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CAPACITY AND MEASURE DENSITIES

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1. Introduction

Let $h\colon [0, \infty)\to[0, \infty)$ be a measure function, i.e. h is continuous, strictly increasing, $h(0)=0$, and $\lim_{t\to\infty} h(t)=\infty$, and let

$$H_h(A) = \inf\left\{\sum_i h(r_i)\colon \bigcup_i \bar{B}^n(x_i, r_i) \supset A\right\}$$

be the h-(outer)measure of $A\subset R^n$. The upper h-measure density of A at $x\in R^n$ is

$$\Theta_h(x, A) = \overline{\lim_{r\to 0}}\, H_h(A \cap \bar{B}^n(x, r))/h(r).$$

Assume that $1<p\leqq n$ and that C is a closed set in R^n. For $x\in R^n$

$$\operatorname{cap}_p(x, C) = \overline{\lim_{r\to 0}}\, r^{p-n} \operatorname{cap}_p\big(B^n(x, 2r), \bar{B}^n(x, r) \cap C\big)$$

defines the upper p-capacity density of C at x. Here cap_p on the right hand side is the ordinary variational p-capacity of a condenser.

The purpose of this note is to compare $\Theta_h(x, C)$ and $\operatorname{cap}_p(x, C)$ for various h and p. Among other things we show that $\operatorname{cap}_p(x, C)=0$ implies $\Theta_h(x, C)=0$ for $h(r)=r^\alpha$, where $\alpha>n-p$. As a byproduct some measure theoretic properties of sets C which satisfy $\operatorname{cap}_p(x, C)=0$ for all $x\in C$ are given. Observe that such a set C need not be of zero p-capacity.

We shall mainly employ the method due to Ju. G. Rešetnjak, cf. [7, 8]. There is an extensive literature on measure theoretic properties of sets of zero p-capacity, see e.g. [1], [6], [7, 8], and [10].

2. Preliminary results

2.1. *Notation.* The open ball centered at $x\in R^n$ with radius $r>0$ is denoted by $B^n(x, r)$. We abbreviate $B^n(r)=B^n(0, r)$ and $S^{n-1}(r)=\partial B^n(r)$. The Lebesgue measure in R^n is denoted by m and $\Omega_n=m(B^n(1))$. We let ω_{n-1} denote the $(n-1)$-measure of $S^{n-1}(1)$. For $p\geqq 1$, L^p is the class of all p-integrable functions in R^n with the norm $\|\ \|_p$.

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If $A\subset R^n$ is open, then $C_0^1(A)$ means the set of continuously differentiable real valued functions with compact support in A. For $u\in C_0^1(A)$, $\nabla u=(\partial_1 u, \dots, \partial_n u)$ is the gradient of u. Each u has the representation, cf. [7, Lemma 3],

$$u(x)=\frac{1}{\omega_{n-1}}\int_{R^n}\frac{\nabla u(y)\cdot(x-y)}{|x-y|^n}\,dm(y). \tag{2.2}$$

If A is open in R^n and $C\subset A$ is compact, then the pair (A, C) is called a condenser and its p-capacity, $1<p\leqq n$, is defined by

$$\operatorname{cap}_p(A, C)=\inf_{u\in W(A,C)}\int_A |\nabla u|^p\,dm$$

where $W(A, C)$ is the set of all non-negative functions $u\in C_0^1(A)$ with $u(x)>1$ for all $x\in C$. Note that for $x\in R^n$ and $0<r_1<r_2$

$$\operatorname{cap}_p\left(B^n(x, r_2), \bar{B}^n(x, r_1)\right)=\begin{cases}\omega_{n-1}\left((r_2^q-r_1^q)/q\right)^{1-p}, & p\in(1, n)\\ \omega_{n-1}\left(\ln(r_2/r_1)\right)^{1-n}, & p=n,\end{cases} \tag{2.3}$$

where $q=(p-n)/(p-1)$. The following subadditivity result for capacities is well-known:

2.4. Lemma. *Suppose that (A, C) is a condenser. If (A_i, C_i), $i=1, 2, \dots$, is a sequence of condensers such that $A\supset A_i$ and $\cup C_i\supset C$, then*

$$\operatorname{cap}_p(A, C)\leqq\sum\operatorname{cap}_p(A_i, C_i).$$

If C is closed in R^n, $x\in R^n$, and $r>0$ we let

$$\operatorname{cap}_p(x, C, r)=r^{p-n}\operatorname{cap}_p\left(B^n(x, 2r), \bar{B}^n(x, r)\cap C\right).$$

The set C is of zero p-capacity, abbreviated $\operatorname{cap}_p C=0$, if for all compact sets $C'\subset C$, $\operatorname{cap}_p(A, C')=0$ for all open $A\supset C'$.

