*

$$\sum_{\gamma \in \Gamma} \operatorname{mes}\,[\partial D \cap \partial\gamma(F)] \leqq \varepsilon .$$

In particular the limit set of Γ *has measure* $\geqq 2\pi - \varepsilon$.

4. Groups of Widom type

Let Γ be a Fuchsian group of convergence type for which 0 is not an elliptic fixed point. We set

$$u(z) = \sum_{\gamma \in \Gamma} |\gamma'(z)| \qquad (z \in \bar{D}) . \tag{4.1}$$

Then $u(z) > 1$ because $\iota \in \Gamma$. It follows from (1.5) that the Green's function $g(z)$ satisfies

$$|g'(z)| = \left| g(z) \sum_{\gamma \in \Gamma} \frac{\gamma'(z)}{\gamma(z)} \right| \leqq u(z) \qquad (u \in D) . \tag{4.2}$$

T h e o r e m 4. *The following three conditions are equivalent:*

(i) $g'(z)$ *is of bounded characteristic;*

(ii) $\displaystyle\int_{\partial D} \log u(z)\, |dz| < \infty$;

(iii) *there exists a character-automorphic function* $g^*(z)$ *with* $g^*(0) \neq 0$ *such that*

$$|g^*(z)| \leqq \frac{|g'(z)|}{u(z)} \leqq 1 \qquad (z \in D) .$$

If (i) *holds then we can choose* g^* *as the inner factor of* g', *so that*

$$g'(z) = g^*(z) \exp\left[\frac{1}{2\pi} \int_{\partial D} \frac{\zeta + z}{\zeta - z} \log u(\zeta)\, |d\zeta|\right] \qquad (z \in D) . \tag{4.3}$$

We say that Γ is *of Widom type* if it satisfies the above equivalent conditions; we shall describe the relation of our definition with Widom's work in Section 5. We deduce first a consequence of (iii):

Theorem 5. *Let Γ be of Widom type and let $g^*(z)$ be the inner factor of $g'(z)$. If v is any character of Γ and if $h(z)$ is analytic and bounded in D then*

$$f(z) = \frac{g(z)}{g'(z)} \sum_{\gamma \in \Gamma} \overline{v(\gamma)}\, g^*(\gamma(z))\, h(\gamma(z)) \frac{\gamma'(z)}{\gamma(z)} \tag{4.4}$$

is analytic in D and satisfies $f(\gamma(z)) \equiv v(\gamma) f(z)$ $(\gamma \in \Gamma)$ and

$$\sup_{z \in D} |f(z)| \leqq \sup_{z \in D} |h(z)|, \quad f(0) = g^*(0)\, h(0). \tag{4.5}$$

Thus (4.4) defines a bounded linear operator from H^∞ into $H^\infty(\Gamma, v)$; compare [7]. If v^* is the character associated with g^* then we can write (4.4) as

$$f(z) = \frac{g(z)\, g^*(z)}{g'(z)} \sum_{\gamma \in \Gamma} \overline{v(\gamma)}\, v^*(\gamma)\, h(\gamma(z)) \frac{\gamma'(z)}{\gamma(z)}. \tag{4.6}$$

Metzger and Rajeswara Rao [12] have shown that this Poincaré theta series is $\not\equiv 0$ if $h(z) \not\equiv 0$ is a polynomial.

We only mention that (4.4) defines a bounded linear operator from the Hardy space H^p into $H^p(\Gamma, v)$ for every $p \geqq 1$ and that

$$\|f\|_p \leqq \|h\|_p \quad (1 \leqq p \leqq \infty);$$

compare Earle–Marden [7] and Widom [21].

