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ON ULTRAPSEUDOCOMPACT AND RELATED SPACES

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Abstract. We investigate topological spaces, which at the same time satisfy certain connection, disconnection and separation conditions. We find the spaces X for which $C(X)$ is a field, an integral domain, a local ring, a semi-local ring, a semisimple ring, a Noetherian ring, an Artinian ring or a principal ideal ring. The spaces in which every open or dense subset, or every subset is C-embedded, are found. Weakly locally connected, semiconnected, locally irreducible and hereditarily connected spaces are defined and studied. Partition spaces are characterized by 18 equivalent conditions. No separation axioms are assumed in the definitions of a strongly 0-dimensional space, an extremally disconnected space and a P-space. The T_1-separation axiom is not a part of the definition of a T_3-, T_4- or T_5-space.

1. Ultrapseudocompact spaces. A topological space is *ultrapseudocompact* if every continuous real-valued function on it is constant.

Theorem 1. *The following statements concerning a topological space X are equivalent:*

(1) *X is ultrapseudocompact.*
(2a) *The only cozero-sets in X are $\emptyset$ and X.*
(2b) *The only zero-sets in X are $\emptyset$ and X.*
(3a) *Every two nonvoid cozero-sets in X intersect.*
(3b) *The union of every two proper zero-sets is proper.*
(4a) *Every two nonvoid zero-sets in X intersect.*
(4b) *The union of every two proper cozero-sets is proper.*

Proof. To show that (1) implies (2a), let A be a cozero-set in X. Then $A = \{x \in X \mid f(x) \neq 0\}$, where f is a continuous real-valued function on X. By (1), f is constant. Thus A is empty or X.

Suppose that $f(p) < f(q)$, where f is a continuous real-valued function on X. If $f(p) < \alpha < f(q)$, then $\{x \mid f(x) < \alpha\}$ and $\{x \mid f(x) > \alpha\}$ are disjoint nonvoid cozero-sets. Thus (3a) implies (1).

Suppose that a continuous function f admits two distinct values α and β.

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Then $\{x|f(x)=\alpha\}$ and $\{x|f(x)=\beta\}$ are disjoint nonvoid zero-sets. Thus (4a) implies (1).

The other implications are trivial.

By definition, a topological space X is *strongly* 0-*dimensional* if it satisfies the following equivalent conditions: (a) If $A\subset U$, where A is a zero-set and U a cozero-set, there exists a clopen set V such that $A\subset V\subset U$; (b) It the subsets A and B of X are completely separated, there exists a separation (U, V) of X such that $A\subset U$ and $B\subset V$, i.e., A and B have disjoint clopen neighborhoods.

Theorem 2. *The following statements about a topological space X are equivalent:*

(1) *X is ultrapseudocompact.*

(5) *X is connected and strongly* 0*-dimensional.*

(6) *Every cozero-set in X is connected.*

(7) *Every zero-set in X is connected.*

Proof. If (1) holds, there are no continuous functions from X onto the discrete space $\{0, 1\}$. Thus X is connected. On the other hand, if $A\subset U$, where A is a zero-set and U a cozero-set, then by (2b), A is empty or X. Thus either $\emptyset$ or X serves as a clopen set V satisfying $A\subset V\subset U$. Hence X is strongly 0-dimensional, and (1) implies (5).

Suppose (5) holds. Let f be a continuous real-valued function on X, and let α and β be two distinct real numbers. Then the sets $A=\{x|f(x)=\alpha\}$ and $B=\{x|f(x)=\beta\}$ are completely separated. Since X is strongly 0-dimensional, there is a separation (U, V) of X such that $A\subset U$ and $B\subset V$. Since X is connected, the separation is trivial. Thus either A or B is empty and, consequently, f is constant. So (5) implies (1).

Let (1) hold. By (5), X is connected. Thus (6) and (7) hold, because of (2a) and (2b).

Let U and V be nonvoid cozero-sets. If (6) holds, then the cozero-set $U\cup V$ is connected. Since U and V are nonvoid open subsets of $U\cup V$, we therefore have $U\cap V\neq\emptyset$, and (3a) holds. In the same way it is seen that (7) implies (4a).

Let X be a topological space and $C(X)$ the ring of all real-valued continuous functions on X. For every x in X, the set $M_x=\{f\in C(X)|f(x)=0\}$ is a maximal ideal of $C(X)$. On the other hand, $\cap\{M_x|x\in X\}=(0)$. Thus $C(X)$ is always a radical free ring.

Theorem 3. *For a topological space X, the following statements are equivalent:*

(1) *X is ultrapseudocompact and nonvoid.*

(8) *$C(X)$ is isomorphic to the field of real numbers.*

(9) *$C(X)$ is a field.*

(10) *$C(X)$ is an integral domain.*

(11) *$C(X)$ is a local ring, i.e., it has a unique maximal ideal.*

Proof. The implication (1)⇒(8) is easy and the implications (8)⇒(9)⇒(10) and (9)⇒(11) are trivial. On the other hand, every local radical free commutative ring is a field, so (11)⇒(9) holds. Thus the proof is completed if we show that (10) implies (3b). Let A and B be zero-sets for which $A\cup B=X$. Then $A=\{x\mid f(x)=0\}$ and $B=\{x\mid g(x)=0\}$, where f and g belong to $C(X)$. From $A\cup B=X$ it follows that $fg=0$. Thus by (10), $f=0$ or $g=0$ and, consequently, $A=X$ or $B=X$.

It is well known that there exist infinite ultrapseudocompact regular spaces ([7]) and that a $T_{3\frac{1}{2}}$-space is ultrapseudocompact if and only if it is indiscrete.

2. Ultraconnected spaces. By definition, a topological space X is *ultraconnected* if every two nonvoid closed subsets of X intersect, i.e. if X is a T_4-space in a trivial way ([16]). Ultraconnected spaces are studied in [11] and [12] under the name of strongly connected spaces. In [16], strongly connected means the same as ultrapseudocompact in the present paper.

Theorem 4. *For a topological space X, the following conditions are equivalent:*

(1) *X is ultraconnected.*

(2) *Every closed subspace of X is ultraconnected.*

(3) *Every closed subspace of X is connected.*

(4) *X is an ultrapseudocompact T_4-space.*

Proof. (1)⇒(2): Let S be a closed subspace of X and let $A\cap B=\emptyset$, where A and B are closed subsets of S. Then A and B are closed in X; hence $A=\emptyset$ or $B=\emptyset$ and S is ultraconnected.

(2)⇒(3) is trivial.

(3)⇒(1): Let A and B be disjoint closed subsets of X. Then (A, B) is a separation of the closed subspace $A\cup B$, which is connected by (3). Hence $A=\emptyset$ or $B=\emptyset$.

(1)⇒(4): From (1) it at once follows that X satisfies the condition (3a) in Theorem 1. On the other hand, X is trivially a T_4-space.

