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ON QUADRATIC DIFFERENTIALS WITH CLOSED TRAJECTORIES ON OPEN RIEMANN SURFACES

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1. Introduction

A holomorphic or meromorphic quadratic differential φ on a Riemann surface R is represented by a system of holomorphic resp. meromorphic function elements $\varphi_\nu(z_\nu)$ in the local parameters z_ν such that the expression $\varphi_\nu(z_\nu)\, dz_\nu^2$ stays invariant under a conformal transformation of the parameter. In the sequel we will always leave the index ν away and just write $\varphi(z)\, dz^2$. The horizontal trajectories of φ are the maximal curves α on R along which $\varphi(z)\, dz^2$ is real and positive. Thus zeroes and poles are excluded by definition on a trajectory; but of course it can tend to such a point in either direction. In this case the trajectory is called critical, otherwise regular.

The closed trajectories of a quadratic differential φ (if there are any) sweep out certain disjoint ringdomains R_i of R which we call the characteristic ringdomains of φ. On compact surfaces there are holomorphic quadratic differentials with the property that all regular horizontal trajectories are closed. The induced geometric structure on R can now be used to characterize the quadratic differentials with closed trajectories (see first part of Section 3).

It is the purpose of this paper to generalize these structure theorems to open (i.e. non compact) Riemann surfaces. But now φ can have infinitely many characteristic ringdomains. It is then said to be of infinite, otherwise of finite (topological) type. It is this last case we are going to deal with. One can roughly say that everything what is true on compact surfaces is also true, in the finite case, on arbitrary open surfaces.

This paper is closely related to the papers [1], [2], [3] of the author. However, we only make use of the main existence theorem on compact Riemann surfaces, which is proved in [1]. This theorem is then first generalized, by the process of doubling, to compact bordered surfaces. The rest

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is done by exhaustion. The case of compact surfaces with punctures is contained in the general result to be proved, but now no branched covering surfaces are needed. Also the quadratic differentials with second order poles, where one has to work with reduced moduli, are treated in the same way. For later generalizations, the extremal property of quadratic differentials with closed trajectories (and finite norm) is proved for arbitrary type.

2. Extremum properties of quadratic differentials with closed trajectories

Definition 1. A meromorphic quadratic differential φ on an arbitrary Riemann surface R is said to have closed trajectories, if its non closed trajectories cover a set of measure zero. (A point set is said to have measure zero, if its intersection with every parameter neighbourhood has area measure zero, in the respective parameter plane.)

A quadratic differential with closed trajectories cannot have poles of higher order than two, and at every pole of order two the leading coefficient must be negative.

The characteristic ringdomains R_i of a quadratic differential are the ringdomains swept out by its closed trajectories. If φ has closed trajectories, its characteristic ringdomains fill out the surface up to a set of measure zero.

Definition 2. A system of finitely or infinitely many Jordan curves γ_i on a Riemann surface R is called admissible, if none of the curves is homotopically trivial (homotope zero) and if, for $i \neq k$, $\gamma_i \cap \gamma_k = \emptyset$ and $\gamma_i \nsim \gamma_k$, where the symbol $\sim$ means free homotopy.

If φ is holomorphic on R and if we pick a closed trajectory α_i from every characteristic ringdomain R_i of φ, we have an admissible curve system $\{\alpha_i\}$. The same is true for a meromorphic φ, if we puncture R at the poles of φ.

Definition 3. A ringdomain R_0 on R is said to be of homotopy type γ, if a Jordan curve $\gamma_0 \subset R_0$ which separates its two boundary components is freely homotopic to γ.

A system of non overlapping ringdomains $R_j \subset R$ is said to be of homotopy type $\{\gamma_i\}$, where $\{\gamma_i\}$ is an admissible curve system, if every R_j is of homotopy type γ_i for exactly one γ_i. It is, however, not required that to every γ_i there really exist a ringdomain R_i of this type. (If there is no ringdomain, we sometimes say that it is degenerate and has modulus zero, which allows us to take the same index for corresponding elements of both sets.)

Finally, a holomorphic quadratic differential φ with closed trajectories is said to be of homotopy type $\{\gamma_i\}$, if its characteristic ringdomains are of this type.

We are now going to prove the basic extremal property of quadratic differentials with closed trajectories and finite norm. For its formulation we need a few notations. We will consider systems of non overlapping ringdomains $\tilde{R}_j$ such that the system $\{\tilde{\gamma}_j\}$, where $\tilde{\gamma}_j$ is a Jordan curve in $\tilde{R}_j$ which separates its boundary components, is admissible. We denote by a_j the infimum of the lengths (in the metric $|\varphi(z)|^{1/2}\,|dz|$) of all closed curves on R which are freely homotopic to $\tilde{\gamma}_j$. For the characteristic ringdomains R_i we have $a_i = \int_{\alpha_i} |\varphi(z)|\,|dz|$ with α_i any closed horizontal trajectory in R_i . $\tilde{M}_j$ and M_i are the moduli of $\tilde{R}_j$ and R_i respectively. The following inequalities hold: $0 \leq a_j < \infty$, $0 < \tilde{M}_j \leq \infty$, $0 < M_i < \infty$.

Theorem 1. *Let φ be a holomorphic quadratic differential with closed trajectories and finite norm*

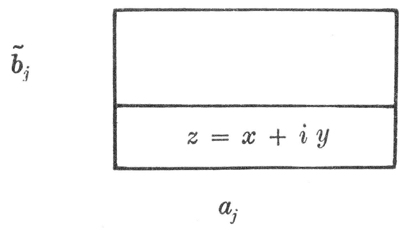
$$\|\varphi\| = \iint_R |\varphi(z)|\, dx\, dy$$

on a Riemann surface R . Assume that the curves $\tilde{\gamma}_j$ are freely homotopic to (some of) the curves α_i or vice versa. Then

$$\sum_j a_j^2 \tilde{M}_j \leq \|\varphi\| = \sum a_i^2 M_i , \tag{1}$$

where the sum on the left hand side goes over all values j with $a_j > 0$. Equality holds if and only if the two systems of ringdomains are identical.

Proof. Let $\tilde{R}_j$ be a ringdomain with $a_j > 0$. We map it conformally, first on an annulus, then, after cutting it along a radius, onto a horizontal rectangle in the z-plane (the concentric circles going into horizontal straight segments) of base a_i and height $\tilde{b}_i$ ($\leq \infty$).