Proof of Theorem 5. Since $g(0) / \gamma(0) = 0$ for $\gamma \neq \iota$ and $= g'(0)$ for $\gamma = \iota$, we have $f(0) = g^*(0)\, h(0)$ because $v(\iota) = 1$. If $|h(z)| \leqq M$ for $z \in D$ then, by (4.6), (4.1) and (iii),

$$|f(z)| \leqq M \left|\frac{g^*(z)}{g'(z)}\right| \sum_{\gamma \in \Gamma} \left|\frac{g(z)}{\gamma(z)}\right| |\gamma'(z)| \leqq M \left|\frac{g^*(z)}{g'(z)}\right| u(z) \leqq M.$$

In particular, we see that the series (4.4) converges absolutely and that $f(z)$ is analytic in D. For $\varphi \in \Gamma$, we obtain from (4.4) and (1.5) that

$$\begin{aligned} f(\varphi(z))\, \varphi'(z) &= \frac{g(z)}{g'(z)} \sum_{\gamma \in \Gamma} \overline{v(\gamma)}\, g^*(\gamma \circ \varphi(z))\, h(\gamma \circ \varphi(z)) \frac{\gamma'(\varphi(z))\, \varphi'(z)}{\gamma(\varphi(z))} \\ &= \frac{g(z)}{g'(z)} \sum_{\chi \in \Gamma} v(\varphi)\, \overline{v(\chi)}\, g^*(\chi(z))\, h(\chi(z)) \frac{\chi'(z)}{\chi(z)} = v(\varphi) f(z). \end{aligned}$$

Proof of Theorem 4. (a) Suppose that (i) holds. Since, by (2.2), $|g'(\zeta)| = u(\zeta)$ for almost all $\zeta \in \partial D$, it follows [6, p. 17] that (ii) holds. Furthermore we can write [6, p. 25]

$$(4.7) \qquad g'(z) = g^*(z)\, w(z) \qquad (z \in D)$$

where the inner factor is $g^*(z)$ and where the outer factor is given by the exponential in (4.3) because $|g'(\zeta)| = u(\zeta)$ for almost all ζ. Hence

$$(4.8) \qquad \log |w(z)| = \frac{1}{2\pi} \int_{\partial D} \frac{1 - |z|^2}{|\zeta - z|^2} \log u(\zeta)\, |d\zeta| .$$

It follows from a well-known identity and from $u(\zeta) = u(\gamma(\zeta))\, |\gamma'(\zeta)|$ that, for $\gamma \in \Gamma$,

$$\log |w(z)| = \frac{1}{2\pi} \int_{\partial D} \frac{1 - |\gamma(z)|^2}{|\gamma(\zeta) - \gamma(z)|^2} |\gamma'(\zeta)|\, [\log u(\gamma(\zeta)) + \log |\gamma'(\zeta)|]\, |d\zeta| .$$

If we substitute $\zeta^* = \gamma(\zeta)$ and use the Poisson integral formula to evaluate the contribution from the second summand, we see that

$$\log |w(z)| = \log |g(\gamma(z))| + \log |\gamma'(z)| .$$

It follows that $w(z)$ is character-automorphic, hence also $g^*(z)$.

We write now $\Gamma = \{ \gamma_k : k = 1, 2, \dots \}$ and

$$(4.9) \qquad v_n(z) = \log \sum_{k=1}^{n} |\gamma_k'(z)| \qquad (n = 1, 2, \dots) .$$

Computation shows that the Laplacian is

$$\Delta v_n = -e^{-2v_n} \left| \sum_{k=1}^{n} |\gamma_k'| \frac{\gamma_k''}{\gamma_k'} \right|^2 + e^{-v_n} \sum_{k=1}^{n} \frac{|\gamma_k''|^2}{|\gamma_k'|} .$$

Hence we obtain from Schwarz's inequality that $\Delta v_n \geqq 0$. Therefore $v_n(z)$ is subharmonic in D, and it follows from (4.9) and (4.8) that

$$v_n(z) \leqq \frac{1}{2\pi} \int_{\partial D} \frac{1 - |z|^2}{|\zeta - z|^2} v_n(\zeta)\, |d\zeta| \leqq \log |w(z)| \qquad (z \in D) .$$

If we let $n \to \infty$ we obtain that $\log u \leqq \log |w|$ and thus, by (4.7), that $u \leqq |w| = |g' / g^*|$. Hence (iii) holds.

(b) Suppose now that (ii) holds. It is easy to deduce from (1.1) that $|\gamma'(r z)| \leqq 4\, |\gamma'(z)|$ for $|z| = 1$, $0 \leqq r < 1$. Hence $u(r z) \leqq 4\, u(z)$ by (4.1). Therefore it follows from (4.2) that, for $0 \leqq r < 1$,

$$\frac{1}{2\pi} \int_{\partial D} \log^+ |g'(r z)|\, |dz| \leqq \log 4 + \frac{1}{2\pi} \int_{\partial D} \log u(z)\, |dz| < \infty .$$

Thus (i) holds. This proof is due to Ahern and Clark [1, p. 118].