(4)⇒(1): Let A and B be disjoint closed subsets of X. Since X is a T_4-space, there exists a continuous function f admitting the value 0 on A and the value 1 on B. Since X is ultrapseudocompact, f is constant. Hence either A or B is empty, and (1) follows.

3. Weakly locally connected spaces

Theorem 5. *The following statements about a topological space X are equivalent:*

(1) *Every point of X has a connected open neighborhood.*

(2) *Every point of X has a connected neighborhood.*

(3) *Every component of X is open.*

(4) *Every quasi-component of X is open.*

(5) *X is the sum of its quasi-components.*
(6) *X is the sum of its components.*
(7) *X is (homeomorphic to) a sum of connected spaces.*
(8) *A subset of X is clopen if and only if it is the union of a family of components of X.*
(9a) *Every union of clopen subsets is closed.*
(9b) *Every intersection of clopen subsets is open.*

Proof. (1)⇒(2)⇒(3) and (6)⇒(7) are trivial. Every quasi-component is the union of a family of components. Thus (3) implies (4). If X is the union of a disjoint family of open subsets, then X is the sum of this family. Thus (4) implies (5).

(5)⇒(6): By (5), every quasi-component is open and, consequently, a component. Hence (6) holds.

(7)⇒(8): From (7) it follows that X is the union of a disjoint family $(U_i)_{i\in I}$ of nonvoid open connected subsets. Each U_i is then a nonvoid clopen connected subset and, consequently, a component. Thus every component of X is open. From this it follows that if A (and thus also $X\setminus A$) is a union of components, then A is clopen. Conversely, a clopen subset is always a union of components.

(8)⇒(9a)⇒(9b) are trivial.

(9b)⇒(4): Every quasi-component is an intersection of clopen sets.

(4)⇒(1): As an open set, each quasi-component is a component. Thus each component is an open connected neighborhood of any of its points.

We say that a topological space is *weakly locally connected* if it satisfies the equivalent conditions of Theorem 5. In a weakly locally connected space the components and the quasi-components are identical. The converse does not hold, as is seen by the following example. Let X be the one point compactification of the discrete countably infinite space. Each quasi-component of X is a singleton and, consequently, a component. Thus the component of the particular point is not open; hence X is not weakly locally connected.

4. Weakly locally ultrapseudocompact spaces. By definition, a topological space X is a *P-space* ([3], [13], [9], [4]) if it satisfies the following equivalent conditions: (a) Every cozero-set is closed. (b) Every countable intersection of cozero-sets is open. (c) Every cozero-set is C-embedded. (d) If $f\in C(X)$, then every point $x\in X$ has a neighborhood on which f is constant. (e) $C(X)$ is a regular ring. (f) If a sequence of continuous real-valued functions on X converges at every point of X to a function f, then f is continuous.

It is well known, that a $T_{3\frac{1}{2}}$-space is a P-space if and only if it satisfies the following condition: (g) Every countable intersection of open subsets is open. In the general case, (g) trivially implies (b). The converse does not hold. In fact, an uncountable set X together with the cofinite topology is ultrapseudocompact and thus trivially satisfies the condition (a). On the other hand, a countably infinite subset of X is not closed, although every singleton is.

In a P-space the concepts of a cozero-set, a zero-set and a clopen set coincide. Thus every P-space is trivially strongly 0-dimensional.

Theorem 6. *The following properties of a topological space* X *are equivalent:*

(1) X *is a weakly locally connected* P*-space.*

(2) X *is weakly locally connected and strongly* 0*-dimensional.*

(3) X *is (homeomorphic to) a sum of ultrapseudocompact spaces.*

(4) *Every component of* X *is open and ultrapseudocompact.*

(5) *Every point of* X *has an ultrapseudocompact clopen neighborhood.*

(6) *Every point of* X *has an ultrapseudocompact neighborhood.*

(7a) *A subset of* X *is a cozero-set if and only if it is a union of components.*

(7b) *A subset of* X *is a zero-set if and only if it is a union of components.*

(8a) *Every intersection of cozero-sets is clopen.*

(8b) *Every union of zero-sets is clopen.*

(9a) *Every intersection of cozero-sets is open.*

(9b) *Every union of zero-sets is closed.*

Proof. (1)⇒(2): Every P-space is strongly 0-dimensional.

(2)⇒(3): By (2) and Theorem 5, X is the sum of its components. As a clopen set, each component is C-embedded in X and thus, by (2), strongly 0-dimensional. As a connected strongly 0-dimensional space, each component is ultrapseudocompact, by Theorem 2.

(3)⇒(4): By (3), X is the union of a disjoint family $(U_i)_{i\in I}$ of nonvoid open ultrapseudocompact subspaces. As an ultrapseudocompact set, each U_i is connected and as a nonvoid clopen connected set, a component of X.

(4)⇒(5)⇒(6), (7a)⇔(7b), (8a)⇔(8b), (9a)⇔(9b) and (7b)⇒(9b) are trivial.

(6)⇒(1): From (6) it follows at once, since an ultrapseudocompact space is connected, that X is weakly locally connected. Let A be a zero-set in X. Then $A=\{x|f(x)=0\}$, where f is a continuous real-valued function. By (6), every point x of A has an ultrapseudocompact neighborhood U in X. Then f is constantly zero on U; hence $U\subset A$ and, consequently, A is open. Thus X is a P-space.

(1)⇒(7a): Since X is a P-space, each cozero-set in X is clopen and, consequently, a union of components. Conversely, let A be a union of components. Then the same holds true for $X\setminus A$. Since X is weakly locally connected, each component is open. Thus A is clopen and, therefore, a cozero-set.

(9b)⇒(8b): From (9b) it follows, since every cozero-set is a union of zero-sets, that every cozero-set is closed and, consequently, every zero-set is open. Thus every union of zero-sets is open and hence clopen, by (9b).

(8a)⇒(1): From (8a) it follows that every intersection of clopen subsets is open. Thus X is weakly locally connected. On the other hand, a countable intersection of cozero-sets is open by (8a), i.e., X is a P-space.

We say that a topological space is *weakly locally ultrapseudocompact* if it satisfies the equivalent conditions of Theorem 6. In the terminology of [11], a topological space is *locally ultrapseudocompact* if it has a basis consisting of ultrapseudocompact subsets. A locally ultrapseudocompact space is a locally connected P-space and a locally connected P-space is weakly locally ultrapseudocompact. For both statements the converse is false. To see this, let (X, τ) be a topological space, p a point not in X, $X'=X\cup\{p\}$ and $\tau'=\tau\cup\{X'\}$. Then (X', τ') is ultraconnected and hence weakly locally ultrapseudocompact. On the other hand, (X', τ') is locally connected or locally ultrapseudocompact if and only if (X, τ) has the same property.