A horizontal interval corresponds to a closed curve which is homotopic to $\tilde{\gamma}_j$ and therefore has φ-length $\geq a_j$. If we denote the representation of φ in terms of the parameter z by $\varphi(z)$, we have

$$a_j \leq \int |\varphi(x + i\,y)|^{1/2}\,dx\,. \tag{2}$$

Integration with respect to y and subsequent application of the Schwarz inequality yield

$$a_j\,\tilde{b}_j \leq \iint_{\tilde{R}_j} |\varphi(z)|^{1/2}\,dx\,dy \tag{3}$$

and

$$a_j\,\tilde{b}_j \leq \iint_{\tilde{R}_j} |\varphi(z)|\,dx\,dy\,, \tag{4}$$

hereby showing that $\tilde{b}_j < \infty$.

Summing over the indices j, with $a_j\,\tilde{b}_j = a_j^2\,(\tilde{b}_j\,/\,a_j) = a_j^2\,\tilde{M}_j$ we get

$$\begin{aligned} \sum_j a_j^2\,\tilde{M}_j &\leq \sum_j \iint_{\tilde{R}_j} |\varphi(z)|\,dx\,dy = \iint_{\bigcup_j \tilde{R}_j} |\varphi(z)|\,dx\,dy \\ &\leq \|\varphi\|_R = \sum_i a_i^2\,M_i\,, \end{aligned} \tag{5}$$

which establishes the inequality (1). We have not made use, so far, of the homotopy assumptions.

Assume now that equality holds. In the first case, every $\tilde{\gamma}_j$ is freely homotopic to a certain closed trajectory α_i, and we can use the same index. We must have equality in (2) for almost all, hence for all y. Therefore the horizontals are going into closed trajectories of R_i, which means that $\tilde{R}_i$ is a subring of R_i swept out by closed trajectories. If $\tilde{R}_i$ were not identical with R_i, an open subring of positive φ-area would be missing, which is impossible. The same argument shows that to every R_i there must be an $\tilde{R}_i$.

In the second case, for every α_i there is a $\tilde{\gamma}_i$ and by the same argument, $\tilde{R}_i$ is a subring of R_i swept out by horizontal trajectories. Let $\tilde{R}_j$ be a remaining ringdomain, which does not correspond to an α_i. The horizontals of the rectangle in the z-plane which corresponds to $\tilde{R}_j$ must go over into closed geodesics on the surface, in order to have equality in (2). Moreover, $\tilde{R}_j$ must have points in common with some R_i. But if the closed geodesics in $\tilde{R}_j$ would not be horizontal, it would intersect $\tilde{R}_i$, which is impossible. On the other hand they cannot be horizontal, otherwise $\tilde{\gamma}_j \sim \tilde{\gamma}_i$ contrary to assumption. Hence there cannot be any $\tilde{R}_j$ left over. But then $\tilde{M}_i$ must be equal to M_i because of (5), hence $\tilde{R}_i = R_i$ for all i.

The next extremal property is a consequence of theorem one, but not equivalent to it. It has, however, the remarkable feature that it can be formulated without making use of the φ-metric.

Theorem 2. *Let* φ *and* $\{\tilde{R}_j\}$ *be as before. If there are curves* α_i *to which there is no corresponding* $\tilde{R}_i$, *we say that this ringdomain is degenerate and has modulus* $\tilde{M}_i = 0$, *and similarly for the curves* $\tilde{\gamma}_j$. *We then use the same index for both sets. With this convention we have*

$$\inf\left\{\frac{\tilde{M}_i}{M_i}\right\} \leq 1, \tag{6}$$

and equality holds if and only if $\tilde{R}_i = R_i$ *for all* i.

Proof. Assume first that the set of free homotopy classes of the curves α_i contains the corresponding set for the curves $\tilde{\gamma}_j$. We then have because of Theorem 1

$$\sum_i a_i^2 \tilde{M}_i \leq \sum_i a_i^2 M_i \tag{7}$$

and hence

$$\sum a_i^2 (\tilde{M}_i - M_i) \leq 0 \tag{8}$$

with $a_i > 0$ for all i. Hence $\inf\{\tilde{M}_i - M_i\} \leq 0$, and since $M_i > 0$ for every i, $\inf\{\tilde{M}_i / M_i\} \leq 1$. Let equality hold. Then $\tilde{M}_i \geq M_i$. Thus equality must hold in (7) and we conclude from Theorem 1 that $\tilde{R}_i = R_i$ for all i.

Let now the set of homotopy classes of the curves α_i be a subset of the set of homotopy classes of the curves $\tilde{\gamma}_j$. We can now write (1) in the form

$$\sum_i a_i^2 \tilde{M}_i + \sum_h a_h^2 \tilde{M}_h \leq \sum_i a_i^2 M_i, \tag{9}$$

where the sum $\sum_h$ goes over the additional ringdomains $\tilde{R}_j$. We conclude that

$$\sum_i a_i^2 (\tilde{M}_i - M_i) \leq 0,$$

hence $\inf\{\tilde{M}_i / M_i\} \leq 1$ and a fortiori $\inf\{\tilde{M}_j / M_j\} \leq 1$. Let equality hold. Then

$$\sum_i a_i^2 \tilde{M}_i = \sum_i a_i^2 M_i$$

and by Theorem 1 again $\tilde{R}_i = R_i$ for all i; in particular, there cannot be any additional ringdomains $\tilde{R}_j$.

3. Existence proof for compact, bordered surfaces

The rest of the paper is based on the following *existence and uniqueness theorem* (see [1]):

Let $\{\gamma_i\}$ *be an admissible curve system on a compact surface* R. *Let* $\{m_i\}$ *be a system of positive numbers. Then there exists a holomorphic quadratic differential* φ *with closed trajectories such that its characteristic ringdomains* R_i *are of homotopy type* $\{\gamma_i\}$ *and have moduli* $M_i = \lambda \cdot m_i$, *for some* $\lambda > 0$. *The differential* φ *is determined up to a positive factor. It is called the solution of the modulus problem for the curves* γ_i *with weights* m_i.

Let now R be a compact, bordered surface, with boundary curves Γ_j, $j = 1, \ldots, q$. Let $\{\gamma_i\}_{i=1,\ldots,p}$ denote an admissible curve system on R, with weights $m_i > 0$. The mirror image of R is denoted by R^*; $\hat{R} = R \cup R^*$ is the double, with corresponding boundary points identified. The symmetric image of γ_i is denoted by γ_i^*. Whenever γ_i is homotopic to some Γ_j, we have $\gamma_i^* \sim \Gamma_j^* = \Gamma_j$. We therefore disregard γ_i^* in this case (and may of course replace γ_i by Γ_j).

