

ON LINEAR RELATIONS IN AN INDEFINITE INNER PRODUCT SPACE

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Introduction

In 1961 R. Arens [2] initiated a study of linear relations, i.e. subspaces of $\mathfrak{H} \oplus \mathfrak{H}$, where \mathfrak{H} is a Hilbert space. Since then there has been a growing interest in this subject; see for instance C. Bennewitz [4], E. A. Coddington [6—9], A. Dijksma and H. S. V. de Snoo [11], H. Langer and B. Textorius [18], Å. Pleijel [19].

Some of these authors have used this theory mainly in studying differential equations which lead to linear relations in a Hilbert space.

On the other hand, there are some recent publications studying differential equations with an indefinite weight function which lead to symmetric operators in an indefinite inner product space; see F. V. Atkinson, W. N. Everitt and K. S. Ong [3], K. Daho and H. Langer [10] and H. Langer [17].

In this context a question arises: Can the theory of linear relations in a Hilbert space be extended to a corresponding theory in an indefinite inner product space? This paper tries to answer this question. In another publication we shall apply this theory to study canonical differential equations with an indefinite weight function.

Chapter 1 summarizes the basic definitions and results of the theory of indefinite inner product spaces mainly because our terminology differs from that used by J. Bognár in [5], which is our main reference in indefinite inner product spaces.

Chapter 2 starts the study of linear relations in an indefinite inner product space: Section 2 contains a detailed investigation of reducing a linear relation to an operator; some of these results may be new also in the Hilbert space case. Section 3 represents a linear relation in a Krein space as a linear relation with a similar structure in a Hilbert space perturbed by an operator; these results extend the known facts of the operator case. In Section 4 we study the Cayley transformation. In Section 5 we generalize the notion of the operator matrix to linear relations.

Chapter 3 analyzes dissipative and maximal dissipative linear relations mostly along the lines of [11]. Some of these results seem to be new also in the operator case; compare with [15].

In Chapter 4 we give a detailed investigation of the basic properties of symmetric linear relations mostly in a Pontrjagin space. Eigenvalues and points of regular type are studied in Section 1. With the help of the relation matrix we characterize in Section 2 self-adjoint extensions of a symmetric linear relation in a Pontrjagin space. Section 3 introduces the deficiency spaces and the defects numbers.

1. Indefinite inner product spaces

1.1. Geometry. An (indefinite, non-degenerate) *inner product space* \mathfrak{H} is a (complex) vector space with a non-degenerate hermitean sesquilinear form $[\cdot|\cdot]$. An element $f \in \mathfrak{H}$ (a subspace $\mathfrak{Q} \subset \mathfrak{H}$) is said to be *positive/non-negative/neutral/non-positive/negative* if $[f|f] > 0/\geq 0/ = 0/\leq 0/ < 0$ (for all $f \in \mathfrak{Q} \setminus \{0\}$).

Two vectors $f, g \in \mathfrak{H}$ (subspaces $\mathfrak{Q}, \mathfrak{M} \subset \mathfrak{H}$) are called *orthogonal*, written $f \perp g$ ($\mathfrak{Q} \perp \mathfrak{M}$), if $[f|g] = 0$ (for all $f \in \mathfrak{Q}, g \in \mathfrak{M}$). For a subspace $\mathfrak{Q} \subset \mathfrak{H}$ we define $\mathfrak{Q}^\perp := \{f \in \mathfrak{H} | f \perp \mathfrak{Q}\}$ and call it the *orthogonal companion* of \mathfrak{Q} in \mathfrak{H} . The set $\mathfrak{Q}^0 := \mathfrak{Q} \cap \mathfrak{Q}^\perp$ is the *isotropic part* of \mathfrak{Q} . If $\mathfrak{Q}^0 \neq \{0\}$, the subspace \mathfrak{Q} is called *degenerate*.

Following [2] we define the *closure* $\overline{\mathfrak{Q}}$ of a subspace \mathfrak{Q} to be $\mathfrak{Q}^{\perp\perp}$. The subspace \mathfrak{Q} is called *closed* if $\overline{\mathfrak{Q}} = \mathfrak{Q}$ and *dense* if $\overline{\mathfrak{Q}} = \mathfrak{H}$.

1.2. Operators. We use the following notations with a (linear) operator $T: \mathfrak{D}(T)$ is the domain, $\mathfrak{R}(T)$ the range and $\mathfrak{N}(T)$ the null space of T .

Let T be an operator in an inner product space \mathfrak{H} . T is *dissipative* if $\text{Im} [Tf|f] \geq 0$ for all $f \in \mathfrak{D}(T)$; T is *symmetric* if $\text{Im} [Tf|f] = 0$ or equivalently $[Tf|g] = [f|Tg]$ for all $f, g \in \mathfrak{D}(T)$. Let \mathfrak{K} be another inner product space and T an operator from \mathfrak{H} into \mathfrak{K} . T is called *contractive* if $[Tf|Tf] \leq [f|f]$ for all $f \in \mathfrak{D}(T)$, and *isometric* if $[Tf|Tf] = [f|f]$ or equivalently $[Tf|Tg] = [f|g]$ for all $f, g \in \mathfrak{D}(T)$. An isometric operator T in \mathfrak{H} is *unitary* if $\mathfrak{D}(T) = \mathfrak{R}(T) = \mathfrak{H}$. An *orthogonal projector* is a symmetric operator T in \mathfrak{H} with the properties $\mathfrak{D}(T) = \mathfrak{H}$ and $T^2 = T$.

If T is a densely defined operator from \mathfrak{H} into \mathfrak{K} , then one can define the *adjoint* T^+ of T : $\mathfrak{D}(T^+)$ is the set of all those vectors $g \in \mathfrak{K}$ for which there exists a vector $h \in \mathfrak{H}$ such that $[Tf|g] = [f|h]$ for all $f \in \mathfrak{D}(T)$, and then $T^+g := h$. A densely defined operator T in \mathfrak{H} is called *self-adjoint* if $T^+ = T$.

1.3. Fundamental decompositions. Let \mathfrak{H} be an inner product space. It is said to be *decomposable* if it can be represented in the form

$$(1.1) \quad \mathfrak{H} = \mathfrak{H}_+ [+] \mathfrak{H}_-,$$

where $\mathfrak{H}_+/\mathfrak{H}_-$ is a positive/negative subspace. Here the symbol $[+]$ denotes a direct and orthogonal sum. Every decomposition of this type is called a *fundamental decomposition* of \mathfrak{H} . The decomposition (1.1) induces so-called *fundamental projectors* P_+ and P_- : $P_\pm f := f_\pm$, where $f = f_+ + f_- \in \mathfrak{H}$ with $f_\pm \in \mathfrak{H}_\pm$. The corresponding *fundamental symmetry* $J := P_+ - P_-$ is then self-adjoint and unitary.

With the help of a fundamental symmetry J one can define J -inner product $(\cdot | \cdot)$:

$$(1.2) \quad (f | g) := [Jf | g] \quad (f, g \in \mathfrak{H}),$$

which is positive definite, i.e. \mathfrak{H} is a pre-Hilbert space with respect to the form $(\cdot | \cdot)$. For the corresponding norm, the so-called J -norm, $\|\cdot\|: f \mapsto (f | f)^{1/2}$ we have

$$(1.3) \quad |[f | g]| \cong \|f\| \|g\| \quad (f, g \in \mathfrak{H}).$$

When necessary, we use the prefix J to denote a property which is defined using a J -inner product instead of an indefinite inner product. Thus we can speak about J -symmetric operators etc. Especially, J -adjoint is denoted by $*$ instead of $+$.

1.4. Krein spaces and Pontrjagin spaces. If a decomposable inner product space \mathfrak{H} has a decomposition (1.1) such that \mathfrak{H}_+ (resp. \mathfrak{H}_-) is a Hilbert space with respect to the form $[\cdot | \cdot]$ ($-[\cdot | \cdot]$), it is called a *Krein space*. In this case \mathfrak{H} is a Hilbert space with respect to every J -inner product and the J -norms are all equivalent. All topological notions in a Krein space are to be understood to refer to this J -norm topology. As the J -norm closure of a subspace \mathfrak{Q} is given by $\mathfrak{Q}^{\perp\perp}$, our earlier terminology is consistent with the agreement just made. Note that a closed subspace \mathfrak{Q} is *ortho-complemented*, i.e. $\mathfrak{Q} + \mathfrak{Q}^{\perp} = \mathfrak{H}$, if and only if \mathfrak{Q} itself is a Krein space.

A Krein space \mathfrak{H} with a fundamental decomposition (1.1) is called a *Pontrjagin space* (with κ negative squares) or a π_{κ} -space if $\dim \mathfrak{H}_- = \kappa$. In a Pontrjagin space a closed subspace is ortho-complemented if and only if it is non-degenerate.

1.5. Product spaces. Let \mathfrak{H} and \mathfrak{K} be inner product spaces. The product space $\mathfrak{H} \oplus \mathfrak{K}$ equipped with the usual linear structure and with the inner product

$$((f, g) | (h, k)) := [f | h] + [g | k] \quad ((f, g), (h, k) \in \mathfrak{H} \oplus \mathfrak{K})$$

is also an inner product space.

Proposition 1.1. 1° *If \mathfrak{H} and \mathfrak{K} are decomposable with decompositions (1.1) and $\mathfrak{K} = \mathfrak{K}_+ [+] \mathfrak{K}_-$, then $\mathfrak{H} \oplus \mathfrak{K}$ is also decomposable and has a fundamental decomposition¹*

$$\mathfrak{H} \oplus \mathfrak{K} = \mathfrak{H}_+ \oplus \mathfrak{K}_+ [+] \mathfrak{H}_- \oplus \mathfrak{K}_-$$

2° *If \mathfrak{H} and \mathfrak{K} are Krein spaces, then $\mathfrak{H} \oplus \mathfrak{K}$ is also a Krein space.*

3° *If \mathfrak{H} is a π_{κ} -space and \mathfrak{K} is a $\pi_{\kappa'}$ -space, then $\mathfrak{H} \oplus \mathfrak{K}$ is a Pontrjagin space with $\kappa + \kappa'$ negative squares.*

The proof is clear from the definitions.

¹ The symbol $+$ means the algebraic sum in the product space.