5. Semiconnected spaces

Theorem 7. *For a topological space X, the following conditions are equivalent:*
(1) *X is the union of a finite family of connected subsets.*
(2) *X has only a finite number of components.*
(3) *X has only a finite number of quasi-components.*
(4) *X has only a finite number of clopen subsets.*
(5) *Every disjoint family of nonvoid clopen subsets is finite.*
(6a) *The family of clopen subsets of X satisfies the maximum condition.*
(6b) *The family of clopen subsets of X satisfies the minimum condition.*
(7) *X is (homeomorphic to) the sum of a finite family of connected spaces.*

Proof. (1)⇒(2): By (1), X is the union of a finite family $(A_i)_{i\in I}$ of nonvoid connected subsets. Choose a point x_i from each A_i and denote the component of x_i by C_i. Then $A_i\subset C_i$ for every $i\in I$ and, consequently, $(C_i)_{i\in I}$ covers X. Thus it is the family of components of X, with possible duplications.

Every quasi-component is a union of components and every clopen set is a union of quasi-components. Thus (2) implies (3) and (3) implies (4). The implication (4)⇒(5) is trivial.

(5)⇒(6a): Suppose that $(U_1, U_2, \ldots)$ is a strictly increasing sequence of clopen subsets. Then $(U_2\setminus U_1, U_3\setminus U_2, \ldots)$ would be an infinite disjoint family of nonvoid clopen sets, in contradiction to (5).

(6a)⇔(6b): The complement of a clopen subset is clopen.

(6)⇒(3): Let x be an arbitrary point of X. The family of clopen neighborhoods of x has, by (6b), a minimal element K. Since the intersection of two clopen subsets is clopen, K is the intersection of all clopen neighborhoods of x, i.e., K is the quasi-component of x. Thus every quasi-component of X is clopen. Suppose $(K_1, K_2, \ldots)$ were an infinite sequence of pairwise distinct quasi-components. Then, by the above result, $(K_1, K_1\cup K_2, \ldots)$ would be a strictly increasing sequence of clopen subsets, in contradiction to (6a). Consequently, (3) holds.

(3)⇒(2): From (3) it immediately follows, since every quasi-component is closed, that every quasi-component is open and, consequently, a component. Thus by (3), X has only a finite number of components.

(2)⇒(7): From (2) it follows that each component is open. Thus X is the sum of its components, and (7) follows.

(7)⇒(1) is trivial.

We say that a topological space is *semiconnected* if it satisfies the equivalent conditions of Theorem 7.

A topological space X is *Z-pseudocompact* if it satisfies the following equivalent conditions: (a) Every continuous integer-valued function on X is bounded. (b) There exist no continuous functions from X onto Z. (c) Every disjoint open cover of X has only a finite number of nonvoid members. (d) Every countable clopen cover of X admits a finite subcover. (See [14] and [15]. In [15] a Z-pseudocompact space is called mildly countably compact.) The next theorem is an improvement of 4.6 in [5]:

Theorem 8. *For a topological space X, the following conditions are equivalent:*

(I) *X is semiconnected.*

(8) *X is weakly locally connected and Z-pseudocompact.*

Proof. (I)⇒(8): From (2) it at once follows that every component of X is open. Hence X is weakly locally connected. Let $(U_i)_{i\in I}$ be a disjoint cover of X by nonvoid open subsets. Then each U_i is clopen; hence I is finite by (5). Thus X is Z-pseudocompact.

(8)⇒(2): Since X is weakly locally connected, each component of X is open (and nonvoid). Since X is Z-pseudocompact, it follows from this that X has only a finite number of components (Postulate (c)).

It is obvious that a weakly locally connected space is not necessarily semiconnected and thus not necessarily Z-pseudocompact. On the other hand, even a compact space may fail to be weakly locally connected. In fact, we have seen that the one point compactification of a countably infinite discrete space is not weakly locally connected.

A pseudocompact space is trivially Z-pseudocompact. The converse does not hold, since every connected space is Z-pseudocompact, by Theorem 8.

6. Semi-ultrapseudocompact spaces

Theorem 9. *The following statements concerning a topological space X are equivalent:*

(1) *Every continuous real-valued function on X assumes only a finite number of values.*

(2) *X is a pseudocompact P-space.*

(3) *X is a Z-pseudocompact P-space.*

(4) *X is semiconnected and strongly 0-dimensional.*

(5) *X has only a finite number of components and they are ultrapseudocompact.*

(6) *X is the union of a finite number of ultrapseudocompact subspaces.*

If these conditions are satisfied, then the number of components of X is the largest *possible number of values of a continuous real-valued function and also the smallest number of members in a cover of X by ultrapseudocompact subspaces.*

Proof. (1)⇒(2): From (1) it immediately follows that X is pseudocompact. Let U be a cozero-set in X. Then $U=\{x|f(x)<0\}$, where f is a continuous real-valued function. By (1), f has the largest negative value, say α. Then $U==\{x|f(x)\leqq\alpha\}$; hence U is closed. This shows that X is a P-space.

(2)⇒(3): Every pseudocompact space is Z-pseudocompact.

(3)⇒(4): Suppose that X has a disjoint infinite sequence $(U_1, U_2, \ldots)$ of nonvoid clopen subsets. Then each $V_n=\bigcup_{k=1}^n U_k$ is clopen, as well as each $X\setminus V_n=W_n$. As a clopen set, each W_n is a zero-set and, consequently, the same holds true for $W=\bigcap_{n=1}^\infty W_n$. From this it follows, since X is a P-space, that W is open. Thus $(W, U_1, U_2, \ldots)$ is an infinite disjoint open cover of X, in contradiction to (3). Hence X is semiconnected. On the other hand, every P-space is strongly 0-dimensional.

(4)⇒(5): By (4), X has only a finite number of components and they are clopen. Every clopen subset of a topological space is C-embedded and every C^*-embedded subspace of a strongly 0-dimensional space is strongly 0-dimensional. Thus each component of X is connected and strongly 0-dimensional, hence ultrapseudocompact, by Theorem 2.

(5)⇒(6) is trivial.

(6)⇒(1): By (6), X is the union of a finite number $n\geqq 0$ of ultrapseudocompact subspaces A_i. If f is a continuous real-valued function on X, then f is constant on each A_i. Hence the number of distinct values of f does not exceed n. On the other hand, if $C_1, \ldots, C_m$ are the components of X and if we set $f(x)=i$ for x in C_i, we get a continuous function f admitting exactly m distinct values. Thus m has the properties stated in our theorem.