The remaining curve system γ_i, γ_i^* is admissible on $\hat{R}$. We have to show that $\gamma_i \nsim \gamma_k$ for $i \neq k$, and that $\gamma_i \nsim \gamma_k^*$ for all i and k (if there is a γ_k^*).

Let $\gamma_i \sim \gamma_k$, $i \neq k$. Let D be a ringdomain bounded by the two curves. D must have points in common with R^*, otherwise γ_i and γ_k would be homotopic on R. But as the boundary $\gamma_i \cup \gamma_k$ of D is in R, D contains all of R^*. R^* must be a subannulus of D, bounded by two curves Γ_1, Γ_2. Then R is an annulus (the symmetric image of R^*), hence $\gamma_i \sim \gamma_k$ on R, a contradiction.

Let $\gamma_i \sim \gamma_k^*$. Again, let D be the ringdomain bounded by γ_i and γ_k^*. D contains points of R as well as of R^*, hence a boundary curve Γ_j. This curve cannot be homotopic to zero and must therefore be homotopic to both boundary curves of D. We have $\gamma_i \sim \Gamma_j$. But then γ_i^* was left out, hence $i \neq k$. As $\gamma_i \sim \gamma_i^*$, $\gamma_i^* \sim \gamma_k^*$, we conclude $\gamma_i \sim \gamma_k$, an impossibility.

We now assign weights to the curves of our system: $\gamma_i \to m_i$ as before, unless $\gamma_i \sim \Gamma_j$, in which case we set $\gamma_i \to 2\,m_i$, and $\gamma_i^* \to m_i$. Let φ be the solution of the modulus problem with these weights. Let R_i, R_i^* be the corresponding ringdomains. The symmetry T of $\hat{R}$ takes the system of curves and weights into itself, hence also the system of ringdomains.

Let R_i be the ringdomain associated with γ_i. First assume $\gamma_i \nsim \Gamma_j$ for all j. We want to show that then $R_i \subset R$. Let $R_i \cap R^* \neq \emptyset$. If $R_i \subset R^*$, pick a closed trajectory $\alpha_i \subset R_i$. As it does not meet γ_i, it bounds, together with γ_i, a ringdomain D, which, in its turn evidently must contain a boundary curve Γ_j. Thus $\gamma_i \sim \Gamma_j$, in contradiction to the hypothesis. So let R_i contain points of R and of R^*, hence a boundary point P. The symmetry T leaves P invariant and takes R_i into R_i^*. Therefore $R_i \cap R_i^* \neq \emptyset$, an impossibility. We conclude: For every curve γ_i which is not homotopic to a boundary curve Γ_j, the corresponding ringdomain R_i lies on R.

Assume now $\gamma_i \sim \Gamma_j$. Then R_i must be symmetric (invariant under T). It therefore contains a boundary curve Γ_j which must be a line of symmetry. The ringdomain $R_i \cap R$ has modulus m_i. We therefore have solved the modulus problem for the original surface R with the given curves γ_i and weights m_i.

Let $P \in \Gamma_j$. If P is in one of the ringdomains on $\hat{R}$, it lies on a trajectory (line of symmetry of some R_i). Otherwise it lies on the boundary of one of the ringdomains, hence on a trajectory or else it is a zero of φ, the restriction of $\hat{\varphi}$ to R.

4. Existence and uniqueness for arbitrary Riemann surfaces

We proceed to prove the existence theorem for arbitrary Riemann surfaces with finite admissible curve systems. The proof is valid, in particular, for compact surfaces with finitely many distinguished points. As a corollary we find that every surface, except for the sphere with less than four boundary points, carries holomorphic quadratic differentials with finite norm. Another corollary is that ringdomains with maximal moduli are uniquely determined, if the moduli are finite.

Let $\{\gamma_i\}_{i=1,\ldots,p}$ be an admissible curve system on an arbitrary Riemann surface R and let $m_i > 0$ be the weights. Moreover, assume that the supremum $M(\gamma_i)$ of the moduli of all the ringdomains on R which are of homotopy type γ_i is finite for every i. We put

$$M = \max \{ M(\gamma_i) \mid 1 \leq i \leq p \} .$$

For a subdomain of the sphere this rules out domains with less than four boundary points, for then the boundary components would have to be points, and any non trivial Jordan curve γ would bound a punctured disk, hence $M(\gamma) = \infty$. On the other hand, let R be a subdomain of the Riemann sphere with at least four boundary points. Then either the boundary contains a continuum or we have a non trivial curve γ which does not bound a puctured disk. For surfaces of positive genus we can pick a curve which does not disect the surface. Then its maximal modulus is finite, as it would otherwise be the boundary of a puctured disk on R, hence clearly disect the surface. This proves the first corollary.

To prove uniqueness, let φ and $\tilde{\varphi}$ be two solutions, with ringdomains R_i and $\tilde{R}_i$ of homotopy type $\{\gamma_i\}$, and moduli M_i and $\tilde{M}_i$ respectively. Then the extremal property of Theorem 2, applied to both φ and $\tilde{\varphi}$, immediately shows that $R_i = \tilde{R}_i$ for all i. As the trajectories are the same in R_i, the quotient $\tilde{\varphi} / \varphi$ must be real and positive, hence a positive constant. Because of the normalization $\|\varphi\| = \|\tilde{\varphi}\| = 1$, we must have $\tilde{\varphi} = \varphi$.

The existence of a solution is shown by a limiting process. Consider an exhaustion of R by compact bordered surfaces $R^{(n)}$, $n = 1, 2, \ldots$. We can assume, without loss of generality, that the curves γ_i are in $R^{(n)}$ for all n. We can therefore solve the modulus problem for the curve system $\{\gamma_i\}$ and the weights m_i on $R^{(n)}$. Let φ_n, with norm $\|\varphi_n\| = 1$, be the solution. The ringdomains, moduli etc. are denoted by R_{in}, $M_{in} = b_{in} / a_{in}$ etc. I claim that the sequence (φ_n) tends in norm (hence locally uniformly) towards a quadratic differential φ on R with closed trajectories, the characteristic ringdomains R_i of which solve the modulus problem with curves γ_i and weights m_i for R.

