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QUASISYMMETRIC FUNCTIONS WITH DILATATION ONE

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In a recent paper [3] Strebel introduced the dilatation of a homeomorphism of a Jordan curve onto another as follows: Let G_j, $j=1, 2$, be Jordan domains and $\varphi:\partial G_1\to\partial G_2$ a sense-preserving homeomorphism. Consider all ring domains $A_j\subset G_j$ such that one boundary component of A_j is ∂G_j, and quasiconformal mappings $f: A_1\to A_2$ such that $f|\partial G_1=\varphi$. The infimum of the maximal dilatations of all such mappings is called the interior dilatation of φ. The exterior dilatation is defined similarly using ring domains in the complements of G_1 and G_2. If ∂G_1 and ∂G_2 are analytic, the interior and exterior dilatations of φ coincide. We then call their common value the dilatation of φ and denote it by $L(\varphi)$.

Assume $L(\varphi)<\infty$ and denote by Q_φ the class of all quasiconformal mappings $g: G_1\to G_2$ such that $g|\partial G_1=\varphi$. Making use of a well-known extension theorem [2, p. 96] one easily concludes that $Q_\varphi\neq\emptyset$. The class Q_φ contains one or more extremals, i.e. mappings with smallest possible maximal dilatation in Q_φ. Denote this dilatation by $K(\varphi)$. It was shown by Strebel that if $L(\varphi)<K(\varphi)$, then Q_φ contains only one extremal, which is a Teichmüller mapping. In particular, then, if $L(\varphi)=1$, the extremal is always unique, and either conformal or a Teichmüller mapping.

Strebel [3, p. 469] obtained a necessary and sufficient condition for φ to have dilatation one in the case $G_1=G_2=$the unit disc. There is, however, some interest in carrying Strebel's characterization over to the case of the upper half-plane, the boundary mappings then being the familiar quasisymmetric functions.

Now let $\varphi:\boldsymbol{R}\to\boldsymbol{R}$ be an increasing homeomorphism. It gives rise to a function $q_\varphi: H\to\boldsymbol{R}_+$, defined by

$$q_\varphi(x+iy)=\frac{\varphi(x+y)-\varphi(x)}{\varphi(x)-\varphi(x-y)}.$$

Thus φ is k-quasisymmetric if q_φ is bounded above by k and below by $1/k$. Whether φ has dilatation one depends on the behavior of q near the real axis:

Theorem 1. *An increasing homeomorphism $\varphi:\boldsymbol{R}\to\boldsymbol{R}$ has dilatation one if and only if $q_\varphi(z)$ tends to one as z tends to the real axis in the spherical metric.*

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Proof. To prove the sufficiency part, we utilize the construction of Beurling and Ahlfors [1]. Given an arbitrary $\eta>1$, there exist positive numbers m, M such that

$$\eta^{-1} < q_\varphi(z) < \eta$$

for all z in the subset E of H whose elements satisfy $|\operatorname{Re} z|>M$, $\operatorname{Im} z<m$, or $\operatorname{Im} z>M$. For all $z=x+iy\in H$ set

$$\alpha_j(z) = \int_0^1 \varphi(x+(-1)^j yt)\, dt, \quad j = 0, 1.$$

Then α_j is differentiable, and its partial derivatives are

$$(1) \qquad (\alpha_j)_x(z) = (-1)^j(\varphi(x+(-1)^j y)-\varphi(x)),$$

$$(2) \qquad (\alpha_j)_y(z) = \int_0^1 (-1)^j t\, d\varphi(x+(-1)^j yt).$$

Now set

$$f(z) = (1/2)(\alpha_0(z)+\alpha_1(z)+i(\alpha_0(z)-\alpha_1(z)).$$

It follows from the hypothesis and the continuity of q_φ that φ is k-quasisymmetric for some k. By [1], f is a quasiconformal homeomorphism of H, agreeing with φ on $\boldsymbol{R}$. The dilatation quotient D of f at z satisfies

$$(3) \qquad D+D^{-1} = \frac{5(1+\xi_0^2)\zeta+5(1+\xi_1^2)/\zeta+6(\xi_0\xi_1-1)}{4(\xi_0+\xi_1)},$$

where

$$\zeta = (\alpha_1)_x(z)/(\alpha_0)_x(z), \quad \xi_j = (-1)^j(\alpha_j)_y(z)/(\alpha_j)_x(z),$$