If h is a measure function, then in addition to the measure H_h defined in the introduction we use the h-Hausdorff measure

$$H_h^*(A)=\liminf_{t\to 0}\left\{\sum h(r_i)\colon \cup\bar{B}^n(x_i, r_i)\supset A,\ r_i\leqq t\right\}$$

for $A\subset R^n$. For $h(r)=r^\alpha$, $\alpha>0$, this defines the usual α-dimensional Hausdorff measure on all Borel sets in R^n and

$$\dim_H A=\inf\{\alpha>0\colon H_h^*(A)=0,\ h(r)=r^\alpha\}$$

denotes the Hausdorff dimension of A.

2.5. *Preliminary lemmas.* The first two lemmas are well-known.

Suppose that h is a measure function and that σ is a finite measure in R^n defined on all Borel sets. For $x\in R^n$ and $r>0$ write $\sigma(x, r)=\sigma\left(B^n(x, r)\right)$.

2.6. Lemma. (Cf. [3, pp. 196—204].) *If* $\lambda>0$ *and*

$$A_\lambda=\{x\in R^n\colon \sigma(x,r)\leqq h(r)/\lambda \text{ for all } r>0\},$$

then $H_h(R^n\setminus A_\lambda)\leqq c_n\lambda\sigma(R^n)$, *where* $c_n>0$ *depends only on* n.

2.7. Lemma. [7, Lemma 4] *If* $F\colon (0,\infty)\to R$ *is decreasing, absolutely continuous on compact subintervals, and* $\lim_{r\to\infty}F(r)=0$, $\lim_{r\to 0}F(r)=\infty$, *then*

$$\int_{R^n}F(|x-y|)\,d\sigma(y)=-\int_0^\infty F'(r)\sigma(x,r)\,dr.$$

In order to estimate the upper h-measure density an interpolation lemma of type [7, Lemma 6] is needed:

2.8. Lemma. *Suppose that* $u\in L^p$, $p>1$, *is non-negative and that* $u|\complement B^n(r_0)=0$. *Then for all* $\alpha>0$

$$H_h\left(\left\{x\in R^n\colon v(x)>\Omega_n^{1-1/p}\left(\frac{n-1}{\alpha}\int_0^{r_0}h(t)^{1/p}t^{-n/p}\,dt+r_0^{1-n/p}\|u\|_p\right)\right\}\right)\leqq c_n(\alpha\|u\|_p)^p,$$

where

$$v(x)=\int_{R^n}u(y)\,|x-y|^{1-n}\,dm(y)$$

and c_n *is the constant of Lemma* 2.6.

Proof. For $x\in R^n$, $r>0$, and non-negative measurable w we let

$$Q(w,x,r)=\int_{B^n(x,r)}w\,dm.$$

By Hölder's inequality

$$Q(u,x,r)\leqq \Omega_n^{1-1/p}r^{n-n/p}Q(u^p,x,r)^{1/p}. \tag{2.9}$$

On the other hand

$$Q(u,x,r)\leqq \Omega_n^{1-1/p}r_0^{n-n/p}\|u\|_p \tag{2.10}$$

since $u|\complement B^n(r_0)=0$.

If $A\subset R^n$ is measurable we let

$$\sigma(A)=\int_A u\,dm.$$

Now $\sigma(R^n)<\infty$ since the support of u is compact. Setting $F(r)=r^{1-n}$, $r>0$, Lemma 2.7 implies

$$\begin{aligned} v(x)&=\int_{R^n}u(y)\,|x-y|^{1-n}\,dm=\int_{R^n}F(|x-y|)\,d\sigma(y)\\ &=(n-1)\int_0^\infty Q(u,x,r)r^{-n}\,dr\\ &=(n-1)\int_0^{r_0}Q(u,x,r)r^{-n}\,dr+(n-1)\int_{r_0}^\infty Q(u,x,r)r^{-n}\,dr.\end{aligned} \tag{2.11}$$