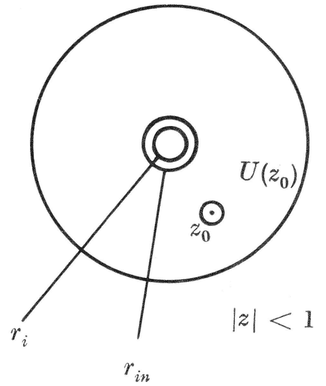
Proof. As the norm of the φ_n is one, we can clearly select, by a diagonal process, a locally uniformly convergent subsequence, which we again denote by (φ_n). Its limit φ is a holomorphic quadratic differential on R with norm $\|\varphi\| \leq 1$. We have $M_{in} = \lambda_n m_i$. As the system of ringdomains R_{in} satisfies the homotopy conditions on $R^{(n+1)}$,

$$\operatorname{Min} \{ M_{in} / M_{i(n+1)} \mid 1 \leq i \leq p \} \leq 1 ,$$

hence $\lambda_n \leq \lambda_{n+1}$. On the other hand, $M_{in} = \lambda_n \cdot m_i \leq M$ for all i and n, hence $\lambda_n \leq M / m$, with $m = \operatorname{Max} \{ m_i \mid 1 \leq i \leq p \}$. We conclude that the coefficients λ_n converge to their supremum λ, moreover $M_{in} \to M_i = \lambda m_i$ for all i. This is of course true for the original sequence.

For fixed i the function $g_{in} = \exp(2\pi i / a_{in}) \, \Phi_n(P)$ with $\Phi_n = \int (\varphi_n(z))^{1/2} \, dz$ is a $1-1$ conformal mapping of R_{in} onto a circular annulus in the z-plane. By choosing the sign of Φ_n and the integration constant properly, we can assume that the orientation induced by γ_i (the Jordan curves γ_i are supposed to be oriented) is taken into the positive orientation of the z-plane and the "outer" boundary component of R_{in} goes into $|z| = 1$. (This only determines the imaginary part of the integration constant, the real part still being free. The mapping g_{in} is determined up to a rotation.) The inverse $f_{in} = g_{in}^{-1}$ is a $1-1$ conformal mapping of $r_{in} < |z| < 1$, $M_{in} = (1 / 2\pi) \log (1 / r_{in})$, onto R_{in}.

By passing, if necessary, to a subsequence, we may assume that the mappings f_{in}, $n \to \infty$, tend locally uniformly to a $1-1$ conformal mapping $f_i \colon r_i < |z| < 1 \to R_i$.



Consider a neighbourhood of a point z_0, $r_i < |z_0| < 1$. We may introduce z as a parameter in this neighbourhood (in fact in all of R_i). The mapping f_i becomes, in terms of this parameter, the identity. Moreover $g_{in}(z) \to z$ uniformly in $U(z_0)$. Let $z_n = g_{in}(z)$. Then the following equations hold:

$$z_n = \exp \frac{2\pi i}{a_{in}} \Phi_n(z) ,$$

$$\log z_n = \frac{2\pi i}{a_{in}} \Phi_n(z) ,$$

$$\frac{1}{z_n} \frac{dz_n}{dz} = \frac{2\pi i}{a_{in}} \Phi'_n(z) ,$$

$$\left(\frac{z'_n}{z_n}\right)^2 = -\left(\frac{2\pi}{a_{in}}\right)^2 \varphi_n(z) .$$

As, for $n \to \infty$, $z_n \to z$, $z'_n \to 1$, $a_{in} \to a_i$ (taking a subsequence again, if necessary), we have

$$\varphi(z) = -\left(\frac{a_i}{2\pi}\right)^2 \frac{1}{z^2} .$$

Thus, for $dz = i \cdot z$, we have $\varphi(z)\, dz^2 = (a_i / 2\pi)^2 dz^2 > 0$. The circles $|z| = \text{const}$ are closed trajectories of φ. Moreover

$$1 \geq \|\varphi\| \geq \sum_i \iint_{R_i} |\varphi(z)|\, dx\, dy = \sum_i a_i^2 M_i$$

$$= \lim_{n \to \infty} \sum_i a_{in} M_{in} = 1 .$$

The set $R \setminus \cup R_i$ has measure zero. Hence φ has closed trajectories and the R_i are the ringdomains of φ. They obviously belong to the curve system $\{\gamma_i\}$.

The sequence (φ_n) is locally bounded (i.e. $\varphi_n(z)$ in terms of a fixed local parameter z is a locally bounded sequence of holomorphic function elements). Therefore, by the Cauchy representation formula, the same is true for the derivatives $\varphi_n'(z)$. The sequence is therefore equicontinuous, and any convergent subsequence converges locally uniformly. If the original sequence (φ_n) would not converge towards φ at some point P, there would be a subsequence with a limit $\lim_{n\to\infty} \varphi_n(z) \neq \varphi(z)$. Starting out with this sequence we would arrive at some quadratic differential $\tilde{\varphi} \neq \varphi$ which is a solution of the same modulus problem, contradicting the uniqueness of the solution.

5. The ringdomains of a given homotopy type

Let $\{\gamma_i\}_{i=1,\ldots,p}$ be a finite admissible curve system, with finite maximal moduli $M(\gamma_i)$ (= supremum of the moduli of all ringdomains of homotopy type γ_i on R), on an arbitrary Riemann surface R. We are looking at the p-tuples of moduli of all p-tuples of non overlapping ringdomains of homotopy type $\{\gamma_i\}$. It is obviously enough to consider, in each direction $\{m_i\}$, the extremal system. To achieve compactness we now allow some of the m_i to be zero.

Let $\vec{m} = (m_1, m_2, \ldots, m_p)$ be an arbitrary unitvector with non negative coordinates $m_i \geq 0$, $i = 1, \ldots, p$. Let $\{R_i\}$ be the extremal system of ringdomains of homotopy type $\{\gamma_i\}$ in the direction $\vec{m}$. This means that $\{R_i\}$ is the system of characteristic ringdomains of a holomorphic quadratic differential φ with closed trajectories and that $M_i = \lambda m_i$ for all i, which we abbreviate to $\vec{M} = \lambda \vec{m}$. Of course, $M_i = 0$ means that there is no ringdomain (no closed trajectory) of homotopy type γ_i.

T h e o r e m 3. *The moduli vector* $\vec{M} = (M_1, M_2, \ldots, M_p)$ *depends continuously on its direction* $\vec{m}$.