$j=0, 1$. The right-hand side of (3) is continuous in ξ_0, ξ_1, ζ and takes the value 2 for $\xi_0=\xi_1=1/2, \zeta=1$. In order to have D arbitrarily close to one it thus suffices to have ξ_0 and ξ_1 sufficiently close to 1/2 and ζ sufficiently close to 1. Now $\zeta=1/q_\varphi(z)$ so that $1/\eta\leqq\zeta\leqq\eta$ holds in E. We next estimate ξ_0. By a lemma of Beurling and Ahlfors [1, p. 137]

$$(4) \qquad \frac{1}{1+\eta} \leqq \xi_0(x+iy) \leqq \frac{\eta}{1+\eta},$$

provided φ is η-quasisymmetric in the interval $(x, x+y)$. This is certainly true if the triangle with vertices $x, x+y, x+y/2+iy/2$ is contained in E. Suppose then that this is not the case. First assume $x\geqq -M$, $y\geqq 2M$. By (1) and (2),

$$\xi_0(x+iy) = \frac{\int_0^1 (\varphi(x+y)-\varphi(x+ty))\, dt}{\varphi(x+y)-\varphi(x)},$$

and since $x+iy\in E$,

$$\eta^{-2}\xi_0(x+iy) \leqq \frac{\int_0^1 (\varphi(x+y+yt)-\varphi(x+y))\,dt}{\varphi(x+2y)-\varphi(x+y)} \leqq \eta^2\xi_0(x+iy)$$

or

$$\eta^{-2}\xi_0(x+iy) \leqq 1-\xi_0(x+y+iy) \leqq \eta^2\xi_0(x+iy).$$

Our assumption implies that $\xi_0(x+y+iy)$ satisfies (4), and hence

$$\frac{1}{\eta^2(1+\eta)} \leqq \xi_0(x+iy) \leqq \frac{\eta^3}{1+\eta}.$$

If $x\leqq -M, y\geqq 2M,$ we may write

$$\xi_0(x+iy) = 1-\frac{\int_0^1 (\varphi(x+yt)-\varphi(x))\,dt}{\varphi(x+y)-\varphi(x)},$$

and a similar argument yields

$$\frac{1+\eta-\eta^3}{1+\eta} \leqq \xi_0(x+iy) \leqq \frac{\eta^3+\eta^2-1}{\eta^2(\eta+1)}.$$

Completely analogous estimates hold for ξ_1. It follows that given $K>1$, we can always find m and M such that f is K-quasiconformal outside the trapezoid with vertices $\pm M+ im$, $\pm(3M-m)+2Mi$. By definition, then, $L(\varphi)=1$.

On the other hand, assume $L(\varphi)=1$. Let $\eta>1$ be arbitrary. By a lemma of Strebel [3, p. 469] there exists a $\delta>0$ such that

$$\eta^{-1} \leqq q_\varphi(x+iy) \leqq \eta \tag{5}$$

as soon as $0<y<\delta$. We thus have to estimate $q_\varphi(z)$ only for $\operatorname{Im} z\geqq\delta$ and $|z|$ large. There is no loss of generality in supposing $x\geqq 0$. By assumption, there exist positive numbers m, M such that φ can be extended to an η-quasiconformal mapping f of the set E considered in the first part of the proof. Further let E_j, $j=0, 1,$ be the simply connected domain obtained from E by removing the closed rectangle with vertices $0, (-1)^j M,$ $(-1)^j M+im,$ $im.$ We consider three cases: (i) $0\leqq x\leqq M$, (ii) $x>M$ and $x>y$, (iii) $x>M$ and $x\leqq y$.