(c) Suppose finally that (iii) holds. Then

$$|g^*(z)| \leqq 1\,, \qquad \left|\frac{g^*(z)}{g'(z)}\right| \leqq \frac{1}{u(z)} \leqq 1 \qquad (z \in D)$$

so that $g'(z)$ is the quotient of two bounded analytic functions and therefore of bounded characteristic [14, p. 189].

Theorem 6. *If there exists a measurable set* $B \subset \partial D$ *containing no two* Γ*-equivalent points such that, with* $l(\gamma) = \operatorname{mes} \gamma(B)$,

$$(4.10) \qquad \sum_{\gamma \in \Gamma} l(\gamma) = 2\pi\,, \qquad \sum_{\gamma \in \Gamma} l(\gamma) \log \frac{2\pi}{l(\gamma)} < \infty$$

then Γ *is of Widom type.*

Proof. We shall verify that condition (ii) of Theorem 4 is satisfied. It follows from the inequality between the geometric and arithmetic means that

$$\exp\left(\frac{1}{l(\gamma)} \int\limits_{\gamma(B)} \log u(z)\,|dz|\right) \leqq \frac{1}{l(\gamma)} \int\limits_{\gamma(B)} u(z)\,|dz|$$
$$= \frac{1}{l(\gamma)} \int\limits_{B} u(\zeta)\,|d\zeta| = \frac{2\pi}{l(\gamma)}$$

where we have used (2.5). Hence, by (4.11),

$$\int\limits_{\partial D} \log u\,|dz| = \sum_{\gamma \in \Gamma} \int\limits_{\gamma(B)} \log u\,|dz| \leqq \sum_{\gamma \in \Gamma} l(\gamma) \log \frac{2\pi}{l(\gamma)} < \infty\,.$$

Remark 1. The conditions (4.10) may be related to Carleson sets [4]. These are closed sets $E \subset \partial D$ for which

$$\sum_n l_n = 2\pi\,, \qquad \sum_n l_n \log \frac{2\pi}{l_n} < \infty$$

where l_n are the lengths of the open arcs of which $\partial D \setminus E$ is composed. The Carleson sets are the zero sets on ∂D of analytic functions with boundary values in Lip α for some $\alpha > 0$. Their zero sets A in $\overline{D}$ satisfy

$$(4.11) \qquad \int\limits_{\partial D} \log \frac{1}{\operatorname{dist}(z\,, A)}\,|dz| < \infty$$

as Taylor and Williams [18] have shown (I want to thank Dr. J. Stegbuchner for this reference). Since

$$|\gamma'(z)| = \frac{1 - |a|^2}{|z - a|^2} \leqq \frac{1 - |\gamma^{-1}(0)|^2}{\operatorname{dist}(z\,, A)^2} \qquad (\gamma \in \Gamma,\ z \in \partial D)$$

it is clear that

$$(4.12) \qquad A = \{\gamma(0) : \gamma \in \Gamma\} \text{ satisfies (4.11)} \Rightarrow \Gamma \text{ is of Widom type.}$$

It will be proved in a forthcoming paper in the Michigan Mathematical Journal that Γ is of Widom type if the limit points of Γ form a Carleson set and if Γ has no elliptic elements.

Remark 2. In a manner similar to Theorem 6, one can show that, for $0 < p < 1$,

$$(4.13) \qquad \sum_{\gamma \in \Gamma} l(\gamma) = 2\pi, \quad \sum_{\gamma \in \Gamma} l(\gamma)^{1-p} < \infty \Rightarrow g' \in H^p.$$

If Γ is finitely generated and of the second kind, Beardon [2] has proved that

$$\sum_{\gamma \in \Gamma} |\gamma'(0)|^{1-p} < \infty \qquad \text{for some } p = p(\Gamma) > 0.$$

It is easily seen that $|\gamma'(z)| \leqq \text{const} \cdot |\gamma'(0)|$ holds on the free sides of F, hence on $B = \partial F \cap \partial D$ except for the parabolic vertices. Hence (4.13) shows that $g' \in H^p$.

