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THE SUBLATTICE OF AN ORTHOGONAL PAIR IN A MODULAR LATTICE

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Introduction

We will be concerned with a modular lattice $\mathscr{L}$ together with an antitone mapping $\perp : \mathscr{L} \to \mathscr{L}$ such that

(1) $$x \leqq x^{\perp\perp} \quad \text{for all} \quad x \in \mathscr{L}.$$

The following rules are easily verified:

(2) $$a^{\perp\perp\perp} = a^{\perp},$$

(3) $$x \leqq y \Rightarrow x^{\perp\perp} \leqq y^{\perp\perp},$$

(4) $$(x \vee y)^{\perp} = x^{\perp} \wedge y^{\perp}.$$

If $x = x^{\perp\perp}$ we call x *closed*; if $x \leqq y^{\perp}$ we write $x \perp y$.

Under the assumption that $f \perp g$ we shall construct the free modular lattice $\mathscr{V}(f, g)$ generated by $\mathscr{V}(f) \cup \mathscr{V}(g)$, where $\mathscr{V}(f)$ is the orthostable lattice generated by $f \in \mathscr{L}$. $\mathscr{V}(f, g)$ is a distributive lattice. We will also give some conditions ensuring that $\mathscr{V}(f, g)$ or a slight modification of $\mathscr{V}(f, g)$ is orthostable. Certain special cases are studied separately because of their importance in geometry.

The value of lattice theoretical computations such as given here rests on the fact that they yield — in conjunction with certain general theorems proved in [3] and [5] — strong results on the classification of subspaces in quadratic spaces, normal bases, decomposition theorems. The role of the lattice theoretic part has been described in detail in Section 3 of [5]. Further applications of this method are given in [4]. Cf. also Remark 5 (iii) at the end.

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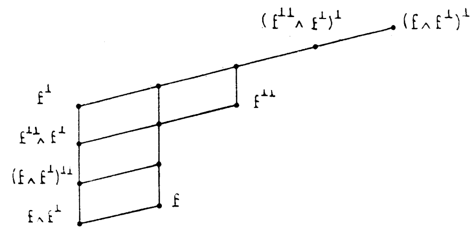
1. The lattice: general case

The $\perp$-stable lattice $\mathscr{V}(f)$ generated by an element $f \in \mathscr{L}$ (modular with $\perp$) is given by the following diagram



Let $\mathscr{I}(f)$ be the ideal generated in $\mathscr{V}(f)$ by $f^{\perp\perp}$; and let the filter generated by $f^{\perp}$ be denoted by $\mathscr{F}(f)$. Note that $\mathscr{V}(f) = \mathscr{I}(f) \cup \mathscr{F}(f)$ and that $\mathscr{F}(f)$ is a chain. Moreover, $\mathscr{V}(f)$ is distributive.

Considering a second element $g \in \mathscr{L}$ we prove:

Lemma 1. *Assume that* $f \perp g$. *Then the lattice* $\mathscr{V}(f, g)$ *generated in* $\mathscr{L}$ *by* $\mathscr{V}(f) \cup \mathscr{V}(g)$ *is distributive.*

Proof. By Theorem 6 of [6] and symmetry it suffices to verify that $(b \vee b') \wedge c = (b \wedge c) \vee (b' \wedge c)$ for all $b, b' \in \mathscr{V}(f)$ and all $c \in \mathscr{V}(g)$. Since $f \perp g$ we have $y \geqq f^{\perp} \geqq g^{\perp\perp} \geqq x$ for all $x \in \mathscr{I}(g)$, $y \in \mathscr{F}(f)$. This and the symmetric fact is expressed by

$$\mathscr{I}(f) \leqq \mathscr{F}(g), \quad \mathscr{I}(g) \leqq \mathscr{F}(f). \tag{5}$$

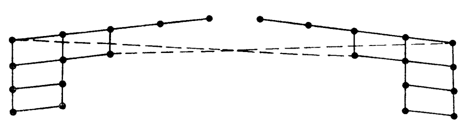
The only elements in $\mathscr{V}(f)$ which are not join-irreducible are $z_1 = f \vee (f \wedge f^{\perp})^{\perp\perp}$, $z_2 = f \vee (f^{\perp} \wedge f^{\perp\perp})$, $z_3 = f \vee f^{\perp}$, $z_4 = f^{\perp\perp} \vee f^{\perp}$. For $i = 3, 4$ we obtain the distributivity of $z_i \wedge y$ using (5) and modularity. The same works for $i = 1, 2$ and $y \in \mathscr{F}(g)$. Finally (5) implies that $y = f^{\perp} \wedge y$ for $y \in \mathscr{I}(g)$; therefore

$$z_1 \wedge y = [f \vee (f \wedge f^{\perp})^{\perp\perp}] \wedge f^{\perp} \wedge y = (f \wedge f^{\perp})^{\perp\perp} \wedge y \leqq (f \wedge y) \vee [(f \wedge f^{\perp})^{\perp\perp} \wedge y]$$

$$z_2 \wedge y = [f \vee (f^{\perp} \wedge f^{\perp\perp})] \wedge f^{\perp} \wedge y = f^{\perp} \wedge f^{\perp\perp} \wedge y \leqq (f \wedge y) \vee [(f^{\perp} \wedge f^{\perp\perp}) \wedge y].$$

This takes care of the remaining cases, bearing in mind the distributive inequality.