We say that a topological space is *semi-ultrapseudocompact* if it satisfies the equivalent conditions of Theorem 9. As to the postulates (2) and (4), it is to be noted that a pseudocompact strongly 0-dimensional space is not necessarily semi-ultrapseudocompact even in the case when it is a compact extremally disconnected T_2-space. To see this we observe that the discrete space $\mathbf{N}$ of natural numbers is extremally disconnected and, consequently, the same holds true for the Stone—Čech compactification $\beta\mathbf{N}$. Since every real-valued function on $\mathbf{N}$ can be extended to a continuous function on $\beta\mathbf{N}$, the space $\beta\mathbf{N}$ is not semi-ultrapseudocompact.

Theorem 10. *For a topological space X, the following conditions are equivalent:*

(I) *X is semi-ultrapseudocompact.*

(7) *X is the union of a finite number of semi-ultrapseudocompact subspaces.*

(8) *Every clopen subspace of X is semi-ultrapseudocompact.*

(9) *The number of cozero-sets in X is finite.*

(10a) *The family of cozero-subsets of X satisfies the maximum condition.*

(10b) *The family of zero-subsets of X satisfies the maximum condition.*

(11a) *Every disjoint family of nonvoid cozero-sets is finite.*

(11b) *Every disjoint family of nonvoid zero-sets is finite.*

Proof. (I)⇒(7)⇒(6), (8)⇒(I), (9)⇒(10a) and (9)⇒(10b) are trivial.

(I)⇒(8): Let U be a clopen subset of X and f a continuous real-valued function on U. Then f can be extended to a continuous function g on X. By (1), g admits only a finite number of values and, consequently, so does f. Hence U is semi-ultrapseudocompact.

(I)⇒(9): By (2) and (4), X is a weakly locally connected P-space. Hence by Theorem 6, every cozero-set is a union of components. By (4), the number of components is finite. Consequently, (9) holds.

(10a)⇒(11a): Let $(U_1, U_2, \dots)$ be a disjoint sequence of cozero-sets. Denote $V_n = U_1 \cup \dots \cup U_n$ for every n. Then $(V_1, V_2, \dots)$ is an increasing sequence of cozero-sets. Hence by (10a), $V_n = V_{n+1} = \dots$ for some n. Then $U_{n+1} = U_{n+2} = \dots = \emptyset$, and (11a) follows. (10b)⇒(11b) is shown in the same way.

(11)⇒(1): Suppose that (1) does not hold. As is readily seen, there then exist a continuous real-valued function f and a sequence $(x_1, x_2, \dots)$ of points such that $f(x_1) < f(x_2) < \dots$. Let $\alpha_1 < f(x_1) < \alpha_2 < f(x_2) < \dots$, and denote $U_k = \{x \mid \alpha_k < f(x) < \alpha_{k+1}\}$ and $A_k = \{x \mid \alpha_{2k} \leqq f(x) \leqq \alpha_{2k+1}\}$ for every k. Then $(U_1, U_2, \dots)$ is a disjoint family of nonvoid cozero-sets and $(A_1, A_2, \dots)$ a disjoint family of nonvoid zero-sets, in contradiction to (11a) and to (11b).

Theorem 11. *Let X be a topological space and $C(X)$ the ring of all continuous real-valued functions on X. Then the following statements are equivalent:*

(I) *X is semi-ultrapseudocompact.*

(12) *$C(X)$ is isomorphic to the product ring $\mathbf{R}^n$ for some nonnegative integer n.*

(13) *$C(X)$ is semisimple, i.e. radical free and Artinian.*

(14) *$C(X)$ is an Artinian ring.*

(15) *$C(X)$ is a Noetherian ring.*

(16) *$C(X)$ is a principal ideal ring.*

(17) *$C(X)$ is a semi-local ring, i.e. it has only a finite number of maximal ideals.*

If X is semi-ultrapseudocompact, then the exponent n in (12) is unique and equal to the number of components of X and to the number of maximal ideals of $C(X)$.

Proof. (I)⇒(12): By (4), the family $(X_i)_{i \in I}$ of components of X is finite and X is the sum of this family. Hence $C(X)$ is isomorphic to the product ring $\prod_{i \in I} C(X_i)$. By (5), each X_i is ultrapseudocompact (and nonvoid). Thus by Theorem 3, each $C(X_i)$ is isomorphic to the field of real numbers. The uniqueness of n follows from the fact that the rings R^m and R^n are isomorphic only if $m = n$.

(12)⇒(13)⇒(14) are trivial.

(14)⇒(15): By Hopkins's theorem, an Artinian ring with identity is Noetherian.

(15)⇒(11a): Suppose that $(U_1, U_2, \dots)$ is a disjoint infinite sequence of nonvoid cozero-sets. Let $U_n = \{x \mid f_n(x) \neq 0\}$, where $f_n \in C(X)$, and let J_n be the ideal of $C(X)$ generated by $\{f_1, \ldots, f_n\}$. By (15), there exists an n such that $J_n = J_{n+1}$. Then $f_{n+1} = g_1 f_1 + \ldots + g_n f_n$, where each $g_k \in C(X)$. As a consequence, $f_{n+1}(x) = 0$ for every x in U_{n+1}, i.e., $U_{n+1} = \emptyset$. This contradiction shows that (11a) holds.

(12)⇒(16): A finite product of principal ideal rings is a principal ideal ring.

(16)⇒(15): Every principal ideal ring is Noetherian.

(12)⇒(17): The ring R^n has precisely n maximal ideals.

(17)⇒(1): Let n be the number of maximal ideals in $C(X)$, f an element of $C(X)$, $\lambda_1, \ldots, \lambda_m$ distinct values of f, and $f(x_k) = \lambda_k$ $(k = 1, \ldots, m)$. Each $M_k = \{g \in C(X) | g(x_k) = 0\}$ is a maximal ideal of $C(X)$. The ideals M_k are mutually distinct, since $f - \lambda_h \in M_k$ if and only if $h = k$. Thus $m \leqq n$, and (1) holds.

7. Extremally disconnected *P*-spaces. By definition, a topological space X is *extremally disconnected* if it satisfies the following equivalent conditions: (a) The closure of every open subset of X is open. (b) Every dense subset of X is C^*-embedded. (c) Every open subset of X is C^*-embedded. (d) Every two disjoint open subsets are completely separated.

For the next lemma we recall that a C^*-embedded subset of a topological space is C-embedded if and only if it is completely separated from every zero-set disjoint from it. (See [4], 1.18.)

Lemma. *A C^*-embedded subset A of a P-space X is C-embedded.*

Proof. Let $A \cap B = \emptyset$, where B is a zero-set in X. Since X is a P-space, B is clopen. Thus the characteristic function of B is continuous and, consequently, A and B are completely separated. From the theorem mentioned above it then follows that A is C-embedded.

Theorem 12. *For a topological space X, the following statements are equivalent:*

(1) *X is an extremally disconnected P-space.*

(2) *Every dense subset of X is C-embedded.*

(3) *Every open subset of X is C-embedded.*

Proof. (1)⇒(2): Let A be a dense subset of X. Since X is extremally disconnected, A is C^*-embedded. From this it follows, by the Lemma, that A is C-embedded.