Theorem 7. *If Γ is of Widom type then*

$$(4.14) \qquad \sum_{\gamma \in \Gamma} |\gamma'(0)| \log \frac{1}{|\gamma'(0)|} < \infty.$$

In a similar manner we can show that $g' \in H^p$ implies $\sum |\gamma'(0)|^{1-p} < \infty$. This estimate is stronger than the estimate of Ahern and Clark [1, p. 120] for general Blaschke products.

Proof. Let $B = \partial F \cap \partial D$. There exists α $(0 < \alpha < 1)$ such that $B_0 = \{z \in B : u(z) > e^{\alpha}\}$ has positive measure because $u(z) > 1$. Since $u(\gamma(z))\, |\gamma'(z)| = u(z)$ we see that

$$(4.15) \qquad \sum_{\gamma \in \Gamma} \int_{B_0} \left(\alpha + \log \frac{1}{|\gamma'(z)|}\right) |\gamma'(z)|\, |dz| \leqq \sum_{\gamma \in \Gamma} \int_{B_0} \log \frac{u(z)}{|\gamma'(z)|} \cdot |\gamma'(z)|\, |dz|$$

$$= \sum_{\gamma \in \Gamma} \int_{\gamma(B_0)} \log u(\zeta)\, |d\zeta| \leqq \int_{\partial D} \log u(\zeta)\, |d\zeta|.$$

We set $\xi(t) = t\,[\alpha + \log(1/t)]$ $(0 < t \leqq 1)$. There is a unique t_0 with $0 < t_0 < 1$ and $\xi(t_0) = \xi(1) = \alpha$. It is easily verified that $\xi(t_1) < \xi(t_2)$ for $t_1 < t_0$, $0 < t_1 \leqq t_2 \leqq 1$. Since $(1/4)\,|\gamma'(0)| \leqq |\gamma'(z)| \leqq 1$ for $z \in B$ and since $(1/4)\,|\gamma'(0)| < t_0$ for all but finitely many $\gamma \in \Gamma$, we deduce that

$$\tfrac{1}{4} |\gamma'(0)| \left(\alpha + \log \frac{4}{|\gamma'(0)|}\right) \leqq |\gamma'(z)| \left(\alpha + \log \frac{1}{|\gamma'(z)|}\right).$$

We integrate over B_0. Since $\operatorname{mes} B_0 > 0$ the assertion (4.14) follows from (4.15).

5. The inner factor of the derivative

Let Γ be a Fuchsian group of convergence type without elliptic elements, so that D is conformally equivalent to the universal covering surface of D / Γ.

We need some results about the Green's function. For $\zeta \in D$, we define the Green's function with respect to ζ by

$$(5.1) \qquad g(z, \zeta) = \prod_{\gamma \in \Gamma} \left[\frac{\gamma(z) - \zeta}{1 - \bar{\zeta}\, \gamma(z)} e^{-i\vartheta(\gamma)} \right], \qquad \vartheta(\gamma) = \arg \frac{\gamma(0) - \zeta}{1 - \bar{\zeta}\, \gamma(0)} .$$

It is character-automorphic and satisfies $g(0, \zeta) > 0$, $|g(z, \zeta)| < 1$ and $g(z, 0) = g(z)$. We easily see that

$$(5.2) \qquad |g(z, \zeta)| = |g(\zeta, z)| \qquad (z, \zeta \in D).$$

In particular $g(0, \zeta) = |g(\zeta)|$.

Let now $\zeta \in \partial D$ be a parabolic fixed point of Γ. Its stabilizer $\Gamma_\zeta = \{ \varphi \in \Gamma : \varphi(\zeta) = \zeta \}$ consists of the elements

$$(5.3) \qquad \varphi_n(z) = \frac{2z + i n \beta(\zeta - z)}{2\zeta + i n \beta(\zeta - z)} \qquad (n = 0, \pm 1, \dots)$$

for some $\beta = \beta(\zeta) > 0$. Let R_ζ denote a complete set of right coset representatives of Γ with respect to Γ_ζ. Thus we can write Γ as the disjoint union

$$(5.4) \qquad \Gamma = \bigcup_{\gamma \in R_\zeta} (\Gamma_\zeta \circ \gamma).$$