Remark 1. Let $\mathscr{D}$ be given by the following diagram



where the broken lines indicate a relation $\leqq$. The proof given above shows that the free modular lattice generated by $\mathscr{D}$ is distributive.

A situation involving $\mathscr{D}$ appears again in the construction of the $\perp$-stable lattice generated by two elements $f, g \in \mathscr{L}, f \perp f, g \perp g$. Here

$$\mathscr{I}_1 = \{f \wedge g^{\perp}, f, (f \wedge g^{\perp})^{\perp\perp}, f \vee (f \wedge g^{\perp})^{\perp\perp}, f^{\perp\perp} \wedge g^{\perp}, f \vee (f^{\perp\perp} \wedge g^{\perp}), f^{\perp\perp}\}$$

and

$$\mathscr{F}_1 = \{g^{\perp}, f \vee g^{\perp}, f^{\perp\perp} \vee g^{\perp}, (f^{\perp} \wedge g^{\perp\perp})^{\perp}, (f^{\perp} \wedge g)^{\perp}\}$$

take the place of $\mathscr{I}(f)$ and $\mathscr{F}(f)$ respectively. If $\mathscr{I}_2$ and $\mathscr{F}_2$ denote the analogous sets with f and g interchanged, then clearly the orthostable lattice generated by f and g must contain the sublattice generated by $\mathscr{I}_1 \cup \mathscr{I}_2 \cup \mathscr{F}_1 \cup \mathscr{F}_2$ which by the proof of Lemma 1 is distributive.

In what follows we construct the free modular lattice generated by $\mathscr{D}$. We will however do it in the setup of $\mathscr{V}(f) \cup \mathscr{V}(g)$ and leave it to the reader to verify that the result has general validity.

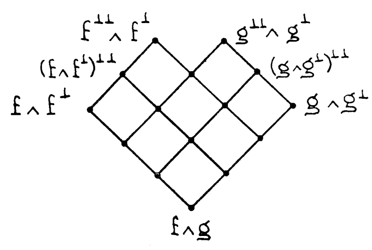
Thanks to the distributivity of $\mathscr{V}(f, g)$ the lattice $\mathscr{V}_2$ generated by $\mathscr{I}(f) \cup \mathscr{I}(g)$ is the join-closure of

$$\mathscr{I}(f) \cup \mathscr{I}(g) \cup \{x \wedge y \mid x \in \mathscr{I}(f),\ y \in \mathscr{I}(g)\}.$$

As $f \perp g$ we have

$$\{x \wedge y \mid x \in \mathscr{I}(f),\ y \in \mathscr{I}(g)\} = \{x \wedge y \mid x \in \mathscr{I}_0(f),\ y \in \mathscr{I}_0(g)\},$$

where $\mathscr{I}_0(f) = \{f \wedge f^{\perp}, (f \wedge f^{\perp})^{\perp\perp}, f^{\perp} \wedge f^{\perp\perp}\}$ and similarly for $\mathscr{I}_0(g)$ (compare the proof of Lemma 1). Therefore, we begin by forming the free modular lattice M generated by the two chains $\mathscr{I}_0(f)$, $\mathscr{I}_0(g)$. M has $8!(4!)^{-2}-2=68$ elements ([1] p. 66) and consists of all joins of elements out of the following diagram



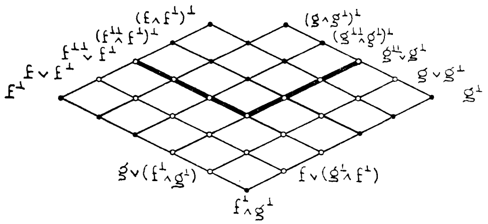
(intersections of lines represent meets of elements).

The next step is to form joins of elements in M with $f, g, f \vee g, f^{\perp\perp}, g^{\perp\perp}, f \vee g^{\perp\perp}, f^{\perp\perp} \vee g, f^{\perp\perp} \vee g^{\perp\perp}$. This produces all elements of $\mathscr{V}_2$ since any element $x \vee y \vee m$, where $x \in \mathscr{I}(f)$, $y \in \mathscr{I}(g)$, $m \in M$ is of the form $x_0 \vee y_0 \vee m_0$ with $x_0 \in \{f, f^{\perp\perp}\}$, $y_0 \in \{g, g^{\perp\perp}\}$, $m_0 \in M$. In an expression like $f \vee x \vee y \vee Vx_i \wedge y_i$, where $x, x_i \in \mathscr{I}_0(f)$, $y, y_i \in \mathscr{I}_0(g)$ one can dispose of terms $x_i \wedge y_i \leqq f$. Thus we may assume that $x, x_i \neq f \wedge f^{\perp}$ and hence $f \vee M = f \vee M_1$, where M_1 is the free modular lattice generated by the 2 chains $\{(f \wedge f^{\perp})^{\perp\perp}, f^{\perp} \wedge f^{\perp\perp}\}$ and $\mathscr{I}_0(g)$. We obtain $7!(3!4!)^{-1}-2=33$ elements or 34 elements if we include f. The same kind of reasoning leads to the following enumeration

M: 68 elements; $f \vee M$: 34 elements (including f); $g \vee M$: 34 elements (including g); $f \vee g \vee M$: 19 elements (including $f \vee g$); $f^{\perp\perp} \vee M$: 4 elements (including $f^{\perp\perp}$); $g^{\perp\perp} \vee M$: 4 elements (including $g^{\perp\perp}$); $f \vee g^{\perp\perp} \vee M$: 3 elements (including $f \vee g^{\perp\perp}$); $g \vee f^{\perp\perp} \vee M$: 3 elements (including $g \vee f^{\perp\perp}$); $f^{\perp\perp} \vee g^{\perp\perp}$: 1 element.