2. Linear relations

2.1. Preliminaries. Throughout this chapter \mathfrak{G} , \mathfrak{H} and \mathfrak{K} are inner product spaces.

A *linear relation from \mathfrak{H} into \mathfrak{K}* is a subspace T of the product space $\mathfrak{H} \oplus \mathfrak{K}$; if $\mathfrak{K} = \mathfrak{H}$, T is said to be a *linear relation in \mathfrak{H}* . A linear relation is *closed* if it is a closed subspace. We recall the following definitions and notations for linear relations T and S from \mathfrak{H} into \mathfrak{K} and R from \mathfrak{G} into \mathfrak{H} :

$$\begin{aligned} \mathfrak{D}(T) &:= \{f \in \mathfrak{H} \mid (f, g) \in T \text{ for some } g \in \mathfrak{K}\}, \\ \mathfrak{R}(T) &:= \{g \in \mathfrak{K} \mid (f, g) \in T \text{ for some } f \in \mathfrak{H}\}, \\ \mathfrak{N}(T) &:= \{f \in \mathfrak{H} \mid (f, 0) \in T\}, \\ T(f) &:= \{g \in \mathfrak{K} \mid (f, g) \in T\} \quad (f \in \mathfrak{D}(T)), \\ T^{-1} &:= \{(g, f) \in \mathfrak{K} \oplus \mathfrak{H} \mid (f, g) \in T\}, \\ zT &:= \{(f, zg) \in \mathfrak{H} \oplus \mathfrak{K} \mid (f, g) \in T\} \quad (z \in \mathbb{C} := \text{complex numbers}), \\ S+T &:= \{(f, g+k) \in \mathfrak{H} \oplus \mathfrak{K} \mid (f, g) \in S, (f, k) \in T\}, \\ SR &:= S \circ R := \{(f, k) \in \mathfrak{G} \oplus \mathfrak{K} \mid (f, g) \in R, (g, k) \in S \text{ for some } g \in \mathfrak{H}\}, \\ T^+ &:= \{(k, h) \in \mathfrak{K} \oplus \mathfrak{H} \mid [f|h] = [g|k] \text{ for all } (f, g) \in T\}. \end{aligned}$$

The only new definition here compared to the Hilbert space case (see [6] and [11]) is the adjoint, which has been formed with respect to the (indefinite) inner product. But because this inner product gives a duality, our definition of the adjoint is a particular case from [2].

The basic algebraic properties of linear relations are given in [2]; we list here only the following facts about the adjoint:

$$\begin{aligned} T^+ &\text{ is closed, } \overline{T} = T^{++}, \\ S \subset T &\text{ implies } T^+ \subset S^+, \\ (zT)^+ &= \bar{z}T^+ \quad (z \in \mathbb{C}), \\ (T^{-1})^+ &= (T^+)^{-1}, \\ \mathfrak{N}(T^+) &= \mathfrak{R}(T)^\perp, \quad T^+(0) = \mathfrak{D}(T)^\perp. \end{aligned}$$

If we identify an operator T from \mathfrak{H} into \mathfrak{K} with its graph in $\mathfrak{H} \oplus \mathfrak{K}$, it is easily seen that a linear relation T is an operator iff (if and only if) $T(0) = \{0\}$. Note that two relations S and T with $\mathfrak{D}(S) = \mathfrak{D}(T)$ and $S(0) = T(0)$ are equal iff $S \subset T$.

As in the case of linear operators the following linear relations T in \mathfrak{H} are of interest:

- (i) *dissipative* linear relations, i.e. $\text{Im}[g|f] \geq 0$ for all $(f, g) \in T$;
- (ii) *symmetric* linear relations, i.e. $\text{Im}[g|f] = 0$ for all $(f, g) \in T$ or equivalently $T \subset T^+$;
- (iii) *self-adjoint* linear relations, i.e. $T = T^+$;
- (iv) *contractive* linear relations, i.e. $[g|g] \leq [f|f]$ for all $(f, g) \in T$;
- (v) *isometric* linear relations, i.e. $[g|g] = [f|f]$ for all $(f, g) \in T$;
- (vi) *unitary* linear relations, i.e. T is isometric and $\mathfrak{D}(T) = \mathfrak{R}(T) = \mathfrak{H}$.

One can define in the usual way a *maximal dissipative* linear relation etc. Of course the definitions (iv) and (v) can be extended to linear relations from \mathfrak{H} into \mathfrak{K} .

If the space \mathfrak{H} is a Hilbert space, the last three linear relations are not interesting as relations because they are all operators, but this is not the case in a general inner product space: Arens [2] has proved that a linear relation T is isometric iff $T^{-1} \subset T^+$, which implies $T(0) \subset \mathfrak{R}(T)^0$ for an isometric relation T . Hence in our case, where the inner product spaces are supposed to be non-degenerate, all unitary relations are operators. On the other hand, an isometric relation is not necessarily an operator. A simple counter-example is given by the relation $T := \{0\} \oplus \mathfrak{L}$, where $\mathfrak{L} \neq \{0\}$ is a neutral subspace in a Pontrjagin space.

We shall need the following result, which is known in the Hilbert space case and also for densely defined operators in a Krein space; see [5].

Proposition 2.1. *Let T be a closed linear relation from a Krein space \mathfrak{H} into a Krein space \mathfrak{K} . Then $\mathfrak{R}(T)$ is closed iff $\mathfrak{R}(T^+)$ is closed.*

Proof. It is enough to prove that $\mathfrak{R}(T^+)$ is closed if $\mathfrak{R}(T)$ is closed, because the converse follows from $T = T^{++}$. On the other hand, $\mathfrak{R}(T)$ is closed (in \mathfrak{H}) iff $\mathfrak{H} \oplus \mathfrak{R}(T) = T \upharpoonright \mathfrak{H} \oplus \{0\}$ is closed (in $\mathfrak{H} \oplus \mathfrak{K}$), and $\mathfrak{R}(T^+)$ is closed iff $T^\perp + (\mathfrak{H} \oplus \{0\})^\perp$ is closed. So it is more than enough to prove the identity

$$T^\perp + S^\perp = (T \cap S)^\perp$$

for linear relations T and S , for which the sum $T \upharpoonright S$ is closed. If one uses the duality given by the inner product and the Hahn—Banach theorem, this result follows by a light modification of the proof of the corresponding fact in a Banach space; see [14], Lemma IV-4.9.

Let T be a linear relation in \mathfrak{H} . Every $z \in \mathbb{C}$ for which $(f, zf) \in T$ with some $f \neq 0$ is called an *eigenvalue* of T ; the corresponding vectors f are *eigenvectors* belonging to the eigenvalue z . The set of all eigenvalues of T is denoted by $\sigma_p(T)$. If for some $z \in \mathbb{C}$ the relation $(T - zI)^{-1}$ is an everywhere defined operator, then z belongs to the *resolvent set* $\rho(T)$ of T . (Here and in the following I denotes the identity operator or the corresponding linear relation.) The *spectrum* $\sigma(T)$ of T is the complement of $\rho(T)$ in \mathbb{C} .

To be able to put some later results in a concise form we introduce the notion of a point of regular type. For this let T be a linear relation in a Krein space \mathfrak{H} .

A complex number z is called a *point of regular type* of the relation T if there exists a constant $c := c(z) > 0$ such that $\|g - zf\| \cong c\|f\|$ for all $(f, g) \in T$. The set of all points of regular type of T is denoted by $r(T)$.

Theorem 2.2. *Let T be a linear relation in a Krein space \mathfrak{H} . Then*

(i) $(T - zI)^{-1}$ is a continuous operator iff $z \in r(T)$; in this case $\|(T - zI)^{-1}\| \cong c(z)^{-1}$;

(ii) if $\Re(T - zI)$ is closed for one $z \in r(T)$, then T is closed;

(iii) $r(T)$ is open.

If in addition T is closed, then for all $z \in r(T)$

(iv) $\Re(T - zI)$ is closed;

(v) $\Re(T - zI) = \Re(T^+ - zI)^\perp$;

(vi) $\Re(T^+ - \bar{z}I) = \mathfrak{H}$.

Proof. (i) and (iii) can be proved in the same way as in the operator case; see e.g. [1], Nr. 100. With a light modification of the proof of Theorem 3.1 in [11] we get the other claims.

2.2. Reduction to an operator. A useful method of studying linear relations is to reduce them to operators. To do this we need first some facts about the (*purely*) *multi-valued part* $T_\infty := \{(0, g) \in T\}$ of a linear relation T .

Proposition 2.3. *If T is a linear relation from \mathfrak{H} into \mathfrak{K} , then*

(i) $T_\infty = \{0\} \oplus T(0)$;

(ii) $\mathfrak{D}(T_\infty) = \{0\}$, $\Re(T_\infty) = T(0)$;

(iii) $(T_\infty)^\perp = \mathfrak{H} \oplus T(0)^\perp$, $(T_\infty)^+ = T(0)^\perp \oplus \mathfrak{H}$;

(iv) $\overline{T(0)} = \mathfrak{D}(T^+)^\perp$, $\overline{\mathfrak{D}(T)} = T^+(0)^\perp$;

(v) T_∞ is closed/ortho-complemented iff $T(0)$ is closed/ortho-complemented;

(vi) T_∞ is non-degenerate iff $T(0)$ is non-degenerate iff $T \cap (T_\infty)^\perp$ is an operator.

If in addition $\mathfrak{K} = \mathfrak{H}$, then

(vii) T_∞ is symmetric;

(viii) $(T - zI)_\infty = T_\infty$, $z \in \mathbb{C}$.

Proof. (i)–(iv) and (vii)–(viii) are obvious from the corresponding Hilbert space results; see [6] and [11]. (v) and (vi) follow from (i), (iii) and from the easily verifiable identities

$$(T_\infty)^{\perp\perp} = \{0\} \oplus T(0)^{\perp\perp},$$

$$(T_\infty)^0 = \{0\} \oplus T(0)^0 = \{0\} \oplus (T \cap (T_\infty)^\perp)(0).$$

Theorem 2.4. *Let T be a linear relation from \mathfrak{H} into \mathfrak{K} with ortho-complemented T_∞ . Then $T_s := T \cap (T_\infty)^\perp$ is an operator with $\mathfrak{D}(T_s) = \mathfrak{D}(T)$ and $\Re(T_s) \subset T(0)^\perp$, and*

$$(2.1) \quad T = T_s[\dot{+}]T_\infty.$$

Furthermore,

- (i) T_s is closed if (iff, in case \mathfrak{H} and \mathfrak{K} are Krein spaces) T is closed;
- (ii) T_s is ortho-complemented iff T is ortho-complemented;
- (iii) $\mathfrak{R}(T) = \mathfrak{R}(T_s)[+]T(0)$.