Using the sin-product one can show that

$$(5.5) \qquad |g(z, r\zeta)| \to \exp\left[-\frac{2\pi}{\beta(\zeta)} \sum_{\gamma \in R_\zeta} \operatorname{Re} \frac{\zeta + \gamma(z)}{\zeta - \gamma(z)} \right]$$

as $r \to 1 - 0$, locally uniformly in D. Hence we are led to define the Green's function with respect to the parabolic fixed point ζ by

$$(5.6) \qquad g(z, \zeta) = \exp\left[-\frac{2\pi}{\beta(\zeta)} \sum_{\gamma \in R_\zeta} \left(\frac{\zeta + \gamma(z)}{\zeta - \gamma(z)} - i \operatorname{Im} \frac{\zeta + \gamma(0)}{\zeta - \gamma(0)} \right) \right].$$

This function is character-automorphic and satisfies $0 < |g(z, \zeta)| < 1$ and $g(0, \zeta) > 0$. It follows from (5.2) and (5.5) that $|g(r\zeta)| = g(0, r\zeta) \to g(0, \zeta)$ as $r \to 1 - 0$. Hence the angular limit $g(\zeta)$ satisfies

$$(5.7) \qquad |g(\zeta)| = g(0, \zeta) = \exp\left[-\frac{2\pi}{\beta(\zeta)} \sum_{\gamma \in R_\zeta} \frac{1 - |\gamma(0)|^2}{|\zeta - \gamma(0)|^2}\right].$$

Since $g'(z) / g(z)$ is of the form (1.5) we can write [8, p. 111]

$$(5.8) \qquad (\zeta - z)^2 \frac{g'(z)}{g(z)} = \sum_{n=m}^{\infty} a_n \exp\left[-\frac{2\pi\, n}{\beta} \frac{\zeta + z}{\zeta - z}\right] \qquad (a_m \neq 0),$$

a power series in the "local uniformizer" $\exp[-(2\pi / \beta)(\zeta + z)/(\zeta - z)]$. The number m is the multiplicity of ζ.

The open set $\{ z \in D : |g(z)| < r \}$ $(0 < r < 1)$ is invariant under Γ. Let $G(r)$ be the component of 0 and let $\Gamma(r) = \{ \gamma \in \Gamma : \gamma(G(r)) = G(r) \}$ be the stabilizer of $G(r)$.

Theorem 8. *Let Γ be a Fuchsian group of convergence type without elliptic elements. Then the following three conditions are equivalent:*

(i) *Γ is of Widom type;*

(ii) *the first Betti number $b(r)$ of $G(r) / \Gamma(r)$ satisfies*

$$\int_0^1 b(r)\, r^{-1}\, dr < \infty ;$$

(iii) *$\partial G(r) \cap \partial D$ consists of only finitely many equivalence classes of parabolic fixed points, and*

$$\prod_k |g(z_k)| > 0$$

where z_k denotes a full system of non-equivalent zeros of $g'(z)$ in D and of non-equivalent parabolic fixed points on ∂D, each with proper multiplicity.

If Γ is of Widom type then the inner factor of $g'(z)$ is given by

$$(5.9) \qquad g^*(z) = \prod_k g(z, z_k).$$

The first Betti number of the Riemann surface $G(r) / \Gamma(r)$ is the rank of the first singular homology group, in other words the maximal number of linearly independent elements in the abelianized group $\Gamma(r)$. H. Widom [21, p. 305] proved that

$$\text{(ii)} \Leftrightarrow H^\infty(\Gamma, v) \neq \{\text{const}\} \text{ for every character } v \text{ of } \Gamma.$$

His results were expressed in terms of cross-sections of unitary line bundles which become character-automorphic functions by uniformization. We shall only need the following easier result:

Lemma 3 (Widom [21, p. 312]). *We have*

$$\exp \int_0^1 b(r)\, r^{-1}\, dr = \sup_v \inf \{ \|f\|_\infty : f \in H^\infty(\Gamma, v) , \ |f(0)| = 1 \}$$

where v *runs through all characters of* Γ .

