

TAYLOR'S FORMULA FOR MAPS: THE REVERSE PROBLEM

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1. Introduction. Suppose that X, Y are Banach spaces and that $\Omega \subset X$ is an open set such that the line segment $[0, x] \subset \Omega$ whenever $x \in \Omega$. Denote by $L_i(X, Y)$ the space of multilinear continuous operators on $X \times \dots \times X$ (i factors) to Y , with the Banach norm, and by $S_i(X, Y)$ the subspace of symmetric operators.

If $f: \Omega \rightarrow Y$ is a C^p -map, $p \geq 1$, then f and its derivatives satisfy the Taylor formulas:

$$(1) \quad f^k(x) = a_k + \dots + \frac{a_{p-1}}{(p-k-1)!} x^{p-k-1} + \int_0^1 \frac{(1-t)^{p-k-1}}{(p-k-1)!} \varphi(tx) x^{p-k} dt,$$

$$k = 0, \dots, p-1; \quad x \in \Omega,$$

where

$$(2) \quad \varphi = f^p: \Omega \rightarrow S_p(X, Y)$$

is the p -th derivative of f , and

$$a_i = f^i(0) \in S_i(X, Y), \quad i = 0, \dots, p-1$$

[1, 2, 3].

Now consider the reverse problem. Suppose that we are given a C^0 -map of Ω into the symmetric continuous multilinear operators of order p

$$\psi: \Omega \rightarrow S_p(X, Y)$$

and symmetric continuous multilinear operators

$$b_i \in S_i(X, Y), \quad i = 0, \dots, p-1;$$

and suppose that we define the map $g: \Omega \rightarrow Y$ by the relations

$$(3) \quad g(x) = b_0 + \dots + \frac{b_{p-1}}{(p-1)!} x^{p-1} + r(x),$$

$$(4) \quad r(x) = \int_0^1 \frac{(1-t)^{p-1}}{(p-1)!} \psi(tx) x^p dt, \quad x \in \Omega.$$

What can we say about the derivatives of g and r ?

Proposition 1 below implies that $r^p = \psi$ if and only if ψ is the p -th derivative of a map. Proposition 3 implies that, if $\psi \in C^1$, then ψ is the p -th derivative of a map if and only if the first derivative ψ^1 , which is a map into $S_1(X, S_p(X, Y))$, is symmetric in its $p+1$ arguments, that is, if and only if $\psi^1(x) \in S_{p+1}(X, Y)$, $x \in \Omega$.

It is worthwhile, I think, to study the behaviour of g and r in the case in which ψ is not a p -th derivative. For example, ψ may be known to be an approximation of the p -th derivative of the unknown map $f: \Omega \rightarrow Y$. We may take g as an approximation of f , based on ψ and the approximate initial values b_0, \dots, b_{p-1} . We would then be interested in the extent to which the derivatives g^k approximate f^k , $k \leq p$. Formulas (22), (23), and (19) below give the errors $g^k - f^k$ in terms of $b_i - a_i$, $i < p$, and $\psi^j - \varphi^j$, $j \leq k$, in the case in which $\psi, \varphi \in C^k$. These errors are appraised in terms of the norms of $b_i - a_i$ and $\psi^j - \varphi^j$ in Section 7.

Formulas (16), (10) give the derivatives r^k , $k \leq p$, in terms of $\psi, \psi^1, \dots, \psi^k$.

2. The case in which ψ is a p -th derivative.

Proposition 1. *If ψ is the p -th derivative of a map on Ω to Y , then $r \in C^p$, $g \in C^p$, and*

$$(5) \quad \begin{cases} r^p = g^p = \psi, \\ r^k(x) = \int_0^1 \frac{(1-t)^{p-k-1}}{(p-k-1)!} \psi(tx) x^{p-k} dt, \quad k < p, \quad x \in \Omega. \end{cases}$$

Conversely, if g or r is in C^p , then the other is also, and

$$(6) \quad \begin{cases} r^p = g^p, & r^p(0) = \psi(0), \\ r^k(0) = 0, & g^k(0) = b_k, \quad k < p; \end{cases}$$

however r^p need not equal ψ .