Proof. As T_∞ is ortho-complemented, it is non-degenerate, and hence T_s is an operator according to Proposition 2.3. The decomposition (2.1) is a consequence of the decomposition $\mathfrak{H} \oplus \mathfrak{K} = T_\infty[+](T_\infty)^\perp$. The relations concerning the domain and the range as well as (i)–(iii) follows from (2.1) and Proposition 2.3. The “iff” in (i) follows from [16], Lemma 5.1.

As to the uniqueness of the decomposition given above we have

Proposition 2.5. *Let T be a linear relation from \mathfrak{H} into \mathfrak{K} and S an operator from \mathfrak{H} into \mathfrak{K} such that $T = S[+]T_\infty$. Then $S = T_s$ iff T_∞ is non-degenerate.*

Proof. 1° If $S = T_s$, then every $(0, k) \in (T_\infty)^0$ belongs to $T \cap (T_\infty)^\perp = T_s = S$ so that $k \in S(0) = \{0\}$; hence T_∞ is non-degenerate.

2° Let T_∞ be non-degenerate. As $S \subset T \cap (T_\infty)^\perp = T_s$, it is enough to prove that every $(f, g) \in T_s(\subset T)$ belongs to S . We have $(f, g) = (f, Sf) + (0, k)$ with $k \in T(0)$; hence

$$\begin{aligned} [k|l] &= [g - Sf|l] = [(f, g) - (f, Sf)|(0, l)] \\ &= [(f, g)|(0, l)] = 0 \end{aligned}$$

for all $l \in T(0)$, because $S \perp T_\infty$ and $T_s \perp T_\infty$. This means that $k \in T(0) \cap T(0)^\perp = \{0\}$ (by Proposition 2.3), but then $(f, g) = (f, Sf) \in S$.

The preceding theorem gives a reduction of a linear relation T in \mathfrak{H} to the operator T_s in \mathfrak{H} , but usually \mathfrak{H} is too “large”. The following result tells us when T_s is an operator in $T(0)^\perp$.

Proposition 2.6. *Let T be a linear relation in \mathfrak{H} with ortho-complemented T_∞ . If $\mathfrak{R}(T_s - zI) \subset T(0)^\perp$ for one $z \in \mathbb{C} \setminus \{0\}$, then $\mathfrak{D}(T_s) \subset T(0)^\perp$. Conversely, if $\mathfrak{D}(T_s) \subset T(0)^\perp$, then $\mathfrak{R}(T_s - zI) \subset T(0)^\perp$ for all $z \in \mathbb{C}$.*

Proof. For all $f \in \mathfrak{D}(T_s)$, $g \in T(0)$ and $z \in \mathbb{C} \setminus \{0\}$ we have

$$[f|g] = -z^{-1}[(T_s - zI)f|g],$$

from which the result follows.

Next we shall study how much information is retained by this reduction to an operator.

Theorem 2.7. *Let T be a linear relation in \mathfrak{H} such that T_∞ is ortho-complemented and $\mathfrak{D}(T) \subset T(0)^\perp$. Regard T_s as an operator in $T(0)^\perp$ and let $z \in \mathbb{C}$. Then*

- (i) $T - zI = (T_s - zI)[\dot{+}]T_\infty$;
- (ii) $\Re(T - zI) = \Re(T_s - zI)[\dot{+}]T(0)$;
- (iii) $\Re(T_s - zI)$ is closed if (iff, in a Krein space) $\Re(T - zI)$ is closed;
- (iv) $\Re(T_s - zI)$ is ortho-complemented iff $\Re(T - zI)$ is ortho-complemented;
- (v) $\Re(T - zI) = \Re(T_s - zI)$;
- (vi) $\sigma_p(T) = \sigma_p(T_s)$ and for $z \notin \sigma_p(T)$ the relation $(T_s - zI)^{-1}$ is an operator, and

$$(T - zI)^{-1} = (T_s - zI)^{-1}[\dot{+}]O_{T(0)},$$

where $O_{T(0)}$ denotes the zero operator in $T(0)$.

In addition let \mathfrak{H} be a Krein space. Then

- (vii) $r(T) = r(T_s)$ in case $T(0) \subset \mathfrak{H}_+$ for a fundamental decomposition (1.1) of \mathfrak{H} ;
- (viii) $\sigma(T) = \sigma(T_s)$ in case T is closed.

The proof is mostly a direct calculation.

The following results show that we can use this reduction to investigate all the interesting linear relations.

Lemma 2.8. *Let T be a dissipative linear relation in \mathfrak{H} . Then $\mathfrak{D}(T) \subset T(0)^\perp$.*

Proof. Let $f \in \mathfrak{D}(T)$ and $k \in T(0)$ be arbitrary and let $g \in T(f)$. Then $zk + g \in T(0) + g = T(f)$ for all $z \in \mathbb{C}$. Hence $(f, zk + g) \in T$ and so

$$0 \leq \operatorname{Im}(z[k|f]) + \operatorname{Im}[g|f].$$

Thus $-\operatorname{Im}(z[k|f])$ is bounded above with the non-negative constant $\operatorname{Im}[g|f]$ for all $z \in \mathbb{C}$, but this is possible only if $[k|f] = 0$, i.e. $T(0) \perp \mathfrak{D}(T)$.

Theorem 2.9. *Let T be a linear relation in \mathfrak{H} such that T_∞ is ortho-complemented. Then*

- (i) T is a dissipative relation iff T_s is a dissipative operator in $T(0)^\perp$;
- (ii) T is a symmetric relation iff T_s is a symmetric operator in $T(0)^\perp$;
- (iii) T is a self-adjoint relation iff T_s is a densely defined self-adjoint operator in $T(0)^\perp$;
- (iv) T is a contractive relation iff $T(0)$ is non-positive and T_s is a contractive operator;
- (v) T is an isometric relation iff T is an isometric operator.

Proof. (i) If T_s is a dissipative operator in $T(0)^\perp$, then

$$\operatorname{Im}[g|f] = \operatorname{Im}[T_s f + k|f] = \operatorname{Im}[T_s f|f] \geq 0$$

for all $(f, g) = (f, T_s f) + (0, k) \in T = T_s[\dot{+}]T_\infty$. Conversely, if T is a dissipative relation, it is clear that T_s is a dissipative operator in $T(0)^\perp$ (see Lemma 2.8).

(ii) Suppose that T_s is symmetric in $T(0)^\perp$. Then

$$T_s \subset (T_s)^+, \quad T_\infty \subset (T_\infty)^+, \quad T_s \subset T(0)^\perp \oplus \mathfrak{H} = (T_\infty)^+$$

(see Proposition 2.3). These together imply that $T = T_s[\dot{+}]T_\infty$ is symmetric. The other half follows from (i) and the fact $T_s \subset T$.

(iii) If T is self-adjoint, then T_s is symmetric by (ii) and $\mathfrak{D}(T_s) = \mathfrak{D}(T) = \mathfrak{D}(\bar{T})$ is dense in $T(0)^\perp$ by Proposition 2.3. Hence the adjoint of T_s exists as an operator in the space $T(0)^\perp$ and $T_s \subset (T_s)^\perp$. For all $h \in \mathfrak{D}((T_s)^\perp)$ and $(T_s f + k, g) \in T = T_s[\dot{+}]T_\infty$ one obtains

$$[T_s f + k | h] = [T_s f | h] = [f | (T_s)^\perp h],$$

and so $(h, (T_s)^\perp h) \in T^+ = T$. This implies $h \in \mathfrak{D}(T) = \mathfrak{D}(T_s)$, i.e. $(T_s)^\perp = T_s$.

Conversely, if T_s is a self-adjoint operator in $T(0)^\perp$, then $T \subset T^+$ by (ii). Furthermore, we have

$$[T_s f + k | h] = [f | l]$$

for all $(h, l) \in T^+$ and $(f, T_s f + k) \in T_s[\dot{+}]T_\infty$. Especially for $k = 0$,

$$[T_s f | h] = [f | l_2],$$

where $l = l_1 + l_2 \in T(0)[\dot{+}]T(0)^\perp$; hence $h \in \mathfrak{D}((T_s)^\perp)$ and $(T_s)^\perp h = T_s h = l_2$. This means that $(h, l) = (h, T_s h + l_1)$ is in $T_s[\dot{+}]T_\infty = T$.

(iv) If T is a contractive relation, then

$$[T_s f + k | T_s f + k] \leq [f | f]$$

for all $f \in \mathfrak{D}(T) = \mathfrak{D}(T_s)$, $k \in T(0)$. By choosing $k = 0$ or $f = 0$ we get the result. The converse follows similarly.

(v) Let T be an isometric relation. Then we can replace in the proof of (iv) the inequality with the equality. This shows that T_s is isometric and $T(0)$ is neutral. But T_∞ is non-degenerate and hence $T(0)$ must be $\{0\}$ by Proposition 2.3.

Note that as $T(0)$, and with it also $T(0)^\perp$, is ortho-complemented, $T(0)^\perp$ is a Krein space/a Pontrjagin space if \mathfrak{H} is a Krein space/a Pontrjagin space.

2.3. Reduction to a relation in a Hilbert space. In the previous section we reduced a linear relation T in \mathfrak{H} to an operator. Here we are going to show another reduction, which is more usable in the Krein space case.

Theorem 2.10. *Let T be a linear relation in a Krein space \mathfrak{H} such that $\mathfrak{D}(T) \supset \mathfrak{H}_-$ for a fundamental decomposition (1.1) of \mathfrak{H} . Let J be the corresponding fundamental symmetry. Define*

$$(2.2) \quad S := TJ = \{(f, g) | (Jf, g) \in T\}.$$

Then S is a linear relation in \mathfrak{H} with $\mathfrak{D}(S) = \mathfrak{D}(T)$, $S(0) = T(0)$ and

$$(2.3) \quad T = S + 2TP_-,$$

where P_- is the fundamental projector belonging to (1.1). Furthermore,

- (i) S is closed iff T is closed;
- (ii) T is dissipative/maximal dissipative/symmetric/maximal symmetric/self-adjoint iff S is J -dissipative/maximal J -dissipative/ J -symmetric/maximal J -symmetric/ J -self-adjoint.