(2)⇒(3): Let U be an open subset of X. As a clopen subset of $A = U \cup (X \setminus \overline{U})$, U is C-embedded in A. As a dense subset of X, A is C-embedded in X by (2). Thus by transitivity, U is C-embedded in X.

(3)⇒(1): By (3), every open subset of X is C^*-embedded and every cozero-set is C-embedded. Thus X is an extremally disconnected P-space.

By definition (see [6] and [8]), a topological space X is a *hyper-T_4-space* if it satisfies the following equivalent conditions: (a) X is an extremally disconnected T_5-space. (b) If the subsets A and B of X are separated, then they have disjoint closed neighborhoods. (c) If the subsets A and B are separated, then they are completely separated. (d) Every subset of X is C^*-embedded. (e) X is a T_4-space and every subspace of X is extremally disconnected. (f) X is a T_4-space and every closed subspace is extremally disconnected. The condition of X being a T_4-space cannot be omitted from (e). This is seen by considering an infinite set together with the cofinite topology.

Theorem 13. *For a topological space X, the following conditions are equivalent:*

(1) *X is a P-space and a hyper-T_4-space.*

(2) *Every subset of X is C-embedded.*

Proof. (1)⇒(2): Let A be a subset of X. Since X is a hyper-T_4-space, A is C^*-embedded. From this it follows, by the above Lemma, that A is C-embedded.

(2)⇒(1): By (2), every cozero-set is C-embedded and every subset is C^*-embedded. Thus X is a P-space and a hyper-T_4-space.

8. Irreducible spaces. By definition (see [1]), a nonvoid topological space X is *irreducible* if it satisfies the following equivalent conditions: (a) Every two nonvoid open subsets of X intersect. (b) X is not the union of a finite family of closed proper subsets. (c) Every nonvoid open subset of X is dense. (d) Every open subset of X is connected. (In [16] an irreducible space is called hyperconnected.)

Theorem 14. *For a nonvoid topological space X, the following conditions are equivalent:*

(1) *X is irreducible.*

(2) *X is ultrapseudocompact and extremally disconnected.*

(3) *X is connected and extremally disconnected.*

(4a) *The only regular open subsets of X are X and $\emptyset$.*

(4b) *The only regular closed subsets of X are X and $\emptyset$.*

Proof. (1)⇒(2): From (a) it immediately follows that X satisfies the condition (3a) in Theorem 1 and it is thus ultrapseudocompact. Let U be an open subset of X. By (c), $\overline{U}$ is either empty or X and hence open. Thus X is extremally disconnected.

(2)⇒(3): Every ultrapseudocompact space is connected.

(3)⇒(4a): Let U be a nonvoid regular open subset of X. Since X is extremally disconnected, $\overline{U}$ is clopen (and nonvoid). Thus $\overline{U}=X$, X being connected. Hence $U=\operatorname{int}\overline{U}=X$.

(4a)⇒(4b) is trivial.

(4b)⇒(1): Let U be a nonvoid open subset of X. Then $\overline{U}$ is a nonvoid regular closed set. Hence U is dense by (4b).

The condition of X being extremally disconnected cannot be omitted from (2) in Theorem 14. In fact, even an ultraconnected T_5-space may fail to be irreducible. An example of this is the three point space $\{a, b, c\}$, in which the nontrivial open subsets are $\{a\}$, $\{b\}$ and $\{a, b\}$. On the other hand, by choosing the same sets as the nontrivial closed subsets, we get an irreducible space which is not ultraconnected.

Theorem 15. *For a nonvoid topological space* X, *the following conditions are equivalent:*

(1) X *is an irreducible* T_4*-space.*

(2) X *is ultraconnected and extremally disconnected.*

(3) X *is a connected extremally disconnected* T_4*-space.*

Proof. (1)⇒(2): Let A and B be disjoint closed subsets of X. Since X is a T_4-space, there exist disjoint open sets $U \supset A$ and $V \supset B$. Since X is irreducible, either U or V is empty; hence either A or B is empty. Thus X is ultraconnected. On the other hand, X is extremally disconnected by Theorem 14.

(2)⇒(3): Every ultraconnected space is a connected T_4-space.

(3)⇒(1) follows from Theorem 14.

Theorem 16. *For a topological space* X, *the following statements are equivalent:*

(1) X *is an irreducible* T_3*-space.*

(2) X *is indiscrete and nonvoid.*

Proof. (1)⇒(2): Let U be a nonvoid open subset of X and x a point of U. Since X is a T_3-space, there exists an open set V such that $x \in V \subset \bar{V} \subset U$. As a nonvoid open subset of an irreducible space V is dense. Thus $U = X$ and, consequently, X is indiscrete.

(2)⇒(1) is clear.

A T_2-space is irreducible if and only if it is a singleton. This does not hold for T_1-spaces, as we see by considering an infinite set together with the cofinite topology. On the other hand, an ultraconnected T_1-space is empty or a singleton.

Irreducible T_5-spaces will be considered later in this paper.

By definition, a point x of a topological space X is *generic* if $\{x\}$ is dense in X. For later reference we state the following

Theorem 17. *Let* X *be a topological space. If* X *has a generic point, then it is irreducible. If the topology of* X *is finite, then* X *is irreducible if and only if it has at least one generic point.*

Proof. If a topological space has a generic point x, then x belongs to every nonvoid open subset, and (a) holds. Conversely, let X be an irreducible space with only a finite number of open subsets. Then the intersection of all nonvoid open subsets of X is nonvoid and every point of this intersection is a generic point of X.

9. Semi-irreducible spaces. By definition ([17]), a topological space X is *semi-irreducible* if it satisfies the following equivalent conditions: (a) Every disjoint family of nonvoid open subsets of X is finite. (b) X is the union of a finite number of irreducible subspaces.

Since every irreducible space is ultrapseudocompact, every semi-irreducible space is semi-ultrapseudocompact. The converse does not hold. In fact, an infinite set together with the excluded point topology (see [16], Example 14) is ultraconnected, but it is not semi-irreducible.

By [17], a topological space X is semi-irreducible if and only if every open subspace of X is semi-irreducible. A related result is

Theorem 18. *For a topological space X, the following statements are equivalent:*
(1) *X is semi-irreducible.*
(2) *Every open subspace of X is $\mathbf{Z}$-pseudocompact.*

Proof. (1)⇒(2): Every open subspace of a semi-irreducible space is semi-irreducible and hence $\mathbf{Z}$-pseudocompact (Theorem 9).

(2)⇒(1): Let $(V_i)_{i\in I}$ be a disjoint family of nonvoid open subsets of X. By (2), $V=\bigcup_{i\in I} V_i$ is $\mathbf{Z}$-pseudocompact. Hence I is finite and, consequently, X is semi-irreducible.