Now by (2.9)

$$\int_0^{r_0} Q(u,x,r)r^{-n}\,dr \leqq \Omega_n^{1-1/p}\int_0^{r_0} Q(u^p,x,r)^{1/p}r^{-n/p}\,dr \tag{2.12}$$

and by (2.10)

$$\int_{r_0}^{\infty} Q(u,x,r)r^{-n}\,dr \leqq \Omega_n^{1-1/p}r_0^{\frac{n}{p}-n/p}\int_{r_0}^{\infty}\|u\|_p r^{-n}\,dr \tag{2.13}$$

$$= \frac{\Omega_n^{1-1/p}}{n-1}r_0^{1-n/p}\|u\|_p.$$

Suppose that $\alpha>0$ and let

$$B_\alpha = \{x\in R^n\colon Q(u^p,x,r)\leqq h(r)/\alpha^p\}.$$

Define $\sigma(A)=\int_A u^p\,dm$ if $A\subset R^n$ is a Borel set and apply Lemma 2.6:

$$H_h(R^n\setminus B_\alpha)\leqq c_n\alpha^p\sigma(R^n) = c_n\alpha^p\|u\|_p^p.$$

If $x\in B_\alpha$, then by (2.12)

$$\int_0^{r_0} Q(u,x,r)r^{-n}\,dr \leqq \Omega_n^{1-1/p}\alpha^{-1}\int_0^{r_0} h(r)^{1/p}r^{-n/p}\,dr \tag{2.14}$$

and hence by (2.11), (2.13), and (2.14) for $x\in B_\alpha$

$$v(x)\leqq K = \Omega_n^{1-1/p}\left[\frac{n-1}{\alpha}\int_0^{r_0} h(r)^{1/p}r^{-n/p}\,dr + r_0^{1-n/p}\|u\|_p\right].$$

This gives $\{x\in R^n\colon v(x)>K\}\subset R^n\setminus B_\alpha$ and the result follows.

3. Upper bounds for measure densities

Suppose that C is a closed set in R^n and $x\in R^n$. If h is a measure function and $r>0$, then we let

$$\Theta_h(x,C,r) = H_h(\bar{B}^n(x,r)\cap C)/h(r).$$

3.1. Theorem. *If* $p\in(1,n]$ *and*

$$\int_0^{2r} h(t)^{1/p}t^{-n/p}\,dt \leqq Ar^{(p-n)/p}h(r)^{1/p} \tag{3.2}$$

for some $A>0$ *and all* $r\in(0,r_0]$, *then*

$$\Theta_h(x,C,r)\leqq c\,\mathrm{cap}_p(x,C,r),\quad r\in(0,r_0].$$

Here the constant c depends only on n, p, and A.

Proof. We may assume that $x=0$, and since $\Theta_h(0, C, r)\leqq 1$ for all $r>0$, we may also assume

(3.3) $$\operatorname{cap}_p(0, C, r) < K = \omega_{n-1}^p \Omega_n^{1-p} 2^{n-2p}$$

for all $r\in(0, r_0]$. Set

$$I(r) = \int_0^{2r} h(t)^{1/p} t^{-n/p}\, dt.$$

Let $\varepsilon>0$ and choose $w\in W(B^n(2r), \bar{B}^n(r)\cap C)$ such that

(3.4) $$\operatorname{cap}_p(B^n(2r), \bar{B}^n(r)\cap C) \geqq \int |\nabla w|^p\, dm - \varepsilon$$

and

(3.5) $$\int |\nabla w|^p\, dm < Kr^{n-p}.$$

By (2.2)

$$w(x) = \omega_{n-1}^{-1} \int |x-y|^{-n} \nabla w(y)\cdot(x-y)\, dm(y)$$
$$\leqq \omega_{n-1}^{-1} \int |x-y|^{1-n} |\nabla w(y)|\, dm(y).$$

Now apply Lemma 2.8 with $u=|\nabla w|/\omega_{n-1}$ and $r_0=2r$. The inequality (3.5) gives

$$\Omega_n^{1-1/p}(2r)^{1-n/p}\|u\|_p \leqq \Omega_n^{1-1/p}(2r)^{1-n/p}(Kr^{n-p})^{1/p}\omega_{n-1}^{-1} = 1/2 < 1,$$

hence we may choose $\alpha>0$ such that

$$\Omega_n^{1-1/p}\left[\frac{n-1}{\alpha} I(r) + (2r)^{1-n/p}\|u\|_p\right] = 1.$$