Proof. Since $\mathfrak{D}(T) \supset \mathfrak{H}_-$ and $J^2 = I$ it follows that $\mathfrak{D}(S) = \mathfrak{D}(T)$ and $S(0) = T(0)$.

Let us prove (2.3). Every $(f, g) \in S + 2TP_-$ is of the form $(h, k + l)$, where $(Jh, k), (2P_-h, l) \in T$; then

$$(f, g) = (h, k + l) = (Jh, k) + (2P_-h, l) \in T.$$

Conversely, if $(f, g) \in T$, then $2P_-f, Jf \in \mathfrak{D}(T)$. Hence $(2P_-f, l) \in T$ or $(f, l) \in 2TP_-$ for some $l \in \mathfrak{H}$ and $(Jf, k) \in T$ or $(f, k) \in S$ for some $k \in \mathfrak{H}$. These relations imply $(f, k + l) = (Jf, k) + (2P_-f, l) \in T$ and further $(0, g - k - l) = (f, g) - (f, k + l) \in T$. So $(Jf, g - l) = (Jf, k) + (0, g - k - l)$ is in T or $(f, g - l)$ is in S . Putting these facts together we see that $(f, g) = (f, (g - l) + l)$ is in $S + 2TP_-$.

(i) follows from the facts that J is continuous and $T = SJ$. (ii) is a direct calculation, which uses relation (1.2) and the basic properties of the fundamental symmetry J .

In case $S_\infty (= T_\infty)$ is J -ortho-complemented, i.e. closed, we can go a step further using Theorem 2.4 and reduce T to the operator S_s :

$$T = (S_s(\dot{+})S_\infty) + 2TP_-;$$

here $(+)$ means a sum which is orthogonal with respect to a J -inner product. We can put this decomposition in a more usable form:

Theorem 2.11. *Let T be a linear relation in a Krein space \mathfrak{H} such that T_∞ is closed and $\mathfrak{D}(T) \supset H_-$ for a fundamental decomposition (1.1) of \mathfrak{H} with the fundamental symmetry $J := P_+ - P_-$. Define $S := TJ$, $S_s := S \cap (S_\infty)^{\perp}$ and $A := -2S_sP_-$. Then*

$$(2.3') \quad T = S + A, \quad T_s = S_s + A.$$

Furthermore, parts (i) and (ii) of Theorem 2.10 remain true.

Proof. We verify the first identity in (2.3'); the second follows similarly. All other claims are obvious in the light of Theorem 2.10. Let $(f, g) \in T$. Then $(Jf, g) \in S = S_s(\dot{+})S_\infty$ and so $(Jf, g) = (h, S_s h) + (0, k)$ with $k \in S(0)$. Because $h = Jf$ and $J - I = -2P_-$, we get

$$g = S_s((J - I) + I)f + k = Af + S_s f + k.$$

Hence $(f, g) = (f, (S_s f + k) + Af)$ with $(f, S_s f + k) \in S_s(\dot{+})S_\infty = S$ and so $(f, g) \in S + A$.

Conversely, let $(f, g) \in S + A$; then $(f, g) = (h, S_s h + k + Ah)$ with $k \in S(0)$. In this case

$$g = S_s f - 2S_s P_- f + k = S_s Jf + k,$$

which implies that the vector $(Jf, g) = (Jf, S_s Jf) + (0, k)$ is in $S_s(\dagger)S_\infty = S$, i.e. $(f, g) \in T$.

Note that this theorem gives us (at least theoretically) the possibility to use the perturbation theory in studying a linear relation T in a Krein space: T can be regarded as the corresponding Hilbert space relation S perturbed by the operator A .

2.4. The Cayley transformation. Following [11] we define for $z \in \mathbb{C}$ the Cayley transform $C_z(T)$ of a linear relation T in \mathfrak{H} by

$$C_z(T) := \{(g - zf, g - \bar{z}f) \mid (f, g) \in T\}$$

and the inverse transform $F_z(T)$ of T by

$$F_z(T) := \{(g - f, zg - \bar{z}f) \mid (f, g) \in T\}.$$

Then $C_z(T)$ and $F_z(T)$ are linear relations in \mathfrak{H} with $\mathfrak{D}(C_z(T)) = \Re(T - zI)$, $\mathfrak{D}(F_z(T)) = \Re(T - I)$ and $\Re(C_z(T)) = \Re(T - \bar{z}I)$, $\Re(F_z(T)) = \Re(zT - \bar{z}I)$. For future use we quote the following result from [11]:

Lemma 2.12. *Let T and S be linear relations in \mathfrak{H} and $z \in \mathbb{C} \setminus \mathbb{R}$. Then*

- (i) $T = C_z(F_z(T)) = F_z(C_z(T))$;
- (ii) $T \subset S$ iff $C_z(T) \subset C_z(S)$ iff $F_z(T) \subset F_z(S)$;
- (iii) $C_{-z}(T) = C_z(-T)$, $F_{-z}(T) = -F_z(T)$;
- (iv) $C_{\bar{z}}(T) = C_z(T)^{-1}$, $F_{\bar{z}}(T) = F_z(T^{-1})$;
- (v) $C_z(T^+) = C_z(T)^+$, $F_z(T^+) = F_z(T)^+$;
- (vi) $C_z(T \dagger S) = C_z(T) \dagger C_z(S)$, $F_z(T \dagger S) = F_z(T) \dagger F_z(S)$ and the sums are direct iff the sum $T \dagger S$ is direct;
- (vii) $C_z(T)(0) = \Re(T - zI)$, $F_z(T)(0) = \Re(T - I)$;
- (viii) $\mathfrak{D}(T) = \Re(C_z(T) - I) = \Re(F_z(T) - zI)$;
- (ix) $\Re(T) = \Re(C_z(T) - \bar{z}z^{-1}I) = \Re(F_z(T) - \bar{z}I)$;
- (x) $T(0) = \Re(C_z(T) - I) = \Re(F_z(T) - zI)$;
- (xi) in case \mathfrak{H} is a Krein space, T is closed iff $C_z(T)$ is closed iff $F_z(T)$ is closed.

Let z be a non-real complex number. We define a mapping c_z in the extended complex plane $\bar{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ as follows:

$$(2.4) \quad c_z(w) := \begin{cases} (w - \bar{z})(w - z)^{-1}, & \text{if } w \in \mathbb{C} \setminus \{z\}, \\ \infty, & \text{if } w = z, \\ 1, & \text{if } w = \infty. \end{cases}$$

We agree to say that ∞ is an eigenvalue of a linear relation T if $0 \in \sigma_p(T^{-1})$, i.e. T is not an operator. Then the spectral mapping theorem is true (compare with [2] and [18]):

Proposition 2.13. *Let T be a linear relation in \mathfrak{S} and $z \in \mathbb{C} \setminus \mathbb{R}$. Then*

$$c_z(\sigma(T)) = \sigma(C_z(T)).$$

Moreover, eigenvalues correspond to eigenvalues.

The proof is clear if one uses Lemma 2.12 and the relation

$$(2.5) \quad \Re(C_z(T) - c_z(w)I) = \Re(T - zI), \quad w \in \mathbb{C}.$$

We denote by \mathbb{C}_+ (\mathbb{C}_-) the open upper (lower) half-plane of the complex plane \mathbb{C} . As to the Hilbert space case of the following two results, see [11].

Theorem 2.14. *Let T be a linear relation in \mathfrak{S} .*

(i) *If T is dissipative, then $C_z(T)$ is contractive for all $z \in \mathbb{C}_-$ and an operator for all $z \in \mathbb{C} \setminus \sigma_p(T)$.*

(ii) *If T is symmetric, then $C_z(T)$ is isometric for all $z \in \mathbb{C}$ and an operator for all $z \in \mathbb{C} \setminus \sigma_p(T)$.*

(iii) *If \mathfrak{H} is a Krein space and T is self-adjoint, then $C_z(T)$ is a unitary operator for all $z \in \varrho(T)$.*

Proof. The identity

$$[g - zf | g - zf] - [g - \bar{z}f | g - \bar{z}f] = -4(\operatorname{Im} z)[g | f]$$

for all $(f, g) \in T$ and Lemma 2.12 imply (i) and (ii).

Because $\Re(T - zI) = \mathfrak{S}$ and $\Re(T - zI) = \{0\}$ in (iii), we have, by Proposition 2.1, $\mathfrak{D}(C_z(T)) = \Re(T - zI) = \mathfrak{S}$ and

$$\begin{aligned} \Re(C_z(T)) &= \Re(T - \bar{z}I) = \Re((T - zI)^+) = \Re((T - zI)^+)^{\perp\perp} \\ &= \Re(T - zI)^{\perp} = \mathfrak{S}; \end{aligned}$$

hence $C_z(T)$ is unitary by (ii).

The most remarkable fact here is that the Cayley transform is not always an operator as in the Hilbert space case. Theorem 2.14 has the following converse.

Theorem 2.15. *Let T be a linear relation in \mathfrak{S} .*

(i) *If T is contractive, then $F_z(T)$ is dissipative for all $z \in \mathbb{C}_-$.*

(ii) *If T is isometric, then $F_z(T)$ is symmetric for all $z \in \mathbb{C}$.*

(iii) *If T is unitary, then $F_z(T)$ is self-adjoint for all $z \in \mathbb{C} \setminus \mathbb{R}$.*

Proof. The identity

$$\operatorname{Im} [zg - \bar{z}f | g - f] = (\operatorname{Im} z)([g | g] - [f | f])$$

for all $(f, g) \in T$ implies (i) and (ii). (iii) follows from (ii) and from the facts

$$\Re(F_z(T) - zI) = \mathfrak{D}(T) = \mathfrak{S} = \Re(T) = \Re(F_z(T) - \bar{z}I)$$

for all $z \in \mathbb{C} \setminus \mathbb{R}$; see Lemma 2.12.