10. Locally irreducible spaces. By definition, a maximal irreducible subspace of a topological space X is an *irreducible component* of X. Every irreducible subspace of X is contained in an irreducible component and, consequently, X is the union of its irreducible components (see [1]).

Theorem 19. *The following statements about a topological space X are equivalent:*
(1) *X has a basis consisting of irreducible subsets.*
(2) *Every point of X has an irreducible neighborhood.*
(3) *Every point of X has an irreducible open neighborhood.*
(4) *X is the sum of its irreducible components.*
(5) *Every component of X is open and irreducible.*
(6) *Every irreducible component of X is open.*
(7) *X is (homeomorphic to) a sum of irreducible spaces.*
(8) *X is weakly locally connected and extremally disconnected.*

Proof. (1)⇒(2) is trivial.

(2)⇒(3): By (2), every x in X has an irreducible neighborhood A. Let $x\in U\subset A$, where U is open. As a nonvoid open subset of the irreducible space A, the set U is irreducible.

(3)⇒(4): For every x in X, let V_x denote the union of all irreducible open neighborhoods of x. By (3), V_x is an open neighborhood of x. On the other hand, since every two neighborhoods of x intersect, each V_x is irreducible. Thus each V_x is a maximal open irreducible subset of X.

If $V_x\cap V_y$ is nonvoid, then $V_x\cup V_y$ is irreducible and open; hence $V_x=V_y$, by the maximality of V_x and of V_y. Thus the family of all distinct sets V_x is a partition of X. Consequently, each V_x is clopen and X is the sum of the family of all distinct subspaces V_x. As a clopen set, each V_x is a union of irreducible components. Hence $x \in A \subset V_x$ for some irreducible component A. By the maximality of A we then have $A = V_x$; hence each V_x is an irreducible component of X.

(4)⇒(5): Each component A of X is the union of a family $\mathscr{S}$ of irreducible components. From (4) it follows that A is open and that it is the sum of the family $\mathscr{S}$. By the connectedness of A, the family $\mathscr{S}$ thus consists of only one set. Hence A is irreducible.

(5)⇒(6): An irreducible component A of X is contained in a component B of X. Since B is irreducible by (5), we then have $A = B$. Hence by (5), A is open.

(6)⇒(7): Since the union of two intersecting open irreducible subsets is irreducible, it follows from (6) that the family of all irreducible components of X is a partition of X. Thus X is the sum of this family.

(7)⇒(8): Every irreducible space is connected and extremally disconnected. Thus it follows from (7) that X is a sum of connected spaces, i.e. weakly locally connected, and that it is a sum of extremally disconnected spaces and hence extremally disconnected.

(8)⇒(3): Since X is weakly locally connected, every point x of X has a connected open neighborhood U. As an open subspace of an extremally disconnected space U is extremally disconnected. As a connected extremally disconnected space U is irreducible.

(3)⇒(1) is clear, since a nonvoid open subset of an irreducible space is irreducible.

We say that a topological space is *locally irreducible* if it satisfies the equivalent conditions of Theorem 19. The equivalence of the postulates (1) and (3) shows that the concept of a "weakly" locally irreducible space is superfluous. A locally irreducible space is locally ultrapseudocompact. The converse does not hold. This is seen if we consider the three point space $\{a, b, c\}$, in which the nontrivial open subsets are $\{a\}$, $\{b\}$ and $\{a, b\}$.

Every locally irreducible space is an extremally disconnected P-space. We could not prove the converse. Presumably it does not hold.

Since a nonvoid open subset of an irreducible space is irreducible, every open subspace of a locally irreducible space is locally irreducible.

Theorem 20. *For a nonvoid topological space X, the following conditions are equivalent:*

(I) *X is irreducible.*

(4) *X is connected and locally irreducible.*

Proof. (I)⇒(4): An irreducible space is connected. On the other hand, X is an irreducible neighborhood of any of its points. Thus it is locally irreducible.

(4)$\Rightarrow$(I): As a nonvoid locally irreducible space X is the sum of a nonvoid family of irreducible spaces. Since X is connected, this family consists of only one space. Thus X is irreducible.

11. Locally semi-irreducible spaces. Let X be a topological space. Since every open subspace of a semi-irreducible space is semi-irreducible, the following statements are equivalent:

(a) X has a basis consisting of semi-irreducible subspaces.

(b) Every point of X has a semi-irreducible open neighborhood.

(c) Every point of X has a semi-irreducible neighborhood.

We say that X is *locally semi-irreducible* if it satisfies these equivalent conditions. On the basis of Exercise 6 of § 4 in [1], we may state the following

Theorem 21. *A locally irreducible space is locally semi-irreducible. For a locally semi-irreducible space X, the following conditions are equivalent:*

(I) *X is locally irreducible.*

(9) *The quasi-components and the irreducible components of X are identical.*

(10) *The components and the irreducible components of X are identical.*

(11) *Every quasi-component of X is irreducible.*

(12) *Every component of X is irreducible.*

(13) *Every two distinct irreducible components of X are disjoint.*

In the general case none of the conditions (9) through (13) implies (I). An example to show this is a nondiscrete space in which every quasi-component is a singleton.

12. Hereditarily connected spaces

Theorem 22. *For a topological space X, the following statements are equivalent:*

(1) *Every subspace of X is connected.*

(2) *Every two-point subspace of X is connected (i.e. nondiscrete).*

(3) *X has no discrete subspace consisting of more than one point.*

(4a) *The family of open subsets of X is totally ordered by inclusion.*

(4b) *The family of closed subsets of X is totally ordered by inclusion.*

(5) *If the subsets A and B of X are separated, then either of them is empty.*

(6) *Every nonvoid subspace of X is irreducible.*

(7) *If the induced topology of a nonvoid subset A of X is finite, then the closure of A is identical with the closure of some point of A.*

(8) *The closure of every nonvoid finite subset of X is identical with the closure of some point of X.*[1]

Proof. (1)$\Rightarrow$(2) is trivial.

(2)$\Rightarrow$(3): Every subspace of a discrete space is discrete.

[1] The equivalence of (4a) and (8) has been proven in [10].

(3)⇒(4a): Let U and V be open subsets of X. Suppose that there exist points $x \in U \setminus V$ and $y \in V \setminus U$. Then $\{x, y\}$ would be a two-point discrete subspace of X, in contradiction to (3). Thus either of the sets U and V is contained in the other.

(4a)⇒(4b) is trivial.

(4b)⇒(5): Let A and B be separated. By (4b) we then have, say, $\bar{A} \subset \bar{B}$ and, consequently, $A = A \cap \bar{A} \subset A \cap \bar{B} = \emptyset$.

(5)⇒(1): Let S be a subspace of X and (A, B) a separation of S. Then A and B are separated in X. Thus by (5), either of the sets A and B is empty; hence S is connected.