Altogether the free modular lattice generated by $\mathscr{I}(f) \cup \mathscr{I}(g)$ has 170 elements.

The lattice $\mathscr{V}_1$ generated by the 2 chains $\mathscr{F}(f)$ and $\mathscr{F}(g)$ is the $\vee$-closure of the elements which are depicted in the following diagram (intersections of lines represent meets):



Observe that the elements marked by circles are of the form $f \vee x$, $f^{\perp\perp} \vee x$, $g \vee y$, $g^{\perp\perp} \vee y$ for $x \in \{f^{\perp}, f^{\perp} \wedge a \mid a \in \mathscr{F}(g)\}$, $y \in \{g^{\perp}, g^{\perp} \wedge b \mid b \in \mathscr{F}(f)\}$. Moreover, for

$$r \in \{f^{\perp}, f^{\perp} \wedge (g \wedge g^{\perp})^{\perp}, f^{\perp} \wedge (g^{\perp\perp} \wedge g^{\perp})^{\perp}\},$$
$$s \in \{g^{\perp}, g^{\perp} \wedge (f \wedge f^{\perp})^{\perp}, g^{\perp} \wedge (f^{\perp\perp} \wedge f^{\perp})^{\perp}\}$$

we have

$$r \vee s = (r \vee f^{\perp\perp}) \vee (s \vee g^{\perp\perp}).$$

As a consequence $\mathscr{V}_1$ is the lattice generated by the two chains

$$\{f^{\perp\perp} \vee f^{\perp}, (f^{\perp\perp} \wedge f^{\perp})^{\perp}, (f \wedge f^{\perp})^{\perp}\}$$

and

$$\{g^{\perp\perp} \vee g^{\perp}, (g^{\perp\perp} \wedge g^{\perp})^{\perp}, (g \wedge g^{\perp})^{\perp}\}$$

together with the 20 elements below the solid line in the diagram. The total number of elements in $\mathscr{V}_1$ is therefore at most $68+20=88$.

Finally we prove that $\mathscr{V}_1 \cup \mathscr{V}_2$ is a lattice by showing that $x \vee y$ and $x \wedge y$ are in $\mathscr{V}_1 \cup \mathscr{V}_2$ whenever $x \in \mathscr{V}_1$, $y \in \mathscr{V}_2$. As for the joins it suffices to show that $x \vee y \in \mathscr{V}_1$ for $x \in \mathscr{V}_1$ and y join-irreducible in $\mathscr{V}_2$, $y \nleq f^{\perp} \wedge g^{\perp}$. Since $f^{\perp} \wedge g^{\perp} \wedge (f^{\perp\perp} \vee g^{\perp\perp}) = (f^{\perp} \wedge f^{\perp\perp}) \vee (g^{\perp} \wedge g^{\perp\perp})$ the only such y are f, $f^{\perp\perp}$, g, $g^{\perp\perp}$; for these, however, the claim is obvious. Owing to distributivity we will now consider only those meets $x \wedge y$ for which $x \in \mathscr{V}_1$, $y \in \mathscr{V}_2$ are join-irreducible with $x \nleq f^{\perp\perp} \vee g^{\perp\perp} \vee (f^{\perp} \wedge g^{\perp})$ and $y \nleq f^{\perp} \wedge g^{\perp}$. This means that

$$x \in \{f^{\perp}, f^{\perp} \wedge (g \wedge g^{\perp})^{\perp}, f^{\perp} \wedge (g^{\perp} \wedge g^{\perp\perp})^{\perp}, f^{\perp} \wedge g^{\perp}, g^{\perp} \wedge (f^{\perp} \wedge f^{\perp\perp})^{\perp},$$
$$g^{\perp} \wedge (f \wedge f^{\perp})^{\perp}, g^{\perp}\}$$

and

$$y \in \{f, f^{\perp\perp}, g, g^{\perp\perp}\}.$$

These verifications are easy.