2.5. The relation matrix. In this section we suppose that \mathfrak{H} is a Krein space. Let T be a linear relation in \mathfrak{H} . Extending the notion of the operator matrix we say that T is represented by the relation matrix

$$(2.6) \quad \begin{bmatrix} T_{++} & T_{-+} \\ T_{+-} & T_{--} \end{bmatrix}$$

with respect to the fundamental decomposition (1.1) of \mathfrak{H} if $T_{\pm\pm}$ is a relation from \mathfrak{H}_{\pm} into \mathfrak{H}_{\pm} and

$$(2.7) \quad T = (T_{++} \uparrow T_{-+}) + (T_{+-} \uparrow T_{--}).$$

The basic properties of a relation matrix are given by the following result:

Theorem 2.16. *Let T be a linear relation in a Krein space \mathfrak{H} such that $\mathfrak{D}(T) \supset \mathfrak{H}_{-}$ for a fundamental decomposition (1.1) of \mathfrak{H} . Define*

$$(2.8) \quad \begin{aligned} T_{++} &:= P_+(T \cap (\mathfrak{H}_+ \oplus \mathfrak{H})) = \{(f, P_+g) \in \mathfrak{H}_+ \oplus \mathfrak{H}_+ \mid (f, g) \in T\}, \\ T_{-+} &:= P_+(T \cap (\mathfrak{H}_- \oplus \mathfrak{H})) = \{(f, P_+g) \in \mathfrak{H}_- \oplus \mathfrak{H}_+ \mid (f, g) \in T\}, \\ T_{+-} &:= P_-(T \cap (\mathfrak{H}_+ \oplus \mathfrak{H})) = \{(f, P_-g) \in \mathfrak{H}_+ \oplus \mathfrak{H}_- \mid (f, g) \in T\}, \\ T_{--} &:= P_-(T \cap (\mathfrak{H}_- \oplus \mathfrak{H})) = \{(f, P_-g) \in \mathfrak{H}_- \oplus \mathfrak{H}_- \mid (f, g) \in T\}, \end{aligned}$$

where P_{\pm} are the fundamental projectors belonging to (1.1). Then T is represented by the relation matrix (2.6), whose components are given by (2.8). Furthermore,

- (i) $\mathfrak{D}(T_{++}) = \mathfrak{D}(T_{-+}) = \mathfrak{D}(T) \cap \mathfrak{H}_+$, $\mathfrak{D}(T_{+-}) = \mathfrak{D}(T_{--}) = \mathfrak{H}_-$;
- (ii) $P_+T(0) = T_{++}(0) = T_{-+}(0)$, $P_-T(0) = T_{+-}(0) = T_{--}(0)$;
- (iii) for every $(f, g) \in T$ there exist $f_{\pm} \in \mathfrak{H}_{\pm}$, $g_{\pm} \in \mathfrak{H}$ and $l \in T(0)$ such that $f = f_+ + f_-$, $g = g_+ + g_- + l$ and

$$\begin{aligned} (f_+, P_+g_+) &\in T_{++}, & (f_-, P_+g_-) &\in T_{-+}, \\ (f_+, P_-g_+) &\in T_{+-}, & (f_-, P_-g_-) &\in T_{--}; \end{aligned}$$

$$(iv) \quad T = (T_{++} \uparrow T_{-+}) \uparrow (T_{+-} \uparrow T_{--}).$$

If in addition $T(0)$ is closed and $T(0) \subset \mathfrak{H}_+$, then

- (v) $(T_{++})_s = (T_s)_{++}$ and $T_{++}(0) = T(0)$;
- (vi) $(T_{-+})_s = (T_s)_{-+}$ and $T_{-+}(0) = T(0)$;
- (vii) T_{+-} is an operator and $T_{+-} = (T_s)_{+-}$;
- (viii) T_{--} is an operator and $T_{--} = (T_s)_{--}$.

Proof. (i) and (ii) are obvious from the definitions. (iii) follows easily if one defines $f_{\pm} := P_{\pm}f$ ($\in \mathfrak{D}(T)$), chooses $g_{\pm} \in \mathfrak{H}$ such that $(f_+, g_+), (f_-, g_-) \in T$, and puts $l := g - g_+ - g_-$.

The verifications of (2.7) and (iv) are direct computations, which use (i)—(iii) and which can be split into the following parts: $T = P_+T + P_-T$, $P_+T = T_{++} \uparrow T_{-+}$ and similarly for P_-T , $T \cap (\mathfrak{H}_+ \oplus \mathfrak{H}) = T_{++} + T_{+-}$ and similarly for $T \cap (\mathfrak{H}_- \oplus \mathfrak{H})$, and finally $T = T \cap (\mathfrak{H}_+ \oplus \mathfrak{H}) \uparrow T \cap (\mathfrak{H}_- \oplus \mathfrak{H})$.

If in addition $T(0)$ is closed and in \mathfrak{H}_+ , then T is ortho-complemented and $T_s = T \cap (T_\infty)^\perp$ an operator (see Proposition 2.3 and Theorem 2.4). From (i) it follows that $T_{++}(0) = T_{-+}(0) = T(0)$ as well as that T_{+-} and T_{--} are operators. From the identities

$$\begin{aligned}(T_s)_{++} &= P_+(T_s \cap (\mathfrak{H}_+ \oplus \mathfrak{H})) = P_+(T \cap (\mathfrak{H}_+ \oplus T(0)^\perp)), \\ (T_{++})_s &= T_{++} \cap [(T_{++})_\infty]^\perp \cap \mathfrak{H}_+ \\ &= [P_+(T \cap (\mathfrak{H}_+ \oplus \mathfrak{H}))] \cap [\mathfrak{H}_+ \oplus (T(0)^\perp \cap \mathfrak{H}_+)]\end{aligned}$$

we see that $(T_s)_{++} \subset (T_{++})_s$. Furthermore, they are both operators with the same domain $\mathfrak{D}(T) \cap \mathfrak{H}_+$; thus $(T_s)_{++} = (T_{++})_s$. (vi)—(viii) are proved similarly.

3. Dissipative and maximal dissipative relations

3.1. Dissipative relations. Almost all the properties of a dissipative linear relation T in a Hilbert space are based on the inequality

$$(3.1) \quad \|g - zf\| \cong (\operatorname{Im} \bar{z}) \|f\|$$

for all $(f, g) \in T$, i.e. $C_- \subset r(T)$; see [11] (note that our definition of dissipativeness differs from that in [11]). In a general inner product space this inequality is not necessarily true even in the case of operators. In order to get an analogous but weaker form of this result we suppose that our space is a Pontrjagin space.

Theorem 3.1. *Let T be a dissipative linear relation in a Pontrjagin space \mathfrak{H} such that $T(0)$ is closed and $\mathfrak{D}(T) \supset \mathfrak{H}_-$ for a fundamental decomposition (1.1) of \mathfrak{H} . Then there exists a constant $c_T \geq 0$ such that*

$$(3.2) \quad \|g - zf\| \cong (\operatorname{Im} \bar{z} - c_T) \|f\|$$

for all $(f, g) \in T$, i.e. $z \in r(T)$ for all $z \in C_-$ with $\operatorname{Im} \bar{z} > c_T$.

Proof. According to Theorem 2.11 we can write $T = S + A$ with $S := TJ$ and $A := -2S_s P_-$. As $\mathfrak{R}(P_-) = \mathfrak{H}_-$ is finite-dimensional the operator A is bounded; we put $c_T := \|A\|$. (Only here we need the assumption that \mathfrak{H} is a Pontrjagin space.) The operator $A + ic_T I$ is J -dissipative; indeed, for all $f \in \mathfrak{D}(A)$ we have

$$\begin{aligned}\operatorname{Im}(Af + ic_T f | f) &= \operatorname{Im}(Af | f) + c_T \|f\|^2 \\ &\cong -\|Af\| \|f\| + \|A\| \|f\|^2 \geq 0.\end{aligned}$$

As S is J -dissipative it follows that $S - (\operatorname{Re} z)I$ is also J -dissipative. In this case the relation

$$T - (\operatorname{Re} z - ic_T)I = (S - (\operatorname{Re} z)I) + (A + ic_T I)$$

is also J -dissipative as a sum of two J -dissipative relations. So we can use the inequality (3.1) to get

$$\begin{aligned} \|g - zf\| \|f\| &\geq \operatorname{Im}(g - zf|f) \\ &= \operatorname{Im}(g - (\operatorname{Re} z - ic_T)f|f) + \operatorname{Im}(\operatorname{Re} z - ic_T - z)\|f\|^2 \\ &\geq (\operatorname{Im} \bar{z} - c_T)\|f\|^2 \end{aligned}$$

for all $(f, g) \in T$; hence the result.

The next result gives a sufficient condition for a dissipative linear relation to have dissipative extensions with a given eigenvalue. Later we shall show the converse for symmetric linear relations.

Theorem 3.2. *Let T be a dissipative linear relation in an inner product space \mathfrak{H} and let $z \in \mathbb{C}$.*

(i) *If $\mathfrak{R}(T - \bar{z}I)^\perp$ contains a non-zero vector f such that $(\operatorname{Im} z)[f|f] \geq 0$, then T has a dissipative extension T' in the original space with $z \in \sigma_p(T')$;*

(ii) *if $\mathfrak{R}(T - \bar{z}I)^\perp$ contains a non-zero vector f such that $(\operatorname{Im} z)[f|f] > 0$, then T has a dissipative extension T' in a larger space \mathfrak{H}' with $z \in \sigma_p(T')$ and at least one eigenvector does not belong to \mathfrak{H} .*

Proof. We can follow the proof given in the operator case; see [5]. In case (i) we define $T' := \langle T, (f, zf) \rangle$, where the symbol $\langle \dots \rangle$ denotes the subspace spanned by the set $\{ \dots \}$. Then $T' \supset T$, $z \in \sigma_p(T')$ and a direct calculation suffices to show that T' is dissipative. In case (ii) we take an element e which does not belong to \mathfrak{H} and define $\mathfrak{H}' := \langle \mathfrak{H}, e \rangle$ with the inner product $[\cdot | \cdot]'$:

$$[g|e]' := \begin{cases} 0 & \text{for } g \in \mathfrak{H}, \\ -[f|f] & \text{for } g = e. \end{cases}$$

Then the relation

$$T' := \langle T, (f + e, z(f + e)) \rangle$$

meets our requirements.