Proof. The proof is given in two steps. First we consider an arbitrary sequence of normalized holomorphic quadratic differentials φ_n ($\|\varphi_n\| = 1$), with closed trajectories and of homotopy type $\{\gamma_i\}$. We can pick a locally uniformly convergent subsequence, which we again denote by (φ_n). Let $\varphi = \lim_{n\to\infty} \varphi_n$. We claim that $\|\varphi_n - \varphi\| \to 0$, that φ has closed trajectories, that it is of type $\{\gamma_i\}$, that $M_{in} \to M_i$ for all $i = 1, \ldots, p$, and finally that $a_{in} \to a_i$ for all i. (The latter follows in the course of the proof for those indices i, for which $M_i > 0$; it will later be shown that it is true for all i.)

As $M_{in} \leq M(\gamma_i)$, $i = 1, \ldots, p$, $n = 1, 2, \ldots$ we can pick a subsequence for which the sequences of moduli M_{in} converge for all i. We denote all subsequences by (φ_n) and will later show that it is in fact true for the original sequence. Let $M_i = \lim_{n\to\infty} M_{in}$. We put $i = k$ for $M_i = 0$, $i = h$ for $M_i > 0$. For all h we set

$$g_{hn}(P) = \exp \frac{2\pi i}{a_{hn}} \Phi_n(P),$$

the imaginary constant of $\Phi_n(P)$ in R_{hn} being chosen in such a way that the outer radius of the image annulus is equal to one, the inner radius is equal to r_{hn}, with $M_{hn} = (1/2\pi)\log(1/r_{hn})$. Let $f_{hn} = g_{hn}^{-1}$. For a properly chosen subsequence the mappings f_{hn} converge locally uniformly to some $1-1$ conformal mappings $f_h : r_h < |z| < 1 \to R_h$, $r_h = \lim_{n\to\infty} r_{hn}$, of the limit annuli onto disjoint ringdomains R_h. R_h is evidently of homotopy type γ_h.

By means of the mapping f_h we can introduce the parameter z in R_h. Then, (g_{hn}) becomes a sequence of mappings which tends locally uniformly in $r_h < |z| < 1$ towards the identity. We have

$$g_{hn}(z) = \exp \frac{2\pi i}{a_{hn}} \Phi_n(z),$$

$$\log g_{hn}(z) = \frac{2\pi i}{a_{hn}} \Phi_n(z)$$

and for the derivative with respect to z

$$\frac{g'_{hn}(z)}{g_{hn}(z)} = \frac{2\pi i}{a_{hn}} \Phi'_n(z).$$

Taking squares and going to the limit we get, as $g'_{hn}(z) \to 1$,

$$\frac{1}{z^2} = -\left(\frac{2\pi}{a_h}\right)^2 \varphi(z).$$

(Here it follows that $a_h = \lim_{n\to\infty} a_{hn}$ exists.)

From this we easily recognize that the circles $|z| =$ const are closed trajectories of φ. Hence R_h is a subring of a characteristic ringdomain of φ, swept out by closed trajectories. The φ-length of these trajectories is obviously a_h, and the φ area of R_h is $a_h^2 M_h$. Since φ is the locally uniform limit of the φ_n, $\|\varphi_n\| = 1$, we have $\|\varphi\| \leq 1$. We get

$$1 \geq \|\varphi\| \geq \sum_h a_h^2 M_h = \lim_{n\to\infty} \sum a_{hn}^2 M_{hn} \leq \lim_{n\to\infty} \sum a_{in}^2 M_{in} = 1.$$

Therefore $\|\varphi\| = \sum a_h^2 M_h = 1$. The ringdomains R_h cover R up to a set of measure zero. Therefore φ has closed trajectories and is of type $\{\gamma_i\}$. Obviously $M_{in} \to M_i$ for all $i = 1, \ldots, p$, and $a_{hn} \to a_h$ (for $M_h > 0$).

An easy argument shows that $\|\varphi_n - \varphi\| \to 0$, as $\varphi_n \to \varphi$ locally uniformly and $\|\varphi_n\| = \|\varphi\| = 1$.

We started with a sequence (φ_n) which converges locally uniformly towards φ. Then there exists a subsequence (φ_{n_ν}) with $\|\varphi_{n_\nu} - \varphi\| \to 0$. If this were not true for the original sequence we could find a subsequence with a limit $\tilde{\varphi} \neq \varphi$. But a subsequence of this would again tend to φ, a contradiction. Therefore $\|\varphi_n - \varphi\| \to 0$. Moreover, as φ determines its characteristic ringdomains, $M_{in} \to M_i$, $a_{hn} \to a_h$ for all i resp. for all h with positive M_h.

The second step of the continuity proof is short. Let $\vec{m}_n \to \vec{m}$. Assume $\vec{M}_n \not\to \vec{M}$. Then we can pick a convergent subsequence, which we denote again by $(\vec{M}_n)$, such that $\vec{M}_n \to \lambda_0 \vec{m} = \vec{M}_0 \neq \vec{M}$. Let φ_n be the quadratic differential, associated with the system $\{\gamma_i\}$ and weight $\vec{m}_n$. A subsequence of this sequence converges towards a quadratic differential φ_0 of homotopy type $\{\gamma_i\}$ with moduli vector $\vec{M}_0$. But $\vec{M}$ is supposed to be the moduli vector in the direction $\vec{m}$, and because of the uniqueness $\vec{M}_0 = \vec{M}$, a contradiction. The continuity is proved.

The surface which is described by $\vec{M}$ as a function of the unit vector $\vec{m}$ is called the surface of moduli associated with $\{\gamma_i\}$: We denote it by $\mathfrak{M} = \mathfrak{M}(\{\gamma_i\})$.

Convexity, tangent plane of $\mathfrak{M}$. Let $\vec{M}_0$ be an arbitrary point of $\mathfrak{M}$. Let φ_0 be the holomorphic quadratic differential with the moduli vector $\vec{M}_0$, $\|\varphi_0\| = 1$. We put $\vec{a}_0 = (a_{01}^2, \ldots, a_{0p}^2)$, with a_{0i} the infinum of the lengths of all closed curves which are homotopic to γ_i, in the φ-metric. Then the following can easily be proved:

(1) *The plane* $(\vec{a}_0, \vec{x} - \vec{M}_0) = 0$ *has only the point* $\vec{M}_0$ *in common with* $\mathfrak{M}$. For let $\vec{M} \in \mathfrak{M}$. The corresponding ringdomains are of homotopy type $\{\gamma_i\}$, hence

$$(\vec{a}_0, \vec{M}) \leq (\vec{a}_0, \vec{M}_0)$$

with equality only for $\vec{M} = \vec{M}_0$. We conclude

$$(\vec{a}_0, \vec{M} - \vec{M}_0) < 0$$

unless $\vec{M} = \vec{M}_0$. The entire surface $\mathfrak{M}$, hence also the set bounded by $\mathfrak{M}$ and the coordinate planes, lie to the left (*) of the plane $(\vec{a}_0, \vec{x} - \vec{M}_0) = 0$.