Proof. Say that $\psi = f^p$. Then $f \in C^p$, since $\psi \in C^0$. By (1) and (4),

$$r(x) = f(x) - a_0 - \dots - \frac{a_{p-1}}{(p-1)!} x^{p-1}, \quad x \in \Omega.$$

Hence

$$g(x) = f(x) + b_0 - a_0 + \dots + \frac{b_{p-1} - a_{p-1}}{(p-1)!} x^{p-1},$$

and

$$g^p(x) = f^p(x) = \psi(x) = r^p(x).$$

Furthermore,

$$r^k(x) = f^k(x) - a_k - \dots - \frac{a_{p-1}}{(p-k-1)!} x^{p-k-1} = \int_0^1 \frac{(1-t)^{p-k-1}}{(p-k-1)!} \psi(tx) x^{p-k} dt, \quad k < p,$$

by (1).

Conversely, if g or $r \in C^p$, then so is the other and $g^p = r^p$, by (3).

By (4),

$$(7) \quad \|r(x)\| \cong \frac{\|\psi, x\| \|x\|^p}{p!}, \quad x \in \Omega,$$

where, in a special notation,

$$\|\psi, x\| = \sup_{t \in [0, 1]} \|\psi(tx)\| < \infty,$$

because the segment $[0, x]$ is compact. On the other hand,

$$(8) \quad r(x) = r(0) + \dots + \frac{r^{p-1}(0)}{(p-1)!} x^{p-1} + \int_0^1 \frac{(1-t)^{p-1}}{(p-1)!} r^p(tx) x^p dt.$$

This with (7) implies that

$$r^k(0) = 0, \quad k < p,$$

and

$$g^k(0) = b_k + r^k(0) = b_k.$$

Furthermore (4) implies that

$$r(x) - \psi(0)x^p/p! = \theta(x)x^p, \quad x \in \Omega,$$

where $\theta(x) \rightarrow 0$ as $x \rightarrow 0$. By (8), a similar relation holds with $\psi(0)$ replaced by $r^p(0)$. Hence $r^p(0) = \psi(0)$, and (6) is established.

That r^p need not equal ψ is to be expected, since the differential of x need not be collinear with the segment $[0, x]$. It is sufficient to cite one instance. Take $X = R^2 = \Omega$, $Y = R$, $p = 1$, and ψ the map of R^2 into $S_1(R^2, R) = L_1(R^2, R)$ for which

$$\psi(u)x = u_1 x_2, \quad u = (u_1, u_2) \in R^2, \quad x = (x_1, x_2) \in R^2.$$

Then

$$r(x) = \int_0^1 t x_1 x_2 dt = x_1 x_2 / 2,$$

and

$$r^1(x)h = (h_1 x_2 + x_1 h_2) / 2, \quad h = dx = (h_1, h_2) \in R^2.$$

On the other hand

$$\psi(x)h = x_1 h_2.$$

Thus $r^1(x) \neq \psi(x)$, unless $x = 0$.

3. The derivatives of ψ . Since ψ is a map into $S_p(X, Y)$, the value $\psi^j(x)$ of the j -th derivative, if existent, is an element of $S_j(X, S_p(X, Y))$ and so may be identified with an element of $L_{j+p}(X, Y)$. It need not be true, however, that $\psi^j(x) \in S_{j+p}(X, Y)$.

Proposition 2. *If for some $x \in \Omega$, the first derivative $\psi^1(x)$ exists and is symmetric in its $p+1$ arguments, then all subsequent derivatives which exist at x are symmetric in all their arguments.*

Proof. By hypothesis, $h_1, x_1, \dots, x_p \in X$ may be permuted arbitrarily in

$$\psi^1(x)h_1x_1 \dots x_p \in Y.$$

If $\psi^2(x)$ exists, it follows that h_1, x_1, \dots, x_p may be permuted arbitrarily in

$$\psi^2(x)h_2(h_1x_1 \dots x_p) \in Y, \quad h_2 = dx \in X.$$

But this element equals

$$\psi^2(x)h_1h_2x_1 \dots x_p,$$

since a derivative is symmetric in its differentials. Thus h_1 and therefore h_2, x_1, \dots, x_p may be permuted arbitrarily.