3.2. Maximal dissipative relations. The most interesting extension of a dissipative linear relation in the original space is the maximal one. The following results give some information about maximal dissipative linear relations; for the Hilbert space case see [11].

Theorem 3.3. *Let T be a maximal dissipative linear relation in a Krein space \mathfrak{H} . Then*

(i) *T is closed;*

(ii) *$\mathfrak{R}(T - \bar{z}I)^\perp$ is positive if $\mathfrak{R}(T - \bar{z}I)$ is non-degenerate for $z \in \mathbb{C}_-$.*

Proof. (i) It is quite clear that the closure of T is a closed dissipative extension of T ; hence $\bar{T} = T$. (ii) If there exists a non-positive vector $f \in \mathfrak{R}(T - \bar{z}I)^\perp$, then $V := \langle C_z(T), (0, f) \rangle$ is a contractive extension of $C_z(T)$. So $F_z(V)$ is a dissipative extension of T , which implies that $F_z(V) = T$ or $V = C_z(T)$. Hence $f \in \mathfrak{R}(C_z(T)) = \mathfrak{R}(T - \bar{z}I)$, but $\mathfrak{R}(T - \bar{z}I)$ is non-degenerate and so $f = 0$.

Theorem 3.4. *Let T be a dissipative linear relation in a Pontrjagin space \mathfrak{H} such that $T(0)$ is closed and $\mathfrak{D}(T) \supset \mathfrak{H}_-$.*

- (i) *If T is maximal, then $\mathfrak{R}(T-zI) = \mathfrak{H}$ for all $z \in \mathbb{C}$ with $\text{Im } \bar{z} > c_T$;*
- (ii) *if $\mathfrak{R}(T-zI) = \mathfrak{H}$ for $z \in \mathbb{C}$ with $\text{Im } \bar{z} > 2c_T$, then T is maximal.*

Proof. (i) According to the theorems 2.10 and 2.11 we can write $T = S + A$, where S is a maximal J -dissipative linear relation and A is a bounded operator with $\|A\| = c_T$. Especially $(S-zI)^{-1}$ is an everywhere defined bounded operator with the norm $\leq 1/\text{Im } \bar{z}$ for $z \in \mathbb{C}_-$; see (3.1) and Theorem 2.2. If $\text{Im } \bar{z} > c_T$, then

$$\|A\| = c_T < \text{Im } \bar{z} \leq 1/\|(S-zI)^{-1}\|,$$

and therefore the operator $I + A(S-zI)^{-1}$ has an everywhere defined bounded inverse. In this case

$$T - zI = (S - zI) + A = (I + A(S - zI)^{-1})(S - zI)$$

or

$$(T - zI)^{-1} = (S - zI)^{-1}(I + A(S - zI)^{-1})^{-1},$$

but the operator on the right side is everywhere defined and so $\mathfrak{R}(T-zI) = \mathfrak{H}$.

(ii) As in (i) $T = S + A$, where S is J -dissipative. Furthermore, by the assumption and by the theorems 2.2 and 3.1 the operator $(T-zI)^{-1}$ is everywhere defined and $\|(T-zI)^{-1}\| \leq 1/(\text{Im } \bar{z} - c_T)$. But then $\|A\| = c_T < \text{Im } \bar{z} - c_T \leq 1/\|(T-zI)^{-1}\|$, and hence we can proceed as in (i) to get

$$(S - zI)^{-1} = (T - zI)^{-1}(I - A(T - zI)^{-1})^{-1},$$

which implies $\mathfrak{R}(S-zI) = \mathfrak{H}$. From [11] it follows now that S is maximal J -dissipative and so T is maximal by Theorem 2.11.

Corollary 3.5. *Let T , \mathfrak{H} and z be as in Theorem 3.4, part (i). Then $z \in \rho(T)$ and $\|R(z)\| \leq (\text{Im } \bar{z} - c_T)^{-1}$ for the resolvent $R(z) := (T - zI)^{-1}$ of T .*

4. Symmetric linear relations

4.1. Eigenvalues, points of regular type. Since every symmetric linear relation T is dissipative, we know from Theorem 3.1 (under the other assumptions made there) that all $z \in \mathbb{C}$ with $\text{Im } \bar{z} > c_T$ and (for reasons of symmetry) with $\text{Im } \bar{z} < -c_T$ are points of regular type of T . However, by using better results of the perturbation theory we can improve this. To do so we first examine the eigenvalues of T .

Theorem 4.1. *Let T be a symmetric linear relation in an inner product space \mathfrak{H} . Then*

- (i) $\mathfrak{R}(T-zI)$ and $\mathfrak{R}(T-wI)$ are orthogonal for $z, w \in \sigma_p(T)$, $z \neq \bar{w}$;
- (ii) $\mathfrak{R}(T-zI)$ is neutral for $z \in \sigma_p(T) \setminus \mathbb{R}$.

In addition, let \mathfrak{H} be a π_α -space and let T_∞ be ortho-complemented. Then

- (iii) the number of the eigenvalues of T belonging to C_+ (resp., C_-) is at most \varkappa ;
- (iv) $\sigma(T) \setminus \mathbf{R} = \sigma_p(T) \setminus \mathbf{R}$ is symmetric with respect to \mathbf{R} for self-adjoint T .

Proof. (i) If $f \in \mathfrak{R}(T - zI)$ and $g \in \mathfrak{R}(T - wI)$, then $(f, zf), (g, wg) \in T$ and so

$$0 = [zf|g] - [f|wg] = (z - \bar{w})[f|g].$$

(ii) follows immediately from (i). (iii)—(iv) are direct consequences of the theorems 2.7 and 2.9 and of the fact that the corresponding results are valid for operators; see [5]. Note that $T(0)^\perp$ is also a $\pi_{\varkappa'}$ -space with $\varkappa' \cong \varkappa$.

We can now prove the converse of Theorem 3.2 for symmetric linear relations.

Theorem 4.2. *Let T be a symmetric linear relation in an inner product space \mathfrak{H} and let $z \in \mathbf{C} \setminus \mathbf{R}$.*

(i) $\mathfrak{R}(T - \bar{z}I)^\perp$ contains a non-zero neutral vector iff T has a symmetric extension T' in the original space with $z \in \sigma_p(T')$.

(ii) $\mathfrak{R}(T - \bar{z}I)^\perp$ contains a non-zero negative vector iff T has a symmetric extension T' in a larger space \mathfrak{H}' , which includes \mathfrak{H} as an ortho-complemented subspace with \mathfrak{H}^\perp positive in \mathfrak{H}' , such that $z \in \sigma_p(T')$ and at least one eigenvector does not belong to \mathfrak{H} .

Proof. The other halves of (i) and (ii) follow as in the proof of Theorem 3.2. To show the converse of (i) we take an eigenvector h belonging to $z \in \sigma_p(T')$. Then h is neutral by Theorem 4.1 and belongs to $\mathfrak{R}(T - \bar{z}I)^\perp$. Indeed, since $(h, zh) \in T'$ and T' is symmetric we have

$$[g - \bar{z}f|h] = [g|h] - [f|zh] = 0$$

for all $(f, g) \in T \subset T'$.

For the converse of (ii) we denote by P the orthogonal projector of \mathfrak{H}' onto \mathfrak{H} and let $h \notin \mathfrak{H}$ be an eigenvector belonging to $z \in \sigma_p(T')$. Then

$$0 = [h|h] = [Ph|Ph] + [(I - P)h|(I - P)h]$$

and $(I - P)h \in \mathfrak{H}^\perp$ is positive; hence Ph is negative. Furthermore, it belongs to $\mathfrak{R}(T - \bar{z}I)^\perp$, because

$$[g - \bar{z}f|Ph] = [g|h] - [f|zh] = 0$$

for all $(f, g) \in T \subset T'$.

We remark that if the space \mathfrak{H} in Theorem 4.2 is a Pontrjagin space, then (ii) in Theorem 4.2 can be given the following form: $\mathfrak{R}(T - \bar{z}I)^\perp$ contains a non-zero negative vector iff T has a regular symmetric extension T' with $z \in \sigma_p(T')$ and at least one corresponding eigenvector does not belong to \mathfrak{H} . Here a *regular extension* means an extension which is defined in a Pontrjagin space $\mathfrak{H}' \supset \mathfrak{H}$ with the same number of negative squares as in the original space \mathfrak{H} . This remark follows from the facts that then \mathfrak{H}^\perp is positive and \mathfrak{H} is ortho-complemented in \mathfrak{H}' ; see [5].

Corollary 4.3. *Let T be a symmetric linear relation in a Pontrjagin space \mathfrak{H} and let $z \in \mathbf{C} \setminus \mathbf{R}$. The subspace $\mathfrak{R}(T - \bar{z}I)^\perp$ is positive iff T does not have any regular symmetric extension T' with $z \in \sigma_p(T')$.*

Theorem 4.4. *Let T be a closed symmetric linear relation in a Pontrjagin space \mathfrak{H} such that $\mathfrak{D}(T) \supset \mathfrak{H}_-$ for a fundamental decomposition (1.1) of \mathfrak{H} . Then all non-real points $z \notin \sigma_p(T)$ are points of regular type of T .*

Proof. By Theorem 2.11 we have $T_s = S_s + A$, where S_s is a closed J -symmetric operator and $A := -2S_s P_-$. As $\mathfrak{R}(P_-) = \mathfrak{H}_-$ is finite-dimensional the operator A is S_s -completely continuous and then by [13], Theorem 9.4, all non-real $z \notin \sigma_p(T_s) = \sigma_p(T)$ are points of regular type for $S_s + A = T_s$. Theorem 2.7 then implies the result.

4.2. The relation matrix. The relation matrices introduced in Section 2.5 are useful especially in the study of symmetric linear relations and their extensions. As to the operator case, see [5] and [15].