(2) *The plane* $(\vec{a}_0, \vec{x} - \vec{M}_0) = 0$ *is the tangent plane.* This is easily proved for interior points of $\mathfrak{M}$ (i.e. $M_{0i} > 0$ for all i). For at such a point the vector $\vec{a}$ is continuous: $\vec{M}_n \to \vec{M}_0 \Rightarrow \vec{a}_n \to \vec{a}_0$. We have the two inequalities

(*) I.e. the same side as the origin.

$$(\vec{a}_0\,,\vec{M}_n) \leq (\vec{a}_0\,,\vec{M}_0)$$

$$(\vec{a}_n\,,\vec{M}_0) \leq (\vec{a}_n\,,\vec{M}_n)\,,$$

hence, with

$$\vec{e}_n = \frac{\vec{M}_n - \vec{M}_0}{|\vec{M}_n - \vec{M}_0|}\,,$$

$$(\vec{a}_0\,,\vec{e}_n) \leq 0\,,$$

$$(\vec{a}_n\,,\vec{e}_n) = (\vec{a}_0\,,\vec{e}_n) + (\vec{a}_n - \vec{a}_0\,,\vec{e}_n) \geq 0\,.$$

We conclude: $-(\vec{a}_n - \vec{a}_0\,,\vec{e}_n) \leq (\vec{a}_0\,,\vec{e}_n) \leq 0$, consequently $\lim_{n\to\infty}(\vec{a}_0\,,\vec{e}_n) = 0$.

At interior points $\mathfrak{M}$ *therefore has a continuously changing tangent plane with normal vector* $\vec{a}$ *.*

To prove the same at a boundary point, let $\vec{M}_0$ be a boundary vector. Let $M_{h0} > 0$, $M_{k0} = 0$. We have

$$-\sum_h (a_{hn}^2 - a_{h0}^2)\cdot e_{hn} - \sum_k (a_{kn}^2 - a_{k0}^2)\cdot e_{kn}$$
$$\leq \sum_i a_{i0}^2 \cdot e_{in} \leq 0\,.$$

For arbitrary index i we have $\overline{\lim}_{n\to\infty}\, a_{in} \leq a_{i0}$:

This is easily seen, for, let γ_i be an arbitrary element of its homotopy class. Then

$$a_{in} \leq \int_{\gamma_i} |\varphi_n(z)|^{1/2}\,|dz| \to \int_{\gamma_i} |\varphi(z)|^{1/2}\,|dz|\,.$$

The last expression becomes smaller than $a_i + \varepsilon$, $\varepsilon > 0$, for properly chosen γ_i.

Now, for all indices h, $a_{hn} \to a_{h0}$. For the indices k we have

$$e_{kn} = \frac{M_{kn} - M_{k0}}{|\vec{M}_{kn} - \vec{M}_{k0}|}\,,$$

hence $0 < e_{kn} \leq 1$.

For given $\varepsilon > 0$ we can find N such that for $n > N$ and all h resp. k

$$|a_{hn}^2 - a_{h0}^2|\,|e_{hn}| < \varepsilon\,,$$
$$(a_{kn}^2 - a_{k0}^2) < \varepsilon$$

and therefore also $(a_{kn}^2 - a_{k0}^2)\,e_{kn} < \varepsilon$. The left hand side of the above inequality then becomes $> -p\,\varepsilon$. We have proved

$$\lim_{n \to \infty} (\vec{a}_0 , \vec{e}_n) = 0$$

also for all boundary points of $\mathfrak{M}$. This surface therefore has a normal vector at each point, namely $\vec{a} = (a_1^2 , a_2^2 , \dots , a_p^2)$.

It is also immediately seen, from the above inequality, that $\vec{a}_n \to \vec{a}_0$ if $\vec{M}_0$ is approached in a fixed direction. For then, the unit vector $\vec{e}$ does not depend on n. We have

$$\sum_h (a_{hn}^2 - a_{h0}^2)\, e_h + \sum_k (a_{kn}^2 - a_{k0}^2)\, e_k \geq 0 .$$

Assuming $e_k > 0$, $\underline{\lim}_{n \to \infty} a_{kn} < a_{k0}$ would lead to a contradiction.

6. Quadratic differentials with second order poles. Extremal property

Let φ be a holomorphic quadratic differential with closed trajectories which is of finite topological type on an arbitrary Riemann surface R. We assume that φ has infinite norm. Then at least one of the characteristic ringdomains of φ must have infinite modulus. We exclude the twice punctured sphere, which means that every ringdomain R_j with infinite modulus can be mapped conformally onto a punctured disk. We can then add the puncture as a point Q_j to the surface R, and consequently introduce a conformal parameter z_j near Q_j, such that Q_j corresponds to $z_j = 0$. The quadratic differential φ has a second order pole at $z_j = 0$ with a real, negative leading coefficient $-A_j$ which is independent of the choice of the parameter. Let M_j be the reduced modulus of R_j with respect to the parameter z_j (i.e. $M_j = (1 / 2\pi) \log r_j$, where r_j is the mapping radius with respect to z_j). We denote by R_k the characteristic ringdomains of φ with finite modulus M_k and use the index i for the characteristic ringdomains of φ without distinction. As before, a_i is the length of any closed trajectory α_i of φ in R_i:

$$a_i = \int_{\alpha_i} |\varphi(z)|^{1/2} \, |dz| .$$

Now let $\{\tilde{R}_i\}$ be a system of nonoverlapping ringdomains on R of homotopy type $\{\alpha_i\}$. For every j, $\tilde{R}_j$ is to be a punctured disk, with reduced modulus $\tilde{M}_j$ taken with respect to the same parameter z_j. We allow $\tilde{R}_k$ to be degenerate (=missing), but not $\tilde{R}_j$. Then

$$(10) \qquad \sum_i a_i^2 \tilde{M}_i \leq \sum_i a_i^2 M_i \qquad (= \text{reduced norm of } \varphi)$$

with equality holding if and only if $\tilde{R}_i = R_i$ for all i.