Iterating, we see that if $\psi^j(x)$ exists, then

$$\psi^j(x)h_1 \dots h_jx_1 \dots x_p, \quad h_1, \dots, h_j, x_1, \dots, x_p \in X,$$

is invariant under permutations of the $j+p$ arguments.

Proposition 3. *Suppose that $\psi \in C^1$. Then a necessary and sufficient condition that $\psi = r^p$ is that the first derivative ψ^1 be symmetric in its $p+1$ arguments.*

Proof. Necessity is immediate: If $\psi = r^p$, then $\psi^1 = r^{p+1}$, being a $(p+1)$ -th derivative, is symmetric in its $p+1$ arguments.

Sufficiency. By Leibnitz' rule, which is valid because the segment $[0, x]$ is compact,

$$r^1(x) = \int_0^1 \frac{(1-t)^{p-1}}{(p-1)!} D_x[\psi(tx)x^p] dt, \quad x \in \Omega.$$

In evaluating the interior derivative, we shall work with differentials, since the position of differentials of x is critical and may not be changed arbitrarily. Note however that p, t , and dt are scalars and commute with all other factors. Now

$$\begin{aligned} (9) \quad d_x[\psi(tx)x^p] &= \psi(tx)d_x x^p + [d_x \psi(tx)]x^p \\ &= p\psi(tx)x^{p-1}h + t\psi^1(tx)hx^p, \quad h = dx \in X, \quad x \in \Omega, \quad 0 \leq t \leq 1. \end{aligned}$$

Our present hypothesis implies that

$$\psi^1(tx)hx^p = \psi^1(tx)x^ph.$$

Hence

$$D_x[\psi(tx)x^p] = p\psi(tx)x^{p-1} + t\psi^1(tx)x^p$$

and

$$r^1(x) = \int_0^1 \frac{(1-t)^{p-1}}{(p-1)!} [p\psi(tx)x^{p-1} + t\psi^1(tx)x^p] dt, \quad x \in \Omega.$$

Transforming the second term by parts, with $d_t \psi(tx) = \psi^1(tx)x dt$, we see that

$$r^1(x) = \begin{cases} \psi(x), & \text{if } p = 1, \\ \int_0^1 \frac{(1-t)^{p-2}}{(p-2)!} \psi(tx)x^{p-1} dt, & \text{if } p \geq 2. \end{cases}$$

It follows that $r^p = \psi$, by induction on p .

4. The transposes. In order to treat the case in which ψ is not a p -th derivative, we introduce $\psi_k^j(u)$, the k -transpose of $\psi^j(u)$, defined as follows.

For $0 \leq j \leq k \leq p$ and $u \in \Omega$, $\psi_k^j(u)$ is the element of $L_{p+j}(X, Y)$ determined by the relation

$$(10) \quad \begin{aligned} & C_{k,j} \psi_k^j(u) x_1 \dots x_{p-k+j} h_1 \dots h_k \\ &= \sum_{1 \leq i_1 < \dots < i_j \leq k} \psi^j(u) h_{i_1} \dots h_{i_j} x_1 \dots x_{p-k+j} h_1 \dots \hat{h}_{i_1} \dots \hat{h}_{i_j} \dots h_k, \end{aligned}$$

$h_1, \dots, h_k, x_1, \dots, x_{p-k+j} \in X$, where the numbers $C_{k,j}$ are binomial coefficients. There are $C_{k,j}$ terms in the summation. In particular,

$$(11) \quad \psi_k^0(u) = \psi^0(u) = \psi(u), \quad k \leq p, \quad u \in \Omega,$$

and

$$\psi_k^k(u) x_1 \dots x_p h_1 \dots h_k = \psi^k(u) h_1 \dots h_k x_1 \dots x_p, \quad h_1, \dots, h_k, x_1, \dots, x_p \in X.$$

Thus $\psi_k^j(u)$ exists if $\psi^j(u)$ exists. Since $\psi^j(u)$ is symmetric in its first j arguments and in its last p arguments, it follows that $\psi_k^j(u)$ is symmetric in its last k arguments and in its first $p-k+j$ arguments.