Theorem 4.5. *Let T be a closed symmetric linear relation in a Krein space \mathfrak{H} with $\mathfrak{D}(T) \supset \mathfrak{H}_-$ and let (2.6) with the components (2.8) be a relation matrix of T . Then*

- (i) T_{++} is a closed J -symmetric linear relation in \mathfrak{H}_+ with $T_{++}(0) = T(0)$;
- (ii) T_{--} is a continuous J -self-adjoint operator in \mathfrak{H}_- ;
- (iii) T_{-+} is a closed linear relation from \mathfrak{H}_- into \mathfrak{H}_+ with $T_{-+}(0) = T(0)$, the operator part $(T_{-+})_s$ is continuous and the J -adjoint $(T_{-+})^*$ is an operator;
- (iv) T_{+-} is a continuous operator which admits a continuous closure $\overline{T_{+-}}$ with the domain $\overline{\mathfrak{D}(T)} \cap \mathfrak{H}_+$, and in addition $\overline{T_{+-}} \subset -(T_{-+})^*$ and $\|T_{+-}\| \leq \|(T_{-+})_s\|$;
- (v) T is self-adjoint in \mathfrak{H} iff T_{++} is J -self-adjoint in \mathfrak{H}_+ ; in this case $\overline{T_{+-}} = -(T_{-+})^*$.

Proof. Because $T(0)$ is closed and

$$T(0) \subset T^+(0) = \mathfrak{D}(T)^\perp \subset (\mathfrak{H}_-)^\perp = \mathfrak{H}_+,$$

we can use Theorem 2.16.

- (i) If $(f, g), (h, k) \in T_{++}$, then they are of the form

$$(f, g) = (f, P_+ l) \quad \text{with } f \in \mathfrak{H}_+, (f, l) \in T,$$

$$(h, k) = (h, P_+ m) \quad \text{with } h \in \mathfrak{H}_+, (h, m) \in T.$$

Furthermore, according to Section 1.3 the J -inner product $(\cdot | \cdot)$ in \mathfrak{H}_+ is equal to the inner product $[\cdot | \cdot]$. So we have

$$\begin{aligned} (g|h) - (f|k) &= [P_+ l|h] - [f|P_+ m] \\ &= [l|h] - [f|m] = 0, \end{aligned}$$

since T is symmetric. Thus T_{++} is J -symmetric in \mathfrak{H}_+ . The closedness of T_{++} follows in the same way as the closedness of T_{-+} ; see (iii).

- (ii) In a similar manner as in (i) it can be seen that T_{--} is J -symmetric. Furthermore, $(T_{--})^*(0) = \mathfrak{D}(T_{--})^{(\perp)} = \{0\}$ and so $(T_{--})^*$ is an operator. But then T_{--} must be J -self-adjoint and continuous.

(iii) As $T_{-+}(0)=T(0)$, the subspace $(T_{-+})_\infty$ is ortho-complemented by Proposition 2.3 and so we have the decomposition $T_{-+}=(T_{-+})_s[\perp](T_{-+})_\infty$. In addition, $(T_{-+})^*(0)=\mathfrak{D}(T_{-+})^\perp=\{0\}$, i.e. $(T_{-+})^*$ is an operator. Let us prove that T_{-+} is closed. We can write

$$T_{-+} = T \cap (\mathfrak{H}_- \oplus \mathfrak{H}) - T_{--},$$

because $T \cap (\mathfrak{H}_- \oplus \mathfrak{H}) = T_{-+} + T_{--}$ and T_{--} is an everywhere defined operator. Let (f_n, g_n) be a sequence in T_{-+} converging to $(f, g) \in \mathfrak{H}_- \oplus \mathfrak{H}_+$. Then $(f_n, g_n) = (f_n, k_n - T_{--}f_n)$ with $(f_n, k_n) \in T$. The convergence of the sequences (f_n) and (g_n) and the continuity of T_{--} imply that $k_n = g_n + T_{--}f_n$ converges to some $k \in \mathfrak{H}$. Then $(f_n, k_n) \rightarrow (f, k) \in T \cap (\mathfrak{H}_- \oplus \mathfrak{H})$ and especially $f \in \mathfrak{D}(T_{--})$. So we get

$$(f_n, g_n) = (f_n, k_n - T_{--}f_n) \rightarrow (f, k - T_{--}f),$$

i.e. $(f, g) = (f, k - T_{--}f) \in T_{-+}$. Thus T_{-+} is closed and by Theorem 2.4 $(T_{-+})_s$ is also closed. Being defined everywhere in \mathfrak{H}_- the operator $(T_{-+})_s$ is continuous.

(iv) If $(f, P_-g) \in T_{+-}$ and $(h, P_+k) \in T_{-+}$, then

$$(P_-g|h) - (f|P_+k) = -[g|h] + [f|k] = 0,$$

because T is symmetric. Hence $T_{+-} \subset -(T_{-+})^*$ and so by (iii) T_{+-} is an operator. To prove the continuity of T_{+-} take an arbitrary vector $f \in \mathfrak{D}(T_{+-})$. Then $T_{+-}f \in \mathfrak{H}_- = \mathfrak{D}(T_{-+})$ and so $(T_{+-}f, g) \in T_{-+}$ for some $g \in \mathfrak{H}_+$. This element has the decomposition

$$(T_{+-}f, g) = (T_{+-}f, (T_{-+})_s T_{+-}f + k)$$

with $k \in T(0)$. As $T_{+-} \subset -(T_{-+})^*$ and $T(0) \perp \mathfrak{D}(T_{+-})$, we have

$$\begin{aligned} \|T_{+-}f\|^2 &= (T_{+-}f|T_{+-}f) = |(f|g)| \\ &= |(f|(T_{-+})_s T_{+-}f)| \leq \|(T_{-+})_s\| \|T_{+-}f\| \|f\|, \end{aligned}$$

which implies $\|T_{+-}\| \leq \|(T_{-+})_s\|$. The properties of the extension $\overline{T_{+-}}$ are obvious.

(v) Suppose that T is self-adjoint. We should prove the inclusion $(T_{++})^* \subset T_{++}$. For this, let $(h, k) \in (T_{++})^*$ be arbitrary. If we can find a vector $g \in \mathfrak{H}_-$ such that $(h, k+g) \in T^+ = T$, then $(h, k) = (h, P_+(k+g))$, i.e. $(h, k) \in T_{++}$. Using (i), Proposition 2.3 and (iv) we get

$$\begin{aligned} \mathfrak{D}((T_{++})^*) &\subset \mathfrak{D}((T_{++})^*)^{\perp\perp} = \overline{T_{++}}(0)^\perp = T(0)^\perp \\ &= \overline{\mathfrak{D}(T)} \cap \mathfrak{H}_+ = \mathfrak{D}(\overline{T_{+-}}) \end{aligned}$$

(here the orthogonal companions are in \mathfrak{H}_+). By choosing $g := \overline{T_{+-}}h$ one can, with the help of (i), (iv) and Theorem 2.16, verify that $(h, k+g) \in T^+$ as we wanted.

Suppose that T_{++} is J -self-adjoint in \mathfrak{H}_+ . We should prove $T^+ \subset T$. Take an arbitrary $(h, k) \in T^+$. Using the assumption, (iv) and some calculations we see that (h, P_+k) is in $T_{++} \perp T_{-+}$. Similarly (h, P_-k) is seen to be in $T_{+-} \perp T_{--}$. By (2.7) the element (h, k) is then in T .

Finally we show that $\overline{T_{+-}} = -(T_{-+})^*$ in case T is self-adjoint. By (iv) it is enough to prove the inclusion $\mathfrak{D}((T_{-+})^*) \subset T(0)^\perp \cap \mathfrak{H}_+$. As above we get

$$\mathfrak{D}((T_{-+})^*) \subset \mathfrak{D}((T_{-+})^*)^{\perp\perp} = \overline{T_{-+}}(0)^\perp = T(0)^\perp.$$

Theorem 4.6. *Let T be a closed symmetric linear relation in a Pontrjagin space \mathfrak{H} with $\mathfrak{D}(T) \supset \mathfrak{H}_-$ and with the relation matrix (2.6) defined by (2.8). A linear relation T' is a regular self-adjoint extension of T iff it is represented by the relation matrix*

$$(4.1) \quad \begin{bmatrix} T'_{++} & T_{-+} + \{0\} \oplus T'_{++}(0) \\ -(T_{-+})^*|_{\mathfrak{D}(T_{-+})} & T_{--} \end{bmatrix},$$

where T'_{++} is a J -self-adjoint extension of T_{++} .

Proof. 1° Suppose that T' is a regular self-adjoint extension of T . Then T' is a self-adjoint relation in a Pontrjagin space $\mathfrak{H}' \supset \mathfrak{H}$, which has as many negative squares as \mathfrak{H} , and T' can be represented by the relation matrix

$$\begin{bmatrix} T'_{++} & T'_{-+} \\ T'_{+-} & T'_{--} \end{bmatrix},$$

where the components are defined as in (2.8). Furthermore, by Theorem 4.5 T'_{++} is J -self-adjoint in $\mathfrak{H}'_+ \supset \mathfrak{H}_+$ and $\overline{T'_{+-}} = -(T'_{-+})^*$. Note that \mathfrak{H}' has a fundamental decomposition $\mathfrak{H}' = \mathfrak{H}'_+ [+] \mathfrak{H}'_-$ with $\mathfrak{H}'_+ \supset \mathfrak{H}_+$ and $\mathfrak{H}'_- = \mathfrak{H}_-$; the corresponding fundamental projectors are denoted by P'_\pm .

We show first that T'_{++} is an extension of T_{++} . For this let $(f, g) \in T_{++} = P_+(T \cap (\mathfrak{H}_+ \oplus \mathfrak{H}))$ be arbitrary. Then $f \in \mathfrak{H}_+$ and $(f, g) = (f, P_+k)$ for some k such that $(f, k) \in T \subset T'$. As $P'_\pm|_{\mathfrak{H}} = P_\pm$, we get

$$(f, g) = (f, P_+k) \in P'_+(T' \cap (\mathfrak{H}'_+ \oplus \mathfrak{H}')) = T'_{++},$$

i.e. $T_{++} \subset T'_{++}$.

As above, it can be seen that $T_{-+} \subset T'_{-+}$, $T_{+-} \subset T'_{+-}$ and $T_{--} \subset T'_{--}$. Furthermore, T_{--} and T'_{--} are operators defined throughout in $\mathfrak{H}_- = \mathfrak{H}'_-$; hence $T_{--} = T'_{--}$.

Let us prove the identity $T'_{-+} = T_{-+} + \{0\} \oplus T'_{++}(0)$. From Theorem 2.16 and from above we get the inclusion \supset . Furthermore, it is easy to see that the linear relations on the left and right side of the desired identity have the same domain \mathfrak{H}_- and their multi-valued parts are equal. These facts imply the equation we wanted.