Lemma 2.8 yields

$$H_h(\bar{B}^n(r)\cap C) \leqq c_n \|u\|_p^p \left[\frac{(n-1)I(r)}{\Omega_n^{1/p-1} - (2r)^{1-n/p}\|u\|_p}\right]^p$$
$$\leqq c_n 2^p \Omega_n^{p-1} \omega_{n-1}^{-p} (n-1)^p I(r)^p \left(\operatorname{cap}_p(B^n(2r), \bar{B}^n(r)\cap C) + \varepsilon\right)$$

where the inequality (3.5) has also been used. By the assumption (3.2), $I(r)\leqq Ar^{1-n/p}h(r)^{1/p}$ and thus

$$H_h(\bar{B}^n(r)\cap C)/h(r) \leqq cr^{p-n}\left(\operatorname{cap}_p(B^n(2r), \bar{B}^n(r)\cap C) + \varepsilon\right)$$

where $c=c_n 2^n \Omega_n^{p-1}\omega_{n-1}^{-p}(n-1)^p A^p$. Letting $\varepsilon\to 0$ gives the required result.

3.6. Corollary. *Suppose that h satisfies the condition* (3.2). *If* $\operatorname{cap}_p(x, C)=0$, *then* $\Theta_h(x, C)=0$.

3.7. Corollary. *If* $1<p\leqq n$ *and* $\operatorname{cap}_p(x, C)=0$, *then* $\Theta_h(x, C)=0$ *for* $h(r)=r^\alpha$ *and* $\alpha>n-p$.

Proof. Let $h(r)=r^\alpha$, $\alpha>n-p$. In view of Corollary 3.6 it suffices to show that h satisfies the condition (3.2). An easy calculation shows that this is true for all $r>0$ with $A=p(\alpha-n+p)^{-1}2^{(\alpha-n+p)/p}$.

3.8. Theorem. *Suppose that* $C\subset R^n$ *is closed and* $1<p\leqq n$. *If* $\operatorname{cap}_p(x, C)=0$ *for all* $x\in C$, *then* $\dim_H C\leqq n-p$.

Proof. If $\alpha > n-p$, then for $h(r)=r^\alpha$ Corollary 3.7 gives $\Theta_h(x, C)=0$ for all $x \in C$. By [2, 2.10.19 (2)], $H_h^*(C)=0$. This shows $\dim_H C \leqq \alpha$ and the result follows.

3.9. Remarks. (a) It is well-known, see e.g. [6, p. 136] and [8, Corollary 2], that $\operatorname{cap}_p C=0$ implies $\dim_H C \leqq n-p$.

(b) Especially for $p=n$ it is interesting to know if the condition (3.2) would allow measure functions h increasing more sharply at 0 than $h(r)=r^\alpha$ for any $\alpha>0$. Unfortunately, for $p \in (1, n]$ the condition (3.2) always implies that $h(r) \leqq cr^\beta$ for some $\beta>0$ and $c>0$ for all $r \in (0, r_0]$. To prove this choose an integer $i_0 \geqq 2$ such that $2^{-i_0} \in (0, r_0]$. Then for $i \geqq i_0$

$$Ah(2^{-i})^{1/p} 2^{i(n/p-1)} \geqq \int_0^{2^{-i+1}} h(t)^{1/p} t^{-n/p}\, dt \tag{3.10}$$

$$\geqq \sum_{j=i}^{\infty} h(2^{-j})^{1/p} 2^{(j-1)n/p} 2^{-j} = \sum_{j=i}^{\infty} h(2^{-j})^{1/p} 2^{j(n/p-1)-n/p}.$$