Note that

$$\|\psi_k^j(u)\| \leq \|\psi^j(u)\|$$

and

$$(12) \quad \|\psi_k^j(u) - \psi^j(u)\| \leq 2\|\psi^j(u)\|, \quad 0 \leq j \leq k \leq p, \quad u \in \Omega.$$

If the first derivative $\psi^1(u)$ is symmetric and if $\psi^j(u)$ exists, then

$$\psi_k^j(u) = \psi^j(u), \quad 0 \leq j \leq k \leq p, \quad u \in \Omega.$$

This is because $\psi^j(u)$ is symmetric, by Proposition 2.

5. The derivatives of r . Suppose that $\psi \in C^k$. By Leibnitz' rule,

$$(13) \quad r^k(x) = \int_0^1 \frac{(1-t)^{p-1}}{(p-1)!} D_x^k [\psi(tx) x^p] dt, \quad x \in \Omega.$$

In evaluating the interior derivative, we again use differentials. Thus (cf. (9)),

$$(14) \quad d_x^k [\psi(tx) x^p] = \sum_{0 \leq j \leq k} \frac{p!}{(p-k+j)!} t^j \psi^j(tx) \sum_{1 \leq i_1 < \dots < i_j \leq k} h_{i_1} \dots h_{i_j} x^{p-k+j} h_1 \dots \hat{h}_{i_1} \dots \hat{h}_{i_j} \dots h_k,$$

$$h_1 = d_1 x \in X, \dots, h_k = d_k x \in X, \quad x \in \Omega, \quad 0 \leq t \leq 1, \quad 1 \leq k \leq p.$$

For $j=0$, the inner sum is the solo term $x^{p-k} h_1 \dots h_k$.

We now introduce the k -transposes, in order to arrange that the differentials h_1, \dots, h_k occur at the extreme right of each term. By (10),

$$d_x^k [\psi(tx) x^p] = \sum_{0 \leq j \leq k} \alpha_{k,j} t^j \psi_k^j(tx) x^{p-k+j} h_1 \dots h_k, \quad x \in \Omega,$$

where

$$(15) \quad \alpha_{k,j} = \frac{p! k!}{(p-k+j)! j! (k-j)!}, \quad 0 \leq j \leq k \leq p.$$

Hence

$$D_x^k[\psi(tx)x^p] = \sum_{0 \leq j \leq k} \alpha_{k,j} t^j \psi_k^j(tx) x^{p-k+j}, \quad 0 \leq k \leq p.$$

Thus, from (13),

$$(16) \quad r^k(x) = \int_0^1 \frac{(1-t)^{p-1}}{(p-1)!} \sum_{0 \leq j \leq k} \alpha_{k,j} t^j \psi_k^j(tx) x^{p-k+j} dt, \quad x \in \Omega, \quad k \leq p,$$

if $\psi \in C^k$.

6. The deviations of r^k . It is convenient to define r_k , the pseudo k -th derivative of r , and s_k , the deviation of r^k from r_k , as follows:

$$(17) \quad \begin{cases} r_p = \psi, \\ r_k(x) = \int_0^1 \frac{(1-t)^{p-k-1}}{(p-k-1)!} \psi(tx) x^{p-k} dt, \quad x \in \Omega, \quad k < p; \end{cases}$$

$$(18) \quad s_k(x) = r^k(x) - r_k(x), \quad x \in \Omega, \quad k \leq p.$$

In the one-dimensional case $X=R$, we have $s_k(x)=0, x \in \Omega, k \leq p$. In all cases, $s_k(0)=0, k \leq p$, by Proposition 1. And

$$s_k(x)h_1 \dots h_k = 0, \quad x \in \Omega, \quad 1 \leq k \leq p,$$

whenever the differentials h_1, \dots, h_k are all collinear with x , that is, whenever $h_i = a_i x, a_i \in R, i=1, \dots, k$. As we do not use this fact, we omit its formal proof.