(1)⇒(6): Let A be a nonvoid subspace of X. By (1), every open subspace of A is connected; hence A is irreducible.

(6)⇒(7): By (6), A is irreducible. From this it follows, by Theorem 17, that $\bar{x} \cap A = A$ for some x in A. Thus $A \subset \bar{x} \subset \bar{A}$ and, consequently, $\bar{A} = \bar{x}$.

(7)⇒(8) is trivial.

(8)⇒(2): Let $A = \{x, y\}$ be a two-point subspace of X. By (8), $\bar{x} \cup \bar{y} = \bar{A} = \bar{z}$ for some z in X. Then, say, $z \in \bar{x}$, i.e., x belongs to every neighborhood of z. Let U be an open neighborhood of y. Since $y \in \bar{z}$, U is a neighborhood of z. Hence $x \in U$ by the above result. Thus x belongs to every neighborhood of y and, consequently, A is connected.

We say that a topological space is *hereditarily connected* if it satisfies the equivalent conditions of Theorem 22.

Theorem 23. *For a topological space X, the following conditions are equivalent:*

(I) *X is hereditarily connected.*

(9) *X is a connected hyper-T_4-space, i.e. a connected extremally disconnected T_5-space.*

If X is nonvoid, then (I) *is also equivalent to the following statement:*

(10) *X is an irreducible T_5-space.*

Proof. Since the empty space satisfies (I) and (9), we may suppose that X is nonvoid. From Theorem 14 it then immediately follows that (9) and (10) are equivalent.

(I)⇒(10): By (4a), every two nonvoid open subsets of X intersect. Thus X is irreducible. On the other hand, it follows from (5) that X is trivially a T_5-space.

(10)⇒(5): Let A and B be separated subsets of X. Since X is a T_5-space, there exist disjoint open sets $U \supset A$ and $V \supset B$. Since X is irreducible, either U or V is empty. Hence either A or B is empty, and (5) holds.

By Theorem 23, a nonvoid hereditarily connected space is an irreducible T_4-space. The converse does not hold. To see this, let X be the four point space $\{a, b, c, d\}$, in which the nontrivial open subsets are $\{a\}$, $\{a, b\}$, $\{a, c\}$ and $\{a, b, c\}$. It is not hereditarily connected, since it does not satisfy (4a). However, it is an irreducible T_4-space, since every two nonvoid open (or closed) subsets intersect.

By Theorem 23, a hereditarily connected space is an ultraconnected T_5-space. The converse does not hold, as one sees by considering the ultraconnected T_5-space mentioned after the proof of Theorem 14.

Theorem 24. *For a topological space X, the following statements are equivalent:*
(I) *X is hereditarily connected.*
(11) *Every subspace of X is hereditarily connected.*
(12) *Every nonvoid closed subspace of X is irreducible.*
(13) *Every subspace of X is ultraconnected.*
(14) *Every open subspace of X is ultraconnected.*
(15) *Every subspace of X is ultrapseudocompact.*

Proof. (I)⇒(11)⇒(12), (11)⇒(13)⇒(14) and (13)⇒(15)⇒(1) are clear. (14)⇒(4b) is seen in the same way as (12)⇒(4a) below.

(12)⇒(4a): Let U and V be two open proper subsets of X. Then $A=(X\setminus U)\cup \cup(X\setminus V)$ is nonvoid and closed; hence A is irreducible by (12). Since $A\cap U\cap V=\emptyset$, we then have $A\cap U=\emptyset$ or $A\cap V=\emptyset$. If, say, $A\cap U=\emptyset$, then $(X\setminus V)\cap U=\emptyset$, i.e. $U\subset V$.

The postulate "Every open subspace is ultrapseudocompact (irreducible)" or "Every closed subspace is ultrapseudocompact (ultraconnected)" cannot be added to the list of Theorem 24. In fact, the former postulate characterizes irreducible spaces and the latter one ultraconnected spaces.

13. Partition spaces

Theorem 25. *For a topological space X, the following conditions are equivalent:*
(1a) *Every open subset of X is closed.*
(1b) *Every closed subset of X is open.*
(2) *If the subsets A and B of X satisfy $\bar{A}\cap B=\emptyset$, then they have disjoint (closed) neighborhoods.*
(3) *If the subsets A and B of X satisfy $\bar{A}\cap B=\emptyset$, then they also satisfy $A\cap\bar{B}=\emptyset$.*
(4a) *The family of all open subsets of X is a Boolean lattice with respect to inclusion.*
(5) *Every component of X is open and indiscrete (i.e., a subset of X is open if and only if it is a union of components).*
(6) *Every quasi-component of X is open and indiscrete.*
(7) *Every irreducible component of X is open and indiscrete.*
(8) *X is (homeomorphic to) a sum of indiscrete spaces.*
(9) *X is the sum of a basis of X, i.e., X has a disjoint basis.*
(10) *The intersection U_x of all neighborhoods of x is open for every x in X and the distinct sets U_x form a partition of X.*
(11) *X has a basis consisting of indiscrete subspaces.*
(12) *Every point of X has an indiscrete neighborhood.*

(13) $\bar{x}$ *is open and connected (and hence the component of* x*) for every* x *in* X.

(14) *If* ϱ *is the equivalence relation on* X *defined by*

$$x\varrho y \text{ if and only if } \bar{x}=\bar{y},$$

then the quotient space X/ϱ *is discrete.*

Proof. (1a)⇒(1b) is trivial.

(1b)⇒(2): If $\bar{A}\cap B=\emptyset$, then $\bar{A}\cap\bar{B}=\emptyset$, since $\bar{A}$ is open by (1b). Thus A and B have disjoint clopen neighborhoods $\bar{A}$ and $\bar{B}$.

(2)⇒(3): Let $\bar{A}\cap B=\emptyset$. By (2), there exists an open set U such that $A\subset U$ and $U\cap B=\emptyset$. Then $U\cap\bar{B}=\emptyset$ and, consequently, $A\cap\bar{B}=\emptyset$.

(3)⇒(1a): Let U be an open subset of X. Then $\overline{X\setminus U}\cap U=\emptyset$ and hence $(X\setminus U)\cap\bar{U}=\emptyset$, by (3). Thus $\bar{U}\subset U$, and U is closed.

(1a)⇔(4a): The topology τ of X, ordered by inclusion, is a distributive lattice, in which $\inf(U, V)=U\cap V$ and $\sup(U, V)=U\cup V$ for all U and V and in which $\emptyset$ is the least and X the greatest element. If $U\cap V=\emptyset$ and $U\cup V=X$, then $V=X\setminus U$, and conversely. Thus τ is a complemented lattice if and only if the complement of every open set is open, i.e. if and only if (1a) holds.