We summarize:

Theorem 1. *The free modular lattice generated by $\mathscr{D}$ has 258 elements.*

2. The lattice: some special cases

We recall that $f \perp g$ is assumed throughout. From this it follows that $f^{\perp\perp} \wedge g^{\perp\perp} = f^{\perp} \wedge f^{\perp\perp} \wedge g^{\perp} \wedge g^{\perp\perp}$. The following condition requires that $f^{\perp\perp} \wedge g^{\perp\perp}$ is even smaller:

(6) $$f^{\perp\perp} \wedge g^{\perp\perp} = (f \wedge f^{\perp})^{\perp\perp} \wedge (g \wedge g^{\perp})^{\perp\perp}.$$

Under this assumption

(7) $$\mathscr{V}_2 = \mathscr{I}(f) \cup \mathscr{I}(g) \cup \{x \vee y \mid x \in \mathscr{I}(f),\ y \in \mathscr{I}(g)\} \cup \mathscr{W},$$

where $\mathscr{W}$ is the set containing the following 17 elements:

1 $f \wedge g$

2 $(f \wedge g^{\perp\perp}) \vee (g \wedge f^{\perp\perp})$

3 $f^{\perp\perp} \wedge g^{\perp\perp}$

4 $(f \wedge f^{\perp}) \vee (g \wedge g^{\perp}) \vee (f^{\perp\perp} \wedge g^{\perp\perp})$

5 $f \vee g \vee f^{\perp\perp} \wedge g^{\perp\perp}$

6 $f \wedge g^{\perp\perp}$

7 $g \wedge f^{\perp\perp}$

8 $(f \wedge f^{\perp}) \vee g \wedge f^{\perp\perp}$

9 $(g \wedge g^{\perp}) \vee f \wedge g^{\perp\perp}$

10 $f \vee (g \wedge f^{\perp\perp})$

11 $g \vee (f \wedge g^{\perp\perp})$

12 $(f \wedge f^{\perp}) \vee (f^{\perp\perp} \wedge g^{\perp\perp})$

13 $(g \wedge g^{\perp}) \vee (f^{\perp\perp} \wedge g^{\perp\perp})$

14 $f \vee (f^{\perp\perp} \wedge g^{\perp\perp})$

15 $g \vee (f^{\perp\perp} \wedge g^{\perp\perp})$

16 $f \vee (g \wedge g^{\perp}) \vee (f^{\perp\perp} \wedge g^{\perp\perp})$

17 $g \vee (f \wedge f^{\perp}) \vee (f^{\perp\perp} \wedge g^{\perp\perp})$.

To prove (7) note that by distributivity $\mathscr{V}_2$ consists of joins $u_1 \vee u_2 \vee u_3 \vee \ldots \vee u_r$, where (a) $u_i \in \mathscr{I}(f) \cup \mathscr{I}(g)$ or (b) u_i is a meet $x \wedge y$ of join-irreducible elements $x \in \mathscr{I}(f)$, $y \in \mathscr{I}(g)$. From (6) and $f \perp g$ we see that the joins of elements of type (b) form the set

$$\mathscr{V} = \{f^{\perp\perp} \wedge g^{\perp\perp},\ (f^{\perp\perp} \wedge g) \vee (g^{\perp\perp} \wedge f),\ f^{\perp\perp} \wedge g,\ g^{\perp\perp} \wedge f,\ f \wedge g\}.$$

Under the assumption (6) $\mathscr{V}_2$ therefore has at most $63+17=80$ elements.

Condition (6), which does not have any bearing on $\mathscr{V}_1$, can be obtained from

(8) $$f^{\perp} \vee g^{\perp} = (f \wedge f^{\perp})^{\perp} \vee (g \wedge g^{\perp})^{\perp}$$

by applying $\perp$. Equation (8) has very strong consequences:

Lemma 2. *Assume that* $f \perp g$ *and that* (8) *holds. Then*

$$(y_1 \wedge y_2) \vee (y_1' \wedge y_2') = (y_1 \vee y_1') \wedge (y_2 \vee y_2') \tag{9}$$

for all $y_1, y_1' \in \mathscr{F}(f)$ *and all* $y_2, y_2' \in \mathscr{F}(g)$. *In particular* $\mathscr{F}(f) \cup \mathscr{F}(g) \cup \{x \wedge y | x \in \mathscr{F}(f),\ y \in \mathscr{F}(g)\} \cup \{f^\perp \vee g^\perp\}$ *is* $\vee$*-closed and hence a sublattice of* $\mathscr{L}$, *i.e. it is* $\mathscr{V}_2$; $\operatorname{card} \mathscr{V}_2 \leqq 36$.

Proof. It is clear that $\leqq$ holds in (9). To obtain the converse inclusion we consider the case $y_1 \leqq y_1'$ and $y_2' \leqq y_2$ (the other cases being trivial). The right hand side of (9) then becomes

$$y_1' \wedge y_2 = y_1' \wedge y_2 \wedge [(f \wedge f^\perp)^\perp \vee (g \wedge g^\perp)^\perp] = y_1' \wedge y_2 \wedge (f^\perp \vee g^\perp)$$
$$= (f^\perp \wedge y_2) \vee (g^\perp \wedge y_1') \leqq (y_1 \wedge y_2) \vee (y_1' \wedge y_2').$$

This proves the lemma.

Theorem 2. *Let* $\mathscr{L}$ *be a modular lattice with a Galois autoconnection* $\perp$. *If* $f \perp g$ *and* (6) *holds then* $\operatorname{card} \mathscr{V}(f, g) \leqq 168$ *and* $\mathscr{V}(f, g) = \mathscr{V}_1 \cup \mathscr{V}_2$, *where* $\mathscr{V}_2$ *is given by* (7) *and* $\mathscr{V}_1$ *is generated by two chains. If instead of* (6) *one assumes* (8), *then* $\mathscr{V}_2$ *is as before,* $\mathscr{V}_1$ *is the product of* 2 *chains and* $\operatorname{card} \mathscr{V}(f, g) \leqq 116$.