Theorem. If $\psi \in C^k$, then for all $x \in \Omega$,

$$(19) \quad \begin{cases} s_0(x) = 0, \\ s_k(x) = \int_0^1 \frac{(1-t)^{p-1}}{(p-1)!} \sum_{1 \leq j \leq k} \alpha_{k,j} t^j [\psi_k^j(tx) - \psi^j(tx)] x^{p-k+j} dt, \quad 1 \leq k \leq p. \end{cases}$$

Furthermore, one may replace ψ in (19) by $\psi - \varphi$, where φ is the p -th derivative of any C^{p+k} -map on Ω to Y .

Proof. That $s_0=0$ is immediate, by (17) and (4).

Suppose that $k \geq 1$. We first show that

$$(20) \quad r_k(x) = \int_0^1 \frac{(1-t)^{p-1}}{(p-1)!} \sum_{0 \leq j \leq k} \alpha_{k,j} t^j \psi^j(tx) x^{p-k+j} dt, \quad x \in \Omega, \quad k \leq p,$$

an identity similar to (16).

One may transform the right member by parts, proceeding from $\psi^k(tx)x dt = d_t \psi^{k-1}(tx)$ to $\psi(tx)$. The boundary terms will in fact vanish and the integrations terminate in (20). Instead of dealing with the sequence of integrations and the boundary terms, however, one may argue as follows.

The relation (20) holds for all ψ which are p -th derivatives, by (5), (16), and (17), since in this case $\psi^j(x) \in S_{p+j}(X, Y)$, $x \in \Omega$. It follows that (20) holds for all ψ , whether p -th derivatives or not, as we now show.

Denote the difference of the two members of (20) by $\delta(x)$. In $\delta(x)$ replace the derivatives ψ^j , $j < k$, by their Taylor expansions about the origin with integral remainder in ψ^k . This, by Fubini's theorem, yields an identity:

$$(21) \quad \delta(x) = c_0 \psi(0) x^{p-k} + \dots + c_{k-1} \psi^{k-1}(0) x^{p-1} + \int_0^1 c(t) \psi^k(tx) x^p dt, \quad x \in \Omega,$$

for all $\psi: \Omega \rightarrow S_p(X, Y)$, where $c_i \in \mathbb{R}$, $i < k$, and $c: [0, 1] \rightarrow \mathbb{R}$ is continuous.

By suitable choices of ψ which are p -th derivatives, we will deduce that $c_i = 0$, $i < k$, and $c = 0$. Thus, take $\psi(x) = a_i x^i$, $x \in \Omega$, where $i = 0, \dots, k-1$, and $0 \neq a_i \in S_{p+i}(X, Y)$. Since ψ is a p -th derivative, (21) reduces to

$$\delta(x) = 0 = c_i a_i x^{p-k+i} i!, \quad x \in \Omega.$$

Hence $c_i = 0$, $i < k$, and the right member of (21) reduces to its integral term.

It remains to show that $c = 0$. Take $0 \neq x_0 \in \Omega$. Put $\tilde{R} = \text{span } x_0$ and $Q =$ a complement of \tilde{R} in X . Thus $X = \tilde{R} + Q$ and \tilde{R} is one-dimensional. Denote the projection of x on \tilde{R} by x' , $x \in X$.

Take $0 \neq a \in Y$. Extend the function c , by putting $c(u) = c(1)$, $u \geq 1$, and $c(u) = c(0)$, $u \leq 0$. Define the map $\theta: \Omega \rightarrow Y$ by the relation

$$\theta(x) = \theta(x') = \int_0^1 \frac{(1-t)^{p+k-1}}{(p+k-1)!} c(tx'/x_0) x'^{p+k} dt a, \quad x \in \Omega.$$

Here we consider elements of \tilde{R} as scalars. Thus tx'/x_0 is a scalar and $\theta(x)$ is a scalar multiple of a , hence an element of Y .

Now

$$\theta^{p+k}(x) h_1 \dots h_{p+k} = c(x'/x_0) a h'_1 \dots h'_{p+k}, \quad h_1, \dots, h_{p+k} \in X.$$

This is immediate, by partial differentiation relative to the decomposition $X = \tilde{R} + Q$. For, from the definition of θ , the pure partials of θ relative to \tilde{R} are obtainable from one-dimensional theory (Proposition 1 applied to maps on \tilde{R}), in particular

$$D_{\tilde{R}}^{p+k} \theta(x) = c(x'/x_0) a,$$

and all other $(p+k)$ -th partials vanish.