Proof of Theorem 8. (i) $\Rightarrow$ (ii). Choosing $h(z) \equiv 1$ in Theorem 5 we obtain a function $f \in H^\infty(\Gamma, v)$ with $|f(z)| < 1$, $f(0) = g^*(0)$. Hence it follows from Lemma 3 that

$$\exp \int_0^1 b(r)\, r^{-1}\, dr \leqq \frac{1}{|g^*(0)|} < \infty .$$

(ii) $\Rightarrow$ (iii). It follows from (ii) that $b(r) < \infty$ for every $r < 1$. Hence $G(r) / \Gamma(r)$ is a compact bordered surface with at most finitely many punctures. The border components of $G(r)$ have to lie in D (and not on ∂D because $|g(z)| = 1$ for almost all $z \in \partial D$); the punctures correspond to parabolic fixed points of Γ , and of these there are only finitely many equivalence classes.

Furthermore $b(r)$ is the number of equivalence classes of critical points and parabolic fixed points for which $|g(z_k)| < r$. Hence

$$\log \prod_k |g(z_k)| = \int_0^1 (\log r)\, db(r) = - \int_0^1 b(r)\, r^{-1}\, dr > -\infty .$$

Thus (iii) holds.

We need a lemma to complete the proof. Let $w_r(z)$ map D conformally onto the simply connected domain $G(r)$ such that $w_r(0) = 0$, $w_r'(0) > 0$. Then

$$\Phi(r) = \{ \varphi = w_r^{-1} \circ \gamma \circ w_r : \gamma \in \Gamma(r) \} \tag{5.10}$$

is a Fuchsian group in D .

Lemma 4. *The Green's function* $g_r(z)$ *of* $\Phi(r)$ *with respect to* 0 *satisfies* $g_r(z) = r^{-1} g(w_r(z))$.

Proof. The function $r^{-1} g(w_r(z))$ is character-automorphic with respect to $\Phi(r)$ and is bounded by 1 . Since the Blaschke product $g_r(z)$ has the same zeros $\varphi(0)$ ($\varphi \in \Phi(r)$) we see that

$$q(z) = r^{-1} g(w_r(z)) / g_r(z) \tag{5.11}$$

satisfies $0 < |q(z)| < 1$. If D_0 is a sufficiently small disk around 0 then the disks $\gamma(D_0)$ ($\gamma \in \Gamma$) are disjoint and $|g(z)| > \alpha > 0$ outside these disks. Since $q(z) \neq 0$ it follows from the minimum principle that

$$|q(z)| > \alpha' > 0 \qquad \text{for } z \in D . \tag{5.12}$$

Finally it follows from (5.11) that $|q(z)| = 1$ for almost all $z \in \partial D$. Hence $q(z)$ is a bounded inner function, and its representation [6, p. 24] shows that (5.12) is impossible unless $|q(z)| \equiv 1$ and therefore $q(z) \equiv 1$.

(iii) $\Rightarrow$ (i). We conclude from (iii) and (5.7) that

$$\prod_k g(0, z_k) = \prod_k |g(z_k)| > 0 .$$

Hence it follows from the choice of (z_k) that the functions

$$\tilde{g}(z) = \prod_k g(z, z_k), \qquad h(z) = \tilde{g}(z) / g'(z) \tag{5.13}$$

are analytic in D. Let now ζ be a parabolic fixed point of multiplicity m. We see from (5.13), (5.6) and (5.8) that

$$|h(z)| \leqq \frac{|g(z, \zeta)|^m}{|g'(z)|} = O(|\zeta - z|^2) \qquad (z \to \zeta). \tag{5.14}$$

We consider again the group $\Phi(r)$ $(0 < r < 1)$ defined by (5.10). Let $F(r)$ denote its normal fundamental domain. Let ξ be a parabolic fixed point of $\Phi(r)$. Then $\zeta = w_r(\xi)$ is a parabolic fixed point of $\Gamma(r)$ and hence of Γ. Since some oricycle at ζ belongs to $G(r)$, the mapping function has a finite non-zero angular derivative $w_r'(\xi)$ by a theorem of Carathéodory [16, p. 308]. Hence we conclude from (5.14) that

$$h(w_r(z)) / w_r'(z) = O(|\xi - z|^2) \quad \text{as } z \to \xi \text{ in every angle.} \tag{5.15}$$