In case (i), take $y\geqq 2M$ and consider the quadrilateral $E_1(x-y, x, x+y, \infty)$. The mapping $\zeta\mapsto(\zeta-x)/y$ transforms it into a quadrilateral $E_z(-1, 0, 1, \infty)$ without changing the conformal module. It is clear that as $y\to\infty$ the distance of any point lying on the side of E_z with endpoints $-1, 0$ from the line segment joining the same points approaches zero. By the continuity of the module (see e.g.

[2, p. 26]), $\mathrm{mod}\, E_z$ tends to $\mathrm{mod}\, H(-1, 0, 1, \infty)=1$. In case (ii) the spherical distance of the side of E_z with endpoints $-\infty, -1$ from the ray $(-\infty, -1)$ tends to zero as $|z|\to\infty$. (Observe that we assume $y\geqq\delta$.) Consequently $\mathrm{mod}\, E_1(x-y, x, x+y, \infty)=\mathrm{mod}\, E_z(-1, 0, 1, \infty)\to 1$. In case (iii), a similar argument yields $\mathrm{mod}\, E_0(x-y, x, x+y, \infty)\to 1$ as $|z|\to\infty$. The same argument also shows that $\mathrm{mod} f(E_j)\big(\varphi(x-y), \varphi(x), \varphi(x+y), \infty\big)$ (where $j=1$ in cases (i) and (ii), and $j=0$ in case (iii)) tends to $\mathrm{mod}\, H\big(\varphi(x-y), \varphi(x), \varphi(x+y), \infty\big)$ as $|z|\to\infty$. Thus for $|z|$ large enough

$$\eta^{-3} \leqq \eta^{-2}\,\mathrm{mod}\, E_j \leqq \eta^{-1}\,\mathrm{mod} f(E_j)$$

$$\leqq \mathrm{mod}\, H\big(\varphi(x-y), \varphi(x), \varphi(x+y), \infty\big)$$

$$\leqq \eta\,\mathrm{mod} f(E_j) \leqq \eta^2\,\mathrm{mod}\, E_j \leqq \eta^3.$$

But this means that for $|z|$ large enough

$$\lambda(\eta^3)^{-1} \leqq q_\varphi(z) \leqq \lambda(\eta^3),$$

where λ is the distortion function defined in [2, p. 81]. As $\lim_{t\to 1}\lambda(t)=1$, and because of (5), the theorem is proved.

It follows at once from the extension theorem of Beurling and Ahlfors or from the above proof that k-quasisymmetric mappings with a small k necessarily have a small dilatation. It is, however, easy to construct examples showing that the converse is not in general true:

Theorem 2. *There exist homeomorphisms* $\varphi\colon \boldsymbol{R}\to\boldsymbol{R}$ *with* $L(\varphi)=1$ *and* $\max_{z\in H} q_\varphi(z)$ *arbitrarily large.*

Proof. Given $M>0$, set

$$\psi(x) = 8M(x^3-x^4), \qquad 0\leqq x\leqq 1/2,$$

$$\psi(x) = M-\psi(1-x), \qquad 1/2< x\leqq 1,$$

$$\psi(x) = \psi\big(1-(x-1)/(3M)\big), \quad 1< x\leqq 3M+1,$$

and $\psi(x)=0$ otherwise. Then ψ has a continuous second derivative, and $\psi'(x)\geqq \geqq\psi'(1+3M/2)=-2/3$. Set $\varphi(x)=x+\psi(x)$. Then $\varphi\colon\boldsymbol{R}\to\boldsymbol{R}$ is increasing, $\varphi(1)=1+M$, $q_\varphi(i)=M+1$. Let K be an upper bound for ψ''. Then $q_\varphi(x+iy)= =\varphi'(\xi_1)/\varphi'(\xi_2)$ is bounded above by $(\varphi'(x)+Ky)/(\varphi'(x)-Ky)$ and below by $(\varphi'(x)-Ky)/(\varphi'(x)+Ky)$, whence $\lim_{y\to 0} q_\varphi(x+iy)=1$, uniformly in x. It is clear that $q_\varphi(z)$ tends to one as $|z|\to\infty$. By Theorem 1, $L(\varphi)=1$.

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