Assume first that $p \in (1, n)$. Fix $\beta>0$ and then an integer $k \geqq 2$ such that $i_0 k > i_0 + k$ and

$$2^{\beta/p} A < 2^{k(n/p-1)-n/p}. \tag{3.11}$$

Now for all $i \geqq i_0$

$$h(2^{-i-k}) \leqq 2^{-\beta} h(2^{-i}) \tag{3.12}$$

since otherwise

$$h(2^{-i}) \geqq h(2^{-i-1}) \geqq \ldots \geqq h(2^{-i-k}) > 2^{-\beta} h(2^{-i})$$

and thus

$$\sum_{j=i}^{\infty} h(2^{-j})^{1/p} 2^{j(n/p-1)-n/p} > 2^{-\beta/p} h(2^{-i})^{1/p} \sum_{j=i}^{i+k} 2^{j(n/p-1)-n/p}$$

$$> 2^{-\beta/p} h(2^{-i})^{1/p} 2^{(i+k)(n/p-1)-n/p}.$$

But this combined with (3.10) and (3.11) gives a contradiction.

If $p=n$, fix $\beta>0$ and an integer $k \geqq 2$ so that $i_0 k > i_0 + k$ and $A < 2^{-\beta/n-1} k$. Then it can be shown similarly that (3.12) holds.

To finish the proof let $r \in (0, 2^{-i_0 k}]$. Choose i such that $r \in (2^{-i-1}, 2^{-i}]$ and then $m \geqq i_0$ so that $mk \leqq i < i+1 \leqq (m+1)k$. Since $i_0 k \geqq i_0 + k$ it follows from (3.12) by induction that

$$h(2^{-jk}) \leqq 2^{-\beta(j-i_0+1)} h(2^{-i_0}), \quad j = i_0, i_0+1, \ldots.$$

Hence

$$h(r)^k \leqq h(2^{-mk})^k \leqq 2^{-\beta(m-i_0+1)k} h(2^{-i_0})^k$$

$$= 2^{\beta i_0 k} h(2^{-i_0})^k 2^{-\beta(m+1)k} \leqq 2^{\beta i_0 k} h(2^{-i_0})^k r^\beta.$$

This gives the required result.

4. Lower bounds for measure densities

Here we only consider measure functions h of well-known type.

4.1. Theorem. *Let*

$$h(r) = r^{n-p} \quad \text{for} \quad p\in(1, n), \quad r > 0, \quad \text{and}$$
$$= (\ln(1/r))^{1-n} \quad \text{for} \quad p = n \quad \text{and} \quad 0 < r < 1/2.$$

If C is a closed set in R^n, then

$$\operatorname{cap}_p(x, C, r) \leqq c\Theta_h(x, C, r)$$

for all $r>0$ if $p\in(1, n)$ and for $r\in(0, 1/2)$ if $p=n$. The constant c depends only on n and p.

Proof. We may assume $x=0$. Consider first the case $1<p<n$. Fix $r>0$ and choose a covering $\bar{B}^n(x_i, r_i)$ of the set $\bar{B}^n(r)\cap C$ where $x_i\in\bar{B}^n(r)$. Assume $2r_i<r$ for all i. Now by Lemma 2.4 and by (2.3)

$$\begin{aligned}(4.2)\qquad \operatorname{cap}_p(0, C, r) &= r^{p-n}\operatorname{cap}_p\big(B^n(2r), \bar{B}^n(r)\cap C\big)\\ &\leqq r^{p-n}\sum_{i=1}^{\infty}\operatorname{cap}_p\big(B^n(x_i, r), \bar{B}^n(x_i, r_i)\cap C\big)\\ &= c_1\sum_{i=1}^{\infty}[(r/r_i)^q-1]^{1-p} \leqq c_1(1-2^{-q})^{1-p}\sum_{i=1}^{\infty}(r_i/r)^{n-p}\end{aligned}$$

where $c_1=\omega_{n-1}q^{p-1}$, $q=(n-p)/(p-1)$, and the inequality $2r_i<r$ is used in the last step. Thus

$$\operatorname{cap}_p(0, C, r) \leqq c\sum h(r_i)/h(r).$$

If $2r_i\geqq r$ for some i, then, since $\operatorname{cap}_p(0, C, r)\leqq\omega_{n-1}((1-2^{-q})/q)^{1-p}$, the result is obvious.

In the case $p=n$ the estimate (4.2) can be written in the form

$$\operatorname{cap}_n(0, C, r) \leqq c\sum\left[\frac{-\ln r_i}{-\ln r}\right]^{1-n}$$

when the conditions $r<1/2$ and $2r_i<r$ are used for the inequality $\big(\ln(r/r_i)\big)^{n-1}\geqq c_1(-\ln r_i)^{n-1}(-\ln r)^{1-n}$, $c_1=(\ln 2)^{n-1}2^{1-n}$ and $c=\omega_{n-1}c_1$. This yields the conclusion as above.