Remark 2. The same considerations are valid in the case $f \perp f$, $g \perp g$ provided (6) is replaced by

$$f^{\perp\perp} \wedge g^{\perp\perp} = (f \wedge g^\perp)^{\perp\perp} \wedge (g \wedge f^\perp)^{\perp\perp} \tag{10}$$

and (8) is replaced by

$$f^\perp \vee g^\perp = (f \wedge g^\perp)^\perp \vee (g \wedge f^\perp)^\perp. \tag{11}$$

It is easily seen that

$$f^\perp \vee g^\perp = (f \wedge g)^\perp \tag{12}$$

implies (8) if $f \perp g$ and also implies (11) if $f \perp f$, $g \perp g$.

3. Orthostability

We want to be sure that $x^\perp \in \mathscr{V}(f, g)$ for all $x \in \mathscr{V}(f, g)$. Since $(a \vee b)^\perp = a^\perp \wedge b^\perp$ and $\mathscr{V}(f, g)$ is a lattice we need only find the orthogonals of join-irreducible elements. If $x \in \mathscr{V}_1$ is join-irreducible, then $x \in \mathscr{F}(f)$, or $x \in \mathscr{F}(g)$, or $x = u^\perp \wedge v^\perp$ for some $u \in \mathscr{I}(f)$, $v \in \mathscr{I}(g)$. In the latter case $x^\perp = (u \vee v)^{\perp\perp}$ and the following condition must be satisfied for $x^\perp$ to belong to $\mathscr{V}(f, g)$

$$(a \vee b)^{\perp\perp} \in \mathscr{V}(f, g) \quad \text{for all} \quad a \in \mathscr{I}(f),\ b \in \mathscr{I}(g). \tag{13}$$

Another problem appears when we check orthogonals of elements in $\mathscr{V}_2$: $f \wedge g = (f \wedge f^\perp) \wedge (g \wedge g^\perp) \leqq (f \wedge f^\perp)^{\perp\perp} \wedge (g \wedge g^\perp)^{\perp\perp}$ and therefore $(f \wedge g)^\perp \geqq [(f \wedge f^\perp)^\perp \vee (g \wedge g^\perp)^\perp]^{\perp\perp}$. Since $(f \wedge f^\perp)^\perp \vee (g \wedge g^\perp)^\perp$ is the largest element of

$\mathscr{V}(f, g)$, this lattice will have to be extended at the top end unless $(f \wedge f^{\perp})^{\perp} \vee (g \wedge g^{\perp})^{\perp}$ is closed and $(f \wedge g)^{\perp\perp} = (f \wedge f^{\perp})^{\perp\perp} \wedge (g \wedge g^{\perp})^{\perp\perp}$. Postponing the problem of such an extension at the moment we consider only $x \in \mathscr{V}_2$ such that $x > (f \wedge f^{\perp})^{\perp\perp} \wedge (g \wedge g^{\perp})^{\perp\perp}$, or $x \geqq f \wedge f^{\perp}$, or $x \geqq g \wedge g^{\perp}$. The join-irreducible ones among them are elements of $\mathscr{I}(f)$ or $\mathscr{I}(g)$ or meets $x = a^{\perp\perp} \wedge b^{\perp\perp}$, where $a \in \mathscr{I}(f)$, $b \in \mathscr{I}(g)$. In the last case $x^{\perp} = (a^{\perp} \vee b^{\perp})^{\perp\perp}$. Thus, a further condition must be satisfied:

(14) For $a \in \mathscr{I}(f)$, $b \in \mathscr{I}(g)$ such that $a^{\perp} \vee b^{\perp} < (f \wedge f^{\perp})^{\perp} \vee (g \wedge g^{\perp})^{\perp}$ the closure of $a^{\perp} \vee b^{\perp}$ also belongs to $\mathscr{V}(f, g)$.

The only join-irreducible elements of $\mathscr{V}_2$ not yet considered are

$$f \wedge g^{\perp\perp},\ f \wedge (g \wedge g^{\perp})^{\perp\perp},\ f \wedge g,\ g \wedge (f \wedge f^{\perp})^{\perp\perp},\ g \wedge f^{\perp\perp}.$$

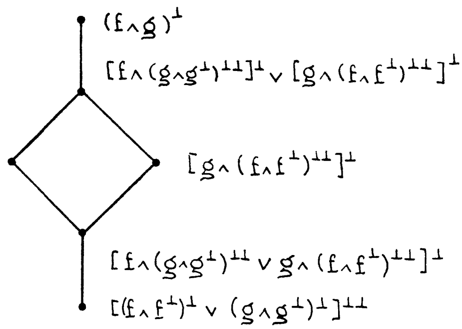
To be able to deal with $f \wedge g^{\perp\perp}$ and $g \wedge f^{\perp\perp}$ we must require that

(15) $(f \wedge g^{\perp\perp})^{\perp\perp}$ and $(g \wedge f^{\perp\perp})^{\perp\perp}$ belong to $\mathscr{V}(f, g)$ and are comparable to $(f \wedge f^{\perp})^{\perp\perp} \wedge (g \wedge g^{\perp})^{\perp\perp}$.

The elements below $(f \wedge f^{\perp})^{\perp\perp} \wedge (g \wedge g^{\perp})^{\perp\perp}$ make it necessary to extend $\mathscr{V}(f, g)$ at the top end; again we must require that

(16) the closures of elements $\leqq (f \wedge f^{\perp})^{\perp\perp} \wedge (g \wedge g^{\perp})^{\perp\perp}$ belong to $\mathscr{V}(f, g)$.

Assuming (16) one can add up to 6 elements at the top end of $\mathscr{V}(f, g)$; the maximum number of 6 is needed if $(f \wedge f^{\perp})^{\perp} \vee (g \wedge g^{\perp})^{\perp}$ is not closed, and all four elements below $(f \wedge f^{\perp})^{\perp\perp} \wedge (g \wedge g^{\perp})^{\perp\perp}$ are closed:



We summarize the result as a theorem:

Theorem 3. *$\mathscr{V}(f, g)$ or a small extension of $\mathscr{V}(f, g)$ is orthostable provided the conditions* (13), (14), (15), *and* (16) *hold. The maximum number of elements in the orthostable lattice is* 264.

The conditions in Theorem 3 are satisfied in the following situation:

(16) closures of elements $\leqq (f \wedge f^{\perp})^{\perp\perp} \wedge (g \wedge g^{\perp})^{\perp\perp}$ belong to $\mathscr{V}(f, g)$.

(17) $(f \wedge g^{\perp\perp})^{\perp\perp} = (f \wedge f^{\perp})^{\perp\perp} \wedge g^{\perp\perp}$; $(g \wedge f^{\perp\perp})^{\perp\perp} = f^{\perp\perp} \wedge (g \wedge g^{\perp})^{\perp\perp}$;

(18) $[(f \wedge f^{\perp})^{\perp\perp} \vee (g \wedge g^{\perp})^{\perp\perp}]^{\perp\perp} = (f \wedge f^{\perp})^{\perp\perp} \vee (g \wedge g^{\perp})^{\perp\perp} \vee (f^{\perp\perp} \wedge g^{\perp\perp})$;

(19) the following are closed:

$$f^{\perp\perp} \vee g^{\perp\perp},\ f^{\perp\perp} \vee (g^{\perp} \wedge g^{\perp\perp}),\ f^{\perp\perp} \vee (g \wedge g^{\perp})^{\perp\perp},\ g^{\perp\perp} \vee (f^{\perp} \wedge f^{\perp\perp}),$$
$$g^{\perp\perp} \vee (f \wedge f^{\perp})^{\perp\perp};$$

(20) the following are closed:

$$(f^{\perp} \wedge f^{\perp\perp})^{\perp} \vee (g \wedge g^{\perp})^{\perp},\ (f \wedge f^{\perp})^{\perp} \vee (g^{\perp} \wedge g^{\perp\perp})^{\perp},\ (f^{\perp} \wedge f^{\perp\perp})^{\perp} \vee (g^{\perp} \wedge g^{\perp\perp})^{\perp}.$$

Remark 3. Note that the right hand side in (18) is closed provided (19) holds, since it is the meet of $f^{\perp\perp} \vee (g \wedge g^{\perp})^{\perp\perp}$ and $g^{\perp\perp} \vee (f \wedge f^{\perp})^{\perp\perp}$. Given (19), the joins $a \vee b$ of closed elements $a \in \mathscr{I}(f)$, $b \in \mathscr{I}(g)$ are closed, being meets of elements listed in (19).

To conclude this section we return to the special cases treated in Section 2 and consider the question of orthostability. As before, $\mathscr{V}_1^{\perp}$ is taken care of by assuming

(13) $\quad \mathscr{V}(f, g)$ contains $(a \vee b)^{\perp\perp}$ for all $a \in \mathscr{I}(f)$, $b \in \mathscr{I}(g)$.

The set of join-irreducible elements of $\mathscr{V}_2$ which are not contained in $\mathscr{I}(f) \cup \mathscr{I}(g)$ is the set $\mathscr{V}$ as defined at the beginning of Section 2. If $[\mathscr{V}^{\perp}]$ is the lattice generated by $\mathscr{V}^{\perp}$, then $\mathscr{V}_0 = \mathscr{V}(f, g) \cup [\mathscr{V}^{\perp}]$ is a lattice because $\mathscr{V}(f, g) \leqq (f^{\perp\perp} \wedge g^{\perp\perp})^{\perp} = (f^{\perp} \vee g^{\perp})^{\perp\perp}$ by (6). If $\mathscr{V}_0$ is to be $\perp$-stable we must have the elements of $[\mathscr{V}^{\perp}]^{\perp}$ in $\mathscr{V}(f, g)$; this will happen precisely when $(\mathscr{V}^{\perp})^{\perp} \subset \mathscr{V}$. This proves

Lemma 3. *Assume* (6). *Then, with the notation introduced above,* $\mathscr{V}(f, g) \cup [\mathscr{V}^{\perp}]$ *is a lattice. This lattice is orthostable if and only if* $(a \vee b)^{\perp\perp} \in \mathscr{V}(f, g)$ *for all* $a \in \mathscr{I}(f)$, $b \in \mathscr{I}(g)$ *and*

(21) $$(\mathscr{V}^{\perp})^{\perp} \subset \mathscr{V}.$$

We now prove

Lemma 4. *For all* $x \in \mathscr{L}$ *with* $f^{\perp} \leqq x^{\perp} \leqq f^{\perp} \vee g^{\perp}$ *we have*

(22) $$[x^{\perp\perp} \vee (g^{\perp\perp} \wedge f^{\perp\perp})]^{\perp\perp} = (x^{\perp\perp} \vee g^{\perp\perp})^{\perp\perp} \wedge f^{\perp\perp}.$$

Proof. We have $x^{\perp} \wedge (f^{\perp} \vee g^{\perp}) = x^{\perp} \wedge (f^{\perp} \vee g^{\perp})^{\perp\perp}$ since both sides reduce to $x^{\perp}$ by the assumption of the lemma. By modularity the left hand side is equal to $f^{\perp} \vee (x^{\perp} \wedge g^{\perp}) = f^{\perp} \vee (x^{\perp} \wedge g^{\perp})^{\perp\perp}$; the right hand side equals $[x^{\perp\perp} \vee (f^{\perp\perp} \wedge g^{\perp\perp})]^{\perp}$. Taking orthogonals on both sides yields the asserted equality.

Remark 4. Obviously, if $f^{\perp} \vee g^{\perp}$ is assumed closed, then by the above proof (22) holds for all $x \in \mathscr{L}$ with $f^{\perp} \leqq x^{\perp}$.

The following lemma elaborates on the first condition ennunciated in Lemma 3:

Lemma 5. *Assume that* $f \perp g$ *satisfy* (8) *and the closedness condition*

(23) $$f^{\perp\perp} \vee g^{\perp\perp} = (f \vee g)^{\perp\perp}.$$

Then we have

(24) $$x_1^{\perp\perp} \vee x_2^{\perp\perp} = (x_1 \vee x_2)^{\perp\perp}$$

for all $x_1 \in \mathscr{I}(f)$, $x_2 \in \mathscr{I}(g)$.

Proof. By (23)

$$(x_1^{\perp\perp} \vee x_2^{\perp\perp})^{\perp\perp} = (x_1^{\perp\perp} \vee x_2^{\perp\perp})^{\perp\perp} \wedge (f^{\perp\perp} \vee g^{\perp\perp}) \leqq (x_1^{\perp\perp} \vee g^{\perp\perp})^{\perp\perp} \wedge (f^{\perp\perp} \vee g^{\perp\perp}).$$

By distributivity and Lemma 4 therefore $(x_1^{\perp\perp} \vee x_2^{\perp\perp})^{\perp\perp} \leqq (x_1^{\perp\perp} \vee (g^{\perp\perp} \wedge f^{\perp\perp}))^{\perp\perp} \vee g^{\perp\perp} = x_1^{\perp\perp} \vee g^{\perp\perp}$ (the last equality by (6)). By a symmetric argumentation $(x_1^{\perp\perp} \vee x_2^{\perp\perp})^{\perp\perp} \leqq (f^{\perp\perp} \vee x_2^{\perp\perp})$ so that $(x_1^{\perp\perp} \vee x_2^{\perp\perp})^{\perp\perp} \leqq (x_1^{\perp\perp} \vee g^{\perp\perp}) \wedge (f^{\perp\perp} \vee x_2^{\perp\perp}) = x_1^{\perp\perp} \vee x_2^{\perp\perp}$ by again using (6). Obviously, if in this proof $f^{\perp} \vee g^{\perp}$ is assumed closed then by Remark 4 we need not assume (8) in order to quote Lemma 4. In other words, we have also proved the

Lemma 5′. *Assume that* $f \perp g$ *has* $f^{\perp\perp} \vee g^{\perp\perp}$ *and* $f^{\perp} \vee g^{\perp}$ *closed. Then* (24) *holds for all* $x_1, x_2 \in \mathscr{L}$ *with* $f^{\perp\perp} \wedge g^{\perp\perp} \leqq x_1^{\perp\perp} \leqq f^{\perp\perp}$, $f^{\perp\perp} \wedge g^{\perp\perp} \leqq x_2^{\perp\perp} \leqq g^{\perp\perp}$.

Another possibility to obtain (24) is to require (23) and condition

(25) $$f^{\perp\perp} \vee (g \wedge g^{\perp})^{\perp\perp},\ g^{\perp\perp} \vee (f \wedge f^{\perp})^{\perp\perp} \quad \text{are closed;}$$

for, simple calculations show that (23) and (25) imply closedness of all spaces $x_1^{\perp\perp} \vee x_2^{\perp\perp}$ occuring in (24).

In order to satisfy (21) we may require condition

(26) $$f^{\perp\perp} \wedge g^{\perp\perp} = (f \wedge g)^{\perp\perp}$$

— which means that the lattice $\mathscr{V}^{\perp}$ of Lemma 3 reduces to $\{(f^{\perp} \vee g^{\perp})^{\perp\perp}\}$ — or

(27) $$f \wedge g,\ f \wedge g^{\perp\perp},\ g \wedge f^{\perp\perp},\ (f \wedge g^{\perp\perp}) \vee (g \wedge f^{\perp\perp}) \quad \text{are closed,}$$

which means that the elements of $\mathscr{V}$ are closed so that $\perp : \mathscr{V} \to \mathscr{V}^{\perp}$ is a bijection. Notice that (26) implies (6). We summarize:

Theorem 4. *Let* $\mathscr{L}$ *be a modular lattice equipped with a Galois autoconnection* $\perp$. *Assume that* $f, g \in \mathscr{L}$ *satisfy* $f \perp g$. *Let* $\mathscr{V}(f, g)$ *be the sublattice generated by the set* $\mathscr{V}(f) \cup \mathscr{V}(g)$, *where* $\mathscr{V}(f)$, $\mathscr{V}(g)$ *are the* $\perp$*-stable sublattices generated by* f *and* g *respectively. In order that the* $\perp$*-stable lattice* $\mathscr{V}(f, g, \perp)$ *generated by* $\mathscr{V}(f) \cup \mathscr{V}(g)$ *(i.e. the* $\perp$*-stable lattice generated by* $\{f, g\}$*) is finite and distributive either of the following four conditions is sufficient:* (26) & (23) & (25), (8) & (23) & (26), (6) & (23) & (25) & (27), (8) & (23) & (27). *We then have* $\mathscr{V}(f, g, \perp) = \mathscr{V}(f, g) \cup \{(f^{\perp} \vee g^{\perp})^{\perp\perp}\}$ *in the first two cases and*

$$\mathscr{V}(f, g, \perp) = \mathscr{V}(f, g) \cup \{(f^{\perp} \vee g^{\perp})^{\perp\perp}, (f \wedge g^{\perp\perp})^{\perp} \wedge (g \wedge f^{\perp\perp})^{\perp}, (f \wedge g^{\perp\perp})^{\perp}, (g \wedge f^{\perp\perp})^{\perp}, [f \wedge (g \wedge g^{\perp})^{\perp\perp}]^{\perp} \vee [g \wedge (f \wedge f^{\perp})^{\perp\perp}]^{\perp}, (f \wedge g)^{\perp}\}$$

in the last two cases. Upper bounds for the cardinality of $\mathscr{V}(f, g, \perp)$ *in the four cases listed are respectively* 169, 117, 174, 122; *they are attained in the "free" cases.*

Theorem 5. *Assume that* $f \perp g$ *has* $f^{\perp\perp} \vee g^{\perp\perp}$ *and* $f^{\perp} \vee g^{\perp}$ *closed. Then* $(f^{\perp\perp}, g^{\perp\perp})$ *is a modular and dual modular pair in the lattice* $\mathscr{L}_{\perp\perp}$ *of all closed elements of* $\mathscr{L}$. *If in addition* (8) *and* (26) *resp.* (8) *and* (27) *are assumed, then* $\mathscr{V}(f, g, \perp)$ *is distributive and has at most* 116 *resp.* 121 *elements.*

Proof. By Remark 3 $(f^{\perp\perp}, g^{\perp\perp})$ is a modular pair in $\mathscr{L}_{\perp\perp}$; in order to show that it is a dual modular pair we have to prove that $((z\wedge f^{\perp\perp})\vee g^{\perp\perp})^{\perp\perp} = z\wedge(f^{\perp\perp}\vee g^{\perp\perp})^{\perp\perp}$ for all $z\geqq g^{\perp\perp}$ in $\mathscr{L}_{\perp\perp}$. Since $f^{\perp\perp}\vee g^{\perp\perp}$ is closed and $\mathscr{L}$ is modular the right hand side is $(z\wedge f^{\perp\perp})\vee g^{\perp\perp}$. In order to show that this is closed we quote Lemma 5′ with $x_2 = g^{\perp\perp}$, $x_1 = z\wedge f^{\perp\perp}$. Cardinalities for $\mathscr{V}(f, g, \perp)$ follow from Theorem 4.

Remark 5. (i) See Theorem (33.4) in [7] for modular and dual modular pairs in hermitean spaces. (ii) We have constructed sesquilinear spaces E with subspaces F, G such that (23) & (26) & (8) resp. (23) & (27) & (8) is satisfied and such that all 117 resp. 122 elements of $\mathscr{V}(F, G, \perp)$ are different. (iii) Let E be a vector space equipped with a non degenerate alternate form, $\dim E = \aleph_0$ and F, G subspaces with $F\cap G = (0)$, $F^{\perp\perp} + G^{\perp\perp}$ closed and $F^{\perp} + G^{\perp} = E$. Brand [2] gave a recursive construction for an orthogonal decomposition of E, $E = E_1 \oplus E_2$, such that $F\subset E_1$, $G\subset E_2$. From this geometric result it follows readily that the lattice $\mathscr{V}(F, G, \perp)$ is given by $\mathscr{I}(F)\cup\mathscr{I}(G)\cup(\mathscr{I}(F)\vee\mathscr{I}(G))\cup\mathscr{F}(F)\cup\mathscr{F}(G)\cup(\mathscr{F}(F)\wedge\mathscr{F}(G))$, in particular $\mathscr{V}(F, G, \perp)$ is distributive and has 98 elements. The fruitfulness of the method hinted at in Introduction is based on a *reversal* of steps: First $\mathscr{V}(F, G, \perp)$ is computed, then the theorems of [3] are applied in order to conclude that E must split in the manner indicated.

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