Finally, we must show $T'_{+-} = -(T_{-+})^*|_{\mathfrak{D}(T_{-+})}$, where the adjoint is taken in the extension space. A little calculation gives

$$\begin{aligned} (T'_{-+})^* &= (T_{-+})^* \cap (T'_{++}(0)^\perp \oplus \mathfrak{H}_-) \\ &= (T_{-+})^* \cap (\overline{\mathfrak{D}(T_{++})} \oplus \mathfrak{H}_-) = (T_{-+})^*|_{\overline{\mathfrak{D}(T_{++})}}. \end{aligned}$$

On the other hand, $\overline{T'_{+-}} = -(T'_{-+})^*$, and putting these together one gets the desired result.

2° Suppose T'_{++} is a J -self-adjoint extension of T_{++} in a Hilbert space \mathfrak{H}'_+ extending \mathfrak{H}_+ . Then the space $\mathfrak{H}' := \mathfrak{H}'_+ \oplus \mathfrak{H}_-$ equipped with the usual linear and inner product structure is a π_x -space with a fundamental decomposition

$$(4.2) \quad \mathfrak{H}' := \mathfrak{H}'_+ [+] \mathfrak{H}_-$$

We should show that the linear relation T' defined via the relation matrix (4.1) is a self-adjoint extension of T .

We verify first that $-(T_{-+})^*|_{\mathfrak{D}(T'_{++})}$ is an operator extension of T_{+-} . As $\mathfrak{D}(T_{-+}) = \mathfrak{H}_-$ we see that $(T_{-+})^*$ is an operator. Let $f \in \mathfrak{D}(T_{+-})$; then $(k|f) = -(h|T_{+-}f)$ for all $(h, k) \in T_{-+}$ by Theorem 2.16. This means $(f, -T_{+-}f) \in (T_{-+})^*$, i.e. $T_{+-} \subset -(T_{-+})^*$. Furthermore, $\mathfrak{D}(T_{+-}) = \mathfrak{D}(T_{++}) \subset \mathfrak{D}(T'_{++})$, and so we get the desired result.

To prove the inclusion $T \subset T'$, let $(f, g) \in T$ be arbitrary. Then by Theorem 2.16 $g = g_1 + g_2$ and

$$(f, g_1) = (f_1, g_{11}) + (f_2, g_{12}),$$

$$(f, g_2) = (f_1, g_{21}) + (f_2, g_{22})$$

with $(f_1, g_{11}) \in T_{++} \subset T'_{++}$, $(f_2, g_{12}) \in T_{-+}$, $(f_1, g_{21}) \in T_{+-} \subset -(T_{-+})^*|_{\mathfrak{D}(T'_{++})}$ and $(f_2, g_{22}) \in T_{--}$; hence

$$(f, g) = (f, g_1 + g_2) \in (T'_{++} \uparrow T_{-+}) + (-(T_{-+})^*|_{\mathfrak{D}(T'_{++})} \uparrow T_{--}) \subset T'.$$

In order to show the symmetry of T' , we decompose an arbitrary $(f, g) \in T'$ analogously to get

$$\begin{aligned} [g|f] &= (g_{11}|f_1) + (g_{12}|f_1) - (g_{21}|f_2) - (g_{22}|f_2) \\ &= (f_1|g_{11}) - (f_2|g_{21}) + (f_1|g_{12}) - (f_2|g_{22}) \\ &= [f|g]; \end{aligned}$$

i.e. T' is symmetric. Here we have used the following facts: $T'_{++} \subset (T_{++})^+$, $(f_2, g_{12}) \in T_{-+}$, $(f_1, -g_{21}) \in -(T_{-+})^*$ and $T_{--} \subset (T_{--})^*$.

Finally, we must show the inclusion $T'^+ \subset T'$. For this we need the identity $\mathfrak{D}(T'_{++}) = \mathfrak{D}((T'_{-+})^*|_{\mathfrak{D}(T'_{++})})$, for which in turn we need the fact that $\mathfrak{D}((T'_{-+})^*)$ is closed. To prove that fact, one can argue as in the proof of Theorem 4.5 to show that $\|(T_{-+})^*f\| \cong \|(T_{-+})_s\| \|f\|$, i.e. $(T_{-+})^*$ is continuous, which is enough. Then

$$\mathfrak{D}((T_{-+})^*) = \mathfrak{D}((T_{-+})^*)^{\perp\perp} = T_{-+}(0)^\perp = T(0)^\perp \supset T'(0)^\perp \supset \mathfrak{D}(T'_{++}),$$

from which the identity mentioned above follows.

Let P'_\pm be the fundamental projectors belonging to the decomposition (4.2) and take an arbitrary $(h, k) \in T'^+$. Then a little calculation shows that $(h, P'_+k) \in T'_{++} \uparrow T_{-+}$ and analogously $(h, P'_-k) \in -(T_{-+})^*|_{\mathfrak{D}(T'_{++})} \uparrow T_{--}$, i.e. $(h, k) \in T'$. This completes the proof.

Corollary 4.7. *Every closed symmetric linear relation T in a Pontrjagin space \mathfrak{H} with $\mathfrak{D}(T) \supset \mathfrak{H}_-$ admits regular self-adjoint extensions.*

We can put the previous theorem in a perturbational form.

Theorem 4.8. *Let T and \mathfrak{H} be as in Theorem 4.6. Every regular self-adjoint extension T' of T is of the form $T' = S' + A$, where S' is a J -self-adjoint linear relation and A is a continuous operator with $\|A\| \leq c_T$.*

Proof. From Theorem 4.6 we know that T' is represented by the relation matrix (4.1). Define $S' := T'_{++} + T'_{--}$; as T'_{++} and T'_{--} are J -self-adjoint it is easy to see that S' has the same property. Let

$$A := -(T'_{-+})^*|_{\mathfrak{D}(T'_{-+})} + (T'_{-+})_s.$$

Then A is an operator with $\mathfrak{D}(A) = \mathfrak{D}(T')$.

To prove the continuity of A , let $f \in \mathfrak{D}(T)$ be arbitrary. Then

$$Af = -(T'_{-+})^*f_1 + (T'_{-+})_s f_2$$

with $f_1 \in \mathfrak{D}(T'_{++})$ and $f_2 \in \mathfrak{D}(T'_{-+})$. This implies

$$\begin{aligned} \|Af\|^2 &= \|(T'_{-+})^*f_1\|^2 + \|(T'_{-+})_s f_2\|^2 \\ &\leq \|(T'_{-+})_s\|^2 (\|f_1\|^2 + \|f_2\|^2) \\ &= \|(T'_{-+})_s\|^2 \|f\|^2 \end{aligned}$$

(see the proof of the preceding theorem). So we must prove that $\|(T'_{-+})_s\| \leq 2\|S_s P_- \| =: c_T$ (for the definition of S_s , see Theorem 2.11). With Theorem 2.16 and a little calculation we derive

$$(T'_{-+})_s = (T_s)_{-+} = (S_s J)_{-+} = P_+ S_s J|_{\mathfrak{H}_-} \subset -P_+ S_s P_-,$$

which implies the desired inequality.

Finally, to verify that $T' = S' + A$ one can proceed as follows: With the help of the relation matrix of T' it is seen that $T' \subset S' + A$. Furthermore, $\mathfrak{D}(S') = \mathfrak{D}(T') = \mathfrak{D}(A)$ and $T'(0) = (S' + A)(0)$. These facts together imply the result.

Corollary 4.9. *Let T and \mathfrak{H} be as in Theorem 4.6 and let T' be an arbitrary regular self-adjoint extension of T . Then*

- (i) $\sigma(T') \subset C_T := \{z \in \mathbb{C} \mid |\operatorname{Im} z| \leq c_T\}$;
- (ii) $\|(T' - zI)^{-1}\| \leq (\operatorname{Im} z - c_T)^{-1}$ for $z \notin C_T$;
- (iii) $\Re(T - zI)^\perp$ is positive for $z \notin C_T$.

For the proof, see Corollary 3.5 and Corollary 4.3.

4.3. The deficiency spaces. Let T be a relation in an inner product space \mathfrak{H} . Define $\mathfrak{N}_z := \mathfrak{N}_z(T) := \Re(T - \bar{z}I)^\perp$ and $M_z := M_z(T) := \{(h, k) \in T^+ \mid k = zh\}$ for $z \in \mathbb{C}$; the latter is called a *deficiency space* of T in [11]. In studying these spaces we restrict ourselves to Pontrjagin spaces.

Theorem 4.10. *Let T be a closed symmetric linear relation in a Pontrjagin space \mathfrak{H} with $\mathfrak{D}(T) \supset \mathfrak{H}_-$ for a fundamental decomposition (1.1) of \mathfrak{H} . Then*

- (i) M_z is a continuous operator with $\mathfrak{D}(M_z) = \mathfrak{R}(M_z) = \mathfrak{R}_z$ for all $z \in \mathbb{C} \setminus \mathbb{R}$;
- (ii) M_z and \mathfrak{R}_z are Hilbert spaces with respect to $[\cdot | \cdot]$ for all $z \notin \mathbb{C}_T$;
- (iii) $\dim M_z = \dim \mathfrak{R}_z$ is constant for all $z \in \mathbb{C}_+ \setminus \sigma_p(T)$ and similarly for all $z \in \mathbb{C}_- \setminus \sigma_p(T)$;
- (iv) $T^+ = T \dot{+} M_z \dot{+} M_{\bar{z}}$ for all $z \notin \mathbb{C}_T$.

Proof. (i) is obvious and (ii) follows from Corollary 4.9. Clearly $\dim M_z = \dim \mathfrak{R}_z = \dim \mathfrak{R}(T - \bar{z}I)^{(\perp)}$, and the constancy of this dimension follows from Theorem 4.4 as in the case of Hilbert space operators; see [1], Nr. 100. (iv) is a direct calculation.

The cardinal number $n_+ := n_+(T) := \dim M_{\bar{z}}(T)$ for $z \in \mathbb{C}_+ \setminus \sigma_p(T)$ is called the *upper defect number* of T ; similarly $n_- := n_-(T) := \dim M_z(T)$ for $z \in \mathbb{C}_- \setminus \sigma_p(T)$ is called the *lower defect number* of T