4.3. Corollary. *Suppose $1<p\leqq n$ and let h be as in Theorem* 4.1. *If $C\subset R^n$ is a closed set, then $\Theta_h(x, C)=0$ implies $\operatorname{cap}_p(x, C)=0$.*

4.4. Remark. It is well-known, see e.g. [6, Theorem 7.2] or [10], that if h is as in Theorem 4.1, then $H_h^*(C)<\infty$ gives $\operatorname{cap}_p C=0$.

4.5. Corollary. *If C is a closed set in R^n and* $\operatorname{cap}_p(x, C)=0$ *for all* $x \in C$, *then* $\operatorname{cap}_q C=0$ *for* $q \in (1, p)$.

Proof. By Theorem 3.8 $\dim_H C \leqq n-p$. Consequently $H_h^*(C)=0$ for $h(r)=r^\alpha$, $\alpha > n-p$. By Remark 4.4 $\operatorname{cap}_q C=0$ for $q \in (1, p)$.

4.6. Corollary. *Suppose that* $1<p\leqq n$. *If C is a closed set in R^n such that* $\operatorname{cap}_p C>0$ *and* $\operatorname{cap}_p(x, C)=0$ *for all* $x \in C$, *then* $\dim_H C=n-p$. *Moreover,* $H_h^*(C)=\infty$, $h(r)=r^{n-p}$.

Proof. The case $p=n$ has been handled by Theorem 3.8. If $\alpha=n-p>0$ and if $H_h^*(C)<\infty$, $h(r)=r^\alpha$, then by Remark 4.4 $\operatorname{cap}_p C=0$. Consequently, $\dim_H C \geqq n-p$ and the opposite inequality follows from Theorem 3.8.

4.7. *Example.* Here we construct for $p=n$ a compact set $C \subset R^n$ such that $\operatorname{cap}_n C>0$ but $\operatorname{cap}_n(x, C)=0$ for all $x \in R^n$. In fact we shall show that even the condition $M(x, C)<\infty$ for all $x \in C$ holds. The condition $M(x, C)<\infty$, cf. [4] and [5], means that there exists a non-degenerate continuum $K \subset \complement C \cup \{x\}$ such that $x \in K$ and the n-modulus of the curve family joining K and C is finite. By [5, Theorem 3.1] $M(x, C)<\infty$ implies $\operatorname{cap}_n(x, C)=0$. A set of this type is of function theoretic interest, see [11].

To this end let $k \in (1, 2)$ and define $l_i'=\exp(-k^{ni/(n-1)})$, $i=0, 1, \ldots$. Fix i_0 such that $4\sqrt{n}\, l_{i+1}'<l_i'$ for $i \geqq i_0$ and write $l_i=l_{i+i_0}'$, $i=0, 1, \ldots$. Let Δ_0 be a closed interval of length l_0 and set $E_0=\Delta_0 \times \ldots \times \Delta_0$ (n times). Denote by F_1 the union of two closed intervals Δ_1^1 and Δ_1^2 of length l_1 lying in Δ_0 and containing both ends of Δ_0. Set $E_1=F_1 \times \ldots \times F_1$ and carry out the same operations in the intervals Δ_1^1 and Δ_1^2 using l_2 instead of l_1. Four intervals Δ_2^i, $i=1, 2, 3, 4$, are obtained. Let their union be F_2 and set $E_2=F_2 \times \ldots \times F_2$. This process can be continued and define $C=\bigcap_{i=0}^\infty E_i$. Each set E_i consists of 2^{in} closed cubes Q_i^j, $j=1, \ldots, 2^{in}$, with sides of length l_i.

The set C is of positive n-capacity since

$$\sum_{i=1}^\infty 2^{ni/(1-n)} \ln(l_i/l_{i+1}) < \infty,$$

cf. [6, Theorem 7.4 and the following Remark]. For relations between the capacity used in [6] and the variational capacity used in this paper see [8, Theorems 6.1 and 6.2].