(1)⇒(5): Let C be a component of X. By (1b), C is open. Let U be a nonvoid open subset of C. Since C is open, U is open in X and hence clopen by (1a). Since C is connected, we thus have $U=C$. Hence C is indiscrete.

(5)⇔(6)⇔(7): If the component or the quasi-component or some irreducible component of a point of X is open, then all three sets are identical.

(5)⇒(8): Since every component of X is open by (5), X is the sum of its components.

(8)⇒(9): By (8), X is the sum of a family $\mathscr{U}$ of indiscrete subspaces. Let V be an open subset of X. If $U\cap V\neq\emptyset$ for some U in $\mathscr{U}$, then $U\subset V$ by the indiscreteness of U. Thus $\mathscr{U}$ is a basis of X.

(9)⇒(10): By (9), X is the sum of a basis $\mathscr{B}$ of X. Let x be a point of X. Then x belongs to a unique B in $\mathscr{B}$. Since B is an open neighborhood of x, we have $U_x\subset B$. Let U be an open neighborhood of x. Since $\mathscr{B}$ is a disjoint basis of X, we then have $x\in B\subset U$. Hence $B\subset U_x$ and, consequently, $U_x=B$. Thus (10) holds.

(10)⇒(11): Let $x\in U$, where U is open. Then $U_x\subset U$. Since each U_x is open by (10), it follows from this that the family of the sets U_x is a basis of X. Let $y\in V\subset U_x$, where V is open in U_x and thus in X. Then $U_y\subset V\subset U_x$ and hence $U_y=V=U_x$ by (10). Thus each U_x is indiscrete.

(11)⇒(12) is trivial.

(12)⇒(1a): Let U be an open subset of X and let x be a point of $X\setminus U=A$. By (12), x has an indiscrete neighborhood V. Since $A\cap V$ is a nonvoid closed subset of V, we then have $x\in V\subset A$. Thus A is open and U is closed.

(5)⇒(13): Let C be the component of x. Since C is closed, we have $\bar{x}\subset C$ and hence $\bar{x}=C$, because C is indiscrete by (5). Thus $\bar{x}$ is connected and open by (5).

(13)⇒(14): From (13) it follows that the equivalence class of x is $\bar{x}$. In fact, if $y\in\bar{x}$, then $\bar{y}\subset\bar{x}$ and hence $\bar{y}=\bar{x}$, since $\bar{y}$ is clopen and $\bar{x}$ is connected. Thus X/ϱ is the set of the distinct sets $\bar{x}$ and each $\bar{x}$ is an open subset of X, saturated with respect to ϱ. Since $\{\bar{x}\}$ is the canonical image of the set $\bar{x}$, it follows from this that $\{\bar{x}\}$ is open in X/ϱ. Thus X/ϱ is discrete.

(14)⇒(8): Let f be the canonical map $X\to X/\varrho$. The family of the sets $f^{-1}(x')$, x' in X/ϱ, is a partition of X. By (14), each of these sets is clopen and, consequently, X is their sum. On the other hand, each subspace $f^{-1}(x')=A$ is indiscrete. To see this, let x be a point of A. Then $\bar{x}\subset A$, since A is closed. Conversely, if $y\in A$, then $f(y)=x'=f(x)$ and hence $y\in\bar{y}=\bar{x}$. Thus $\bar{x}=A$, and so A is indiscrete, as asserted.

A topological space is a *partition space*, if it satisfies the equivalent conditions of Theorem 25.

For the next theorem we recall that a T_4-space X is a *perfectly T_4-space* if every closed subset of X is a G_δ-set.

Theorem 26. *For a topological space X, the following conditions are equivalent:*

(I) *X is a partition space.*

(15) *X is a weakly locally connected extremally disconnected T_3-space, i.e. a locally irreducible T_3-space.*

(16) *X is a T_3-space and the intersection of every family of open subsets of X is open.*

(17) *X is a P-space and a perfectly T_4-space.*

Proof. (I)⇒(15): By (5), every component of X is open. Thus X is weakly locally connected. By (1b), the closure of an open subset is open. Thus X is extremally disconnected. By (1a), every open neighborhood of every point is closed. Thus X is a T_3-space.

(15)⇒(5): Let C be a component of X. Since X is weakly locally connected, C is open. Let U be a nonvoid open subset of C and let x be some point of U. Since C is open, U is open in X. Since X is a T_3-space, there exists an open set V such that $x\in V$ and $\bar{V}\subset U\subset C$. Since X is extremally disconnected, $\bar{V}$ is clopen. Since C is connected, we thus have $\bar{V}=C$. Hence $U=C$ and, consequently, C is indiscrete.

(I)⇒(16): By (15), X is a T_3-space. On the other hand, from (1a) and (1b) it immediately follows that the intersection of every family of open subsets of X is open.

(16)⇒(1b): Let U be an open subset of X and x a point of U. Since X is a T_3-space, there exists an open set V such that $x\in V\subset\bar{V}\subset U$. Then $X\setminus\bar{V}$

is an open neighborhood of $X \setminus U$ to which x does not belong. Thus $X \setminus U$ is an intersection of open sets and hence open by (16).

(I)⇒(17): From (16) it immediately follows that X is a P-space. If A and B are disjoint closed sets, then they are also disjoint open sets by (1b). Thus X is a T_4-space. On the other hand, every closed subset of X is open and hence a G_δ-set. Thus X is a perfectly T_4-space.

(17)⇒(1b): Let A be a closed subset of X. Since X is a perfectly T_4-space, A is a G_δ-set and X is a $T_{3\frac{1}{2}}$-space. Since X is a P-space, it follows that A is open.

Corollary. *A partition space is a locally irreducible* T_5*-space.*

The converse of the Corollary does not hold. In fact, every irreducible space is locally irreducible. Thus every irreducible T_5-space, i.e. every nonvoid hereditarily connected space is a locally irreducible T_5-space.

In accordance with [2] we call a pseudometric d on a set X a *strong ultrapseudometric* if for all $x, y \in X$, $d(x, y)=0$ or 1. As another corollary of Theorem 26 we get the result of K. A. Broughan in [2]:

Theorem 27. *For a topological space* X*, the following conditions are equivalent:*

(I) X *is a partition space.*

(18) *The topology of* X *can be defined by a strong ultrapseudometric.*

Proof. (8)⇒(18): Let X be the union of a disjoint family $\mathcal{U}$ of open indiscrete subspaces. By setting $d(x, y)=0$ or 1, depending on whether x and y belong to the same set $U \in \mathcal{U}$ or not, we get a strong ultrapseudometric d which defines the topology of X.

(18)⇒(16): Let d be a strong ultrapseudometric compatible with the topology of X and x a point of X. The open ball $\{y \mid d(x, y)<1\}$ is a neighborhood of x contained in every neighborhood of x. From this it immediately follows that the intersection of every family of open subsets is open. On the other hand, as a pseudometric space, X is a T_3-space.

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