Proof. Let ζ_j be the distinguished parameter for the quadratic differential φ near the second order pole Q_j. The representation of φ in terms of this parameter is

$$\varphi(z)\, dz^2 = -\frac{A_j}{\zeta_j^2}\, d\zeta_j^2$$

with $a_j = 2\pi A_j^{1/2}$ the length of the closed trajectories of R_j. The inequality itself can be shown by cutting out circular holes (in terms of the distinguished parameters) around the points Q_j and then applying the inequality for quadratic differentials of finite norm. Letting the radii of the holes tend to zero we arrive at (10). However, in order to discuss the equality sign, we have to be more accurate. So let the domains $\tilde{R}_j$ be mapped onto punctured disks $0 < |\tilde{\zeta}_j| < \tilde{\varrho}_j$ by schlicht functions $\tilde{\zeta}_j(\zeta_j)$ with

$$\frac{d\tilde{\zeta}_j}{d\zeta_j}(Q_j) = 1 .$$

The moduli of the ringdomains $\tilde{R}_j(\varrho) = \tilde{R}_j \setminus \{ |\tilde{\zeta}_j| \leq \varrho \}$ and $R_j(\varrho) = R_j \setminus \{ |\zeta_j| \leq \varrho \}$ for sufficiently small positive ϱ are denoted by $\tilde{M}_j(\varrho)$ and $M(\varrho)$ respectively. Evidently $\tilde{M}_j(\varrho) = (1 / 2\pi) \log (\tilde{\varrho}_j / \varrho)$, and $\tilde{M}_j(\varrho) + (1 / 2\pi) \log \varrho \to \tilde{M}_j$, $M_j(\varrho) + (1 / 2\pi) \log \varrho \to M_j$ as $\varrho \to 0$. We cut the annulus $\varrho < |\tilde{\zeta}_j| < \tilde{\varrho}_j$ along a radius and map it onto a horizontal rectangle with sides a_j, $\tilde{b}_j(\varrho) = a_j \cdot \tilde{M}_j(\varrho) = (a_j / 2\pi) \log (\tilde{\varrho}_j / \varrho)$ in the $z = x + i\, y$ -plane. As usual, we have

$$a_j \leq \int |\varphi(x + i\, y)|^{1/2}\, dx$$

and by the Schwarz inequality

$$a_j \leq \int |\varphi(x + i\, y)|\, dx .$$

Assume that for some j one of the circles $|\tilde{\zeta}_j| = \text{const}$ is not a trajectory of φ. Then there are positive numbers ε and δ such that

$$a_j + \varepsilon \leq \int |\varphi(x + i\, y)|\, dx$$

for all y in the δ-neighbourhood of some y_0. We get by integrating

$$a_j\, \tilde{b}_j(\varrho) + \varepsilon \cdot \delta \leq \iint_{\tilde{R}_j(\varrho)} |\varphi(z)|\, dx\, dy$$

and by summing over the j and k

$$\sum_j a_j^2 \tilde{M}_j(\varrho) + \sum_k a_k^2 \tilde{M}_k + \varepsilon\,\delta \;\le\; \sum_j a_j^2 M_j(\varrho) + \sum_k a_k^2 M_k .$$

Adding $\sum_j a_j^2 \cdot (1 / 2\pi) \log \varrho$ to both sides (here we need that there is an R_j for every j) and letting $\varrho \to 0$ we arrive at a contradiction. (The inequality itself is proved by just dropping the term $\varepsilon\,\delta$.)

Now we know that the circles $|\tilde{\zeta}_j| = \varrho$ are actually trajectories of φ. From the equality

$$\sum_j a_j^2 \tilde{M}_j + \sum_k a_k^2 \tilde{M}_k \;=\; \sum_j a_j^2 M_j + \sum_k a_k^2 M_k$$

we conclude, subtracting $\sum a_j^2 (1 / 2\pi) \log \varrho$,

$$\sum_j a_j^2 \tilde{M}_j(\varrho) + \sum_k a_k^2 \tilde{M}_k \;=\; \sum_j a_j^2 M_j(\varrho) + \sum_k a_k^2 M_k .$$

The right hand side is equal to the norm of φ over the truncated surface $R(\varrho) = R \setminus \bigcup_j \{ |\tilde{\zeta}_j| \le \varrho \}$, because φ is a quadratic differential with closed trajectories on $R(\varrho)$ and the $R_j(\varrho)$, R_k are its characteristic ringdomains. But then by the earlier theorem $\tilde{R}_j(\varrho) = R_j(\varrho)$, $\tilde{R}_k = R_k$ for all j, k. Hence also $\tilde{R}_j = R_j$ for all j. We have proved

T h e o r e m 4. *Let φ be a quadratic differential with closed trajectories and a finite number of characteristic ringdomains on a Riemann surface R. We denote by R_j the punctured disks (ringdomains with infinite modulus) and by R_k the ringdomains with finite modulus. There should be at least one R_j, and the doubly punctured sphere is of course excluded. Let $\tilde{R}_j$, $\tilde{R}_k$ be a system of non overlapping disks and ringdomains, with an $\tilde{R}_j$ for every R_j, $Q_j \in \tilde{R}_j$. Then, with the usual notation, but M_j, $\tilde{M}_j$ the reduced moduli with respect to the same parameters near the second order poles Q_j, the inequalities*

$$\text{(I)} \qquad \sum_j a_j^2 \tilde{M}_j + \sum_k a_k^2 \tilde{M}_k \;\le\; \sum_j a_j^2 M_j + \sum_k a_k^2 M_k$$

(equality if and only if $\tilde{R}_j = R_j$, $\tilde{R}_k = R_k$ for all j and k) and, as an easy consequence

$$\text{(II)} \qquad \operatorname*{Min}_{i=j,\,k} \{\tilde{M}_i - M_i\} \;\le\; 0$$

(equality if and only if $\tilde{R}_i = R_i$ for all i) hold.

7. Existence and uniqueness of the solution of the moduli problem

We treat only the special case where all the ringdomains are punctured disks.