In (21), put $\psi = \theta^p$, a p -th derivative. Then $\delta(x) = 0$, $x \in \Omega$. Put $x = x_0 \in \tilde{R}$. Then $x' = x'_0 = x_0$,

$$\theta^{p+k}(tx_0) x_0^p h_1 \dots h_k = \psi^k(tx_0) x_0^p h_1 \dots h_k = c(t) a x_0^p h'_1 \dots h'_k; \quad h_1, \dots, h_k \in X,$$

and

$$\delta(x_0)h_1 \dots h_k = 0 = \int_0^1 c(t)c'(t)ax_0^p h'_1 \dots h'_k dt.$$

Hence

$$\int_0^1 [c(t)]^2 dt = 0,$$

and $c=0$. Thus (20) is established.

Now (18) follows from (16) and (20), since the term for $j=0$, eliminated from the summation in (19), vanishes.

That one may replace ψ by $\psi-\varphi$, where φ is a p -th derivative, follows from the fact that $\varphi_k^j = \varphi^j$, $0 \leq j \leq k$, since φ^j is a $(p+j)$ -th derivative and therefore symmetric.

For $p=1$, the relation (18) agrees with the formula for $\text{rot } \varphi A$ in [3, p. 144].

7. Comparison of f and g . In this section, we assume that $\psi, \varphi \in C^j$ whenever ψ^j, φ^j appear in a statement.

By (1) and (3),

$$g^k(x) - f^k(x) = b_k - a_k + \dots + \frac{b_{p-1} - a_{p-1}}{(p-k-1)!} x^{p-k-1} + f^k(x) - \int_0^1 \frac{(1-t)^{p-k-1}}{(p-k-1)!} \varphi(tx) x^{p-k} dt, \quad x \in \Omega, \quad k < p.$$

Hence, by (18), (17), and (2),

$$(22) \quad g^k(x) - f^k(x) = b_k - a_k + \dots + \frac{b_{p-1} - a_{p-1}}{(p-k-1)!} x^{p-k-1} + \int_0^1 \frac{(1-t)^{p-k-1}}{(p-k-1)!} [\psi(tx) - \varphi(tx)] x^{p-k} dt + s_k(x), \quad x \in \Omega, \quad k < p,$$

and

$$(23) \quad g^p(x) - f^p(x) = \psi(x) - \varphi(x) + s_p(x), \quad x \in \Omega.$$

These differences are readily appraised. In the first place, writing $\|z, x\|$ for $\sup_{t \in [0,1]} \|z(tx)\|$, we have

$$(*) \quad \|s_k(x)\| \leq \sum_{1 \leq j \leq k} \frac{2p! k!}{(p+j)! (p-k+j)! (k-j)!} \|(\psi^j - \varphi^j), x\| \|x\|^{p-k+j},$$

$$x \in \Omega, \quad 1 \leq k \leq p.$$

This inequality follows from (19) with ψ replaced by $\psi-\varphi$, and from (12), (15), and the fact that

$$\int_0^1 \frac{(1-t)^{p-1}}{(p-1)!} t^j dt = \frac{j!}{(p+j)!}.$$

And, by (22), (23),

$$\begin{aligned}
 (*) \quad \|g^k(x) - f^k(x)\| &\leq \|b_k - a_k\| + \dots + \frac{\|b_{p-1} - a_{p-1}\|}{(p-k-1)!} \|x\|^{p-k-1} \\
 &+ \frac{\|(\psi - \varphi), x\|}{(p-k)!} \|x\|^{p-k} + \|s_k(x)\|, \quad x \in \Omega, \quad k < p,
 \end{aligned}$$

and

$$(*) \quad \|g^p(x) - f^p(x)\| \leq \|\psi(x) - \varphi(x)\| + \|s_p(x)\|, \quad x \in \Omega.$$

Added in proof. The results of the present paper hold for complex as well as real Banach spaces X and Y .

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