We consider now the subharmonic function

$$u_r(z) = \left| \frac{r\, h(w_r(z))}{w_r'(z)} \right| \sum_{\varphi \in \Phi(r)} |\varphi'(z)| . \tag{5.16}$$

Since [19, p. 517] [17, p. 636]

$$\sum_{\varphi \in \Phi(r)} |\varphi'(z)| = O((1 - |z|)^{-1}) \qquad (|z| \to 1)$$

it follows from (5.15) and (5.16) that $u_r(z) \to 0$ as $z \to \xi$, $z \in F(r)$. Hence we conclude from (iii) and (5.16) that $u_r(z)$ is continuous in $\overline{F(r)}$ and that $u_r(\xi) = 0$ for all parabolic fixed points ξ.

Since $u_r(\varphi(z)) = u_r(z)$ $(\varphi \in \Phi(r))$ we deduce that the subharmonic function $u_r(z)$ attains its maximum on the free sides of $F(r)$ where, by Lemma 4 and by (5.13),

$$u_r(z) = \left| \frac{r\, h(w_r(z))}{w_r'(z)} \right| |g_r'(z)| = |h(w_r(z))\, g'(w_r(z))| = |\tilde{g}(w_r(z))| \leqq 1 .$$

Hence $u_r(z) \leqq 1$ for $z \in D$ and therefore, by (5.16),

$$(5.17) \qquad \sum_{\varphi \in \Phi(r)} |\varphi'(z)| \leqq \left| \frac{w_r'(z)}{r\, h(w_r(z))} \right| \qquad (z \in D)\,.$$

We keep $z \in D$ fixed and let $r \to 1 - 0$. Since the left-hand side of (5.17) contains only non-negative terms and since $w_r(z) \to z$, $w_r'(z) \to 1$ we see from (5.10) and (5.13) that

$$(5.18) \qquad u(z) = \sum_{\gamma \in \Gamma} |\gamma'(z)| \leqq \frac{1}{|h(z)|} = \left| \frac{g'(z)}{\tilde{g}(z)} \right| \qquad (z \in D)\,.$$

Hence condition (iii) of Theorem 4 is satisfied, so that Γ is of Widom type.

To prove (5.9) we write the inner factor of $g'(z)$ in the form [6, p. 24]

$$(5.19) \qquad g^*(z) = g_0(z) \exp\left(- \frac{1}{2\pi} \int_{|\zeta|=1} \frac{\zeta + z}{\zeta - z} d\mu(\zeta) \right)$$

where $g_0(z)$ is a Blaschke product and μ is a non-negative singular measure because $|g^*| \leqq 1$. It follows from (5.1) and (5.13) that the contribution to $g^*(z)$ from the zeros $z_k \in D$ is equal to $g_0(z)$. We see from (5.8) that $\mu(\{\zeta\}) = 2\pi\, m(\zeta) \,/\, \beta(\zeta)$ where $m(\zeta)$ is the multiplicity of the parabolic fixed point ζ. Hence (5.6) and (5.13) show that the contribution to $g^*(z)$ from the parabolic fixed points is cancelled by a corresponding term in (5.19), and it follows that $|g^*(z)| \leqq |\tilde{g}(z)|$.

On the other hand, we obtain from (5.18) that, for $0 < \varrho < 1$

$$\log \left| \frac{\tilde{g}(0)}{g^*(0)} \right| = \frac{1}{2\pi\varrho} \int_{|z|=\varrho} \log \left| \frac{\tilde{g}}{g^*} \right| |dz| \leqq \frac{1}{2\pi\varrho} \int_{|z|=\varrho} \log \left| \frac{g'}{g^* u} \right| |dz|\,.$$

Since $|g'(z)| = u(z)$ and $|g^*(z)| = 1$ for almost all $z \in \partial D$ and since $u(\varrho z) \leqq 4\, u(z)$ for $z \in \partial D$, it is easy to show that the last integral tends to 0 as $\varrho \to 1 - 0$. It follows that $|\tilde{g}(0) \,/\, g^*(0)| \leqq 1$ and hence from $|g^*| \leqq |\tilde{g}|$ that $|g^*| = |\tilde{g}|$, $g^* = \tilde{g}$.

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