Theorem 5. *Let* $\{\gamma_j\}_{j=1,\dots,p}$ *be an admissible system of Jordan curves on a hyperbolic Riemann surface* R *, with maximal moduli* $M(\gamma_j) = \infty$ *for all* j *. Let* m_j *be arbitrary real numbers, normalized such that* $\sum_j m_j = 0$ *. Then there is a holomorphic quadratic differential* φ *on* R *with closed trajectories, of homotopy type* $\{\gamma_j\}$ *, and such that the reduced moduli* M_j *of its punctured disks satisfy the equalities*

$$M_j = m_j + c$$

for some c *independent of* j *.* φ *is uniquely determined up to a positive constant factor.*

To prove uniqueness, let φ and $\tilde{\varphi}$ be two solutions, with $M_j = m_j + c$, $\tilde{M}_j = m_j + \tilde{c}$ for all j . As the system R_j belongs to a quadratic differential, it satisfies (II) when compared with the system $\tilde{R}_j$ of $\tilde{\varphi}$:

$$\operatorname*{Min}_j \{\tilde{M}_j - M_j\} = \tilde{c} - c \leq 0 .$$

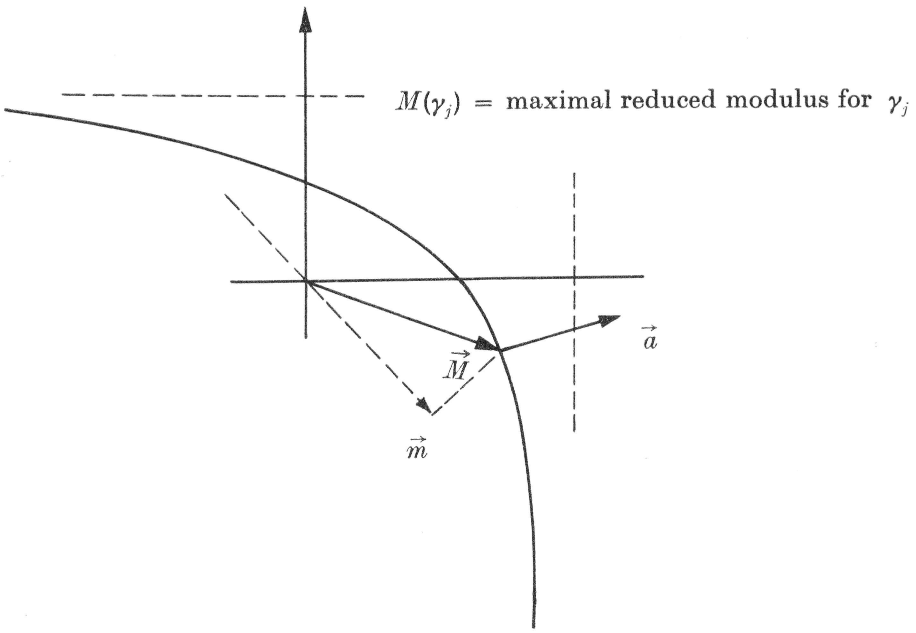
Similarly, starting with $\tilde{\varphi}$, we get $c - \tilde{c} \leq 0$. Therefore $\tilde{c} = c$, hence $\tilde{R}_j = R_j$ for all j . The consequence $\tilde{\varphi} = \text{const} \cdot \varphi$ is immediate.

The existence is easily established by means of the previous existence theorem. We first notice that there are disjoint punctured disks R_j' with reduced moduli $M_j' = m_j + c'$ for some c' . We now make c' as large as we can. By a normal family argument there exists a maximal system R_j . We choose the conformal mappings of the R_j onto punctured disks $0 < |\zeta_j| < \varrho_j$, with $(d\zeta_j/dz_j)(0) = 1$, z_j the given parameters near the points Q_j , as local homeomorphisms. Let $0 < \varrho < \varrho_j$ for all j and denote by $R_j(\varrho)$ the ringdomains corresponding to the annuli $\varrho < |\zeta_j| < \varrho_j$ on R , with moduli $M_j(\varrho)$. This system is extremal, on the truncated surface $R(\varrho)$, in the direction $(M_1(\varrho), M_2(\varrho), \dots, M_p(\varrho))$, in the sense of the earlier existence theorem. For otherwise we would have a system $M_j'(\varrho)$ with $M_j'(\varrho) = (1 + \varepsilon) M_j(\varrho)$ for some $\varepsilon > 0$. Adding the disks $|\zeta_j| \leq \varrho$ to the ringdomains $R_j'(\varrho)$ we would get a system of punctured disks R_j' with

$$M_j' \geq M_j'(\varrho) + \frac{1}{2\pi} \log \varrho > M_j(\varrho) + \frac{1}{2\pi} \log \varrho = M_j .$$

Therefore the system $R_j(\varrho)$ is associated with a quadratic differential φ_ϱ . Its trajectories are the circles $|\zeta_j| = \text{const}$ which are independent of ϱ . But then the φ_ϱ are just the restrictions of a quadratic differential φ on R with the R_j as characteristic punctured disks.

As before, we can now introduce the surfaces of moduli. The following facts can easily be established along the same lines as in the earlier proof.



a) *Continuity.* The vector of (reduced) moduli $\vec{M} = (M_1, \ldots, M_p)$ is a continuous function of the vector $\vec{m}$ of parameters m_j, $\sum m_j = 0$. In fact, for $\vec{m} \to \vec{m}_0$ ($m_j \to m_{j0}$) the quadratic differentials φ converge to φ_0, if properly normalized, in the sense that $\|\varphi - \varphi_0\| \to 0$.

b) *Convexity. Tangent plane.* The plane

$$\sum_j a_j^2 (X_j - M_j) = (\vec{a}, \vec{X} - \vec{M}) = 0,$$

with $\vec{a} = (a_1^2, \ldots, a_p^2)$, is a plane of support, with only the point $\vec{M}$ itself in common with the surface of moduli. It is in fact the tangent plane at this point, and the normal vector $\vec{a}$ varies continuously with $\vec{m}$.

c) *Extremal length problem.* The range of directions of the normal vector $\vec{a}$ is the full open quadrant, i.e. $\vec{a} / |\vec{a}|$ is an arbitrary positive unit vector. We can place a tangent plane with this normal at some well determined point $\vec{M}$ of the surface of moduli and then multiply the quadratic differential by an appropriate factor such that the lengths of its trajectories are the given numbers a_j. The resulting quadratic differential is the solution of the extremal length problem of minimizing the reduced norm with lengths $\geq a_j$ in the classes γ_j ($|\varphi(z)|^{1/2} |dz|$ is the extremal metric). As the leading coefficients of φ at the Q_j in terms of any parameter z_j are $-A_j = -(a_j / 2\pi)^2$, this shows that these coefficients can be arbitrarily given negative numbers. Then there is exactly one quadratic differential φ with closed trajectories the characteristic ringdomains of which are the R_j and for which $a_j = \int_{\alpha_j} |\varphi(z)|^{1/2} |dz|$.

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