Next we consider the condition $M(x_0, C)<\infty$. Fix $x_0 \in C$. For each $i \geqq 1$ choose a cube Q_i in the collection $\{Q_i^j\}$ such that $x_0 \in Q_i$. Now it is easy to construct a continuum $K_{i+1} \subset Q_i$ consisting of line segments L_1, L_2, L_3 in the plane $T=\{x \in R^n\colon x_j=(x_0)_j,\ j=3, \ldots, n\}$ and such that L_1 joins the midpoint of a face of $T \cap Q_i$ to the center of $T \cap Q_i$, L_2 is a part of a similar segment and L_3 is perpendicular to L_2 and joins the midpoint of a face of $T \cap Q_{i+1}$ to the endpoint of L_2. Now $d(K_{i+1}, Q_{i+1}' \setminus Q_{i+1}) \geqq l_i/4$ and $d(K_{i+1}, Q_{i+2}') \geqq l_{i+1}/4$ where $Q_k'=Q_{k-1} \cap \bigcup_j Q_k^j$.

Set $K=\bigcup_{i=1}^{\infty} K_{i+1} \cup \{x_0\}$. Then after a suitable selection of the continua K_{i+1}, K is a non-degenerate continuum with $x_0 \in K$.

If E and F are closed sets in R^n we denote by $\Delta(E, F)$ the family of all paths joining these sets in R^n. For properties of the n-modulus $M(\Delta(E, F))$ of the path family $\Delta(E, F)$ we refer to [9].

It remains to show $M(\Delta(K, C))<\infty$. By [9, Theorem 6.2] for each $i \geqq 1$

$$(4.8) \qquad M(\Delta(K_{i+1}, C)) \leqq M(\Delta(K_{i+1}, Q'_{i+2}))+M\left(\Delta\left(K_{i+1}, \bigcup_{j=1}^{i+1} (Q'_j \setminus Q_j)\right)\right)$$

$$\leqq M(\Delta(K_{i+1}, Q'_{i+2}))+\sum_{j=1}^{i+1} M(\Delta(K_{i+1}, Q'_j \setminus Q_j))$$

and we estimate each term separately.

Fix $1 \leqq j \leqq i$. Now $K_{i+1} \subset Q_i$ and $(Q'_j \setminus Q_j) \cap B^n(x_0, l_{j-1}/2)=\emptyset$, thus

$$(4.9) \qquad M(\Delta(K_{i+1}, Q'_j \setminus Q_j)) \leqq \omega_{n-1}(\ln[(l_{j-1}/2)/(l_i\sqrt{n}/2)])^{1-n}$$

$$= \omega_{n-1} k^{-n(i_0+i)}\left[1-k^{n(i_0+i)/(1-n)} \ln \sqrt{n}-k^{n(j-1-i)/(n-1)}\right]^{1-n} \leqq c_1 k^{-ni}$$

where c_1 depends only on n, k, and i_0.

If $j=i+1$, then because of the quasi-invariance of the n-modulus under bi-Lipschitz mappings, see [9], it is easy to see that there is $c'_2>0$ depending only on n such that

$$(4.10) \qquad M(\Delta(K_{i+1}, Q'_{i+1} \setminus Q_{i+1})) \leqq c'_2 M(\Delta(\bar{B}^n(l_{i+1}), S^{n-1}(l_i)))$$

$$= c'_2 \omega_{n-1}(\ln(l_i/l_{i+1}))^{1-n} \leqq c_2 k^{-ni}$$

and c_2 depends on the same constants as c_1.

As above the estimate

$$(4.11) \qquad M(\Delta(K_{i+1}, Q'_{i+2})) \leqq c'_3 M(\Delta(\bar{B}^n(l_{i+2}), S^{n-1}(l_{i+1}))) \leqq c_3 k^{-n(i+1)}$$

is obtained where c_3 depends on the same constants as c_1.

Finally, the inequalities (4.8)—(4.11) yield

$$M(\Delta(K, C)) \leqq \sum_{i=1}^{\infty} M(\Delta(K_{i+1}, C))$$

$$\leqq \sum_{i=1}^{\infty} [c_3 k^{-n(i+1)}+c_2 k^{-ni}+(i+1) c_1 k^{-ni}]$$

$$\leqq (c_1+c_2+c_3) \sum_{i=1}^{\infty} (i+1) k^{-ni} < \infty.$$

This shows that $M(x_0, C)<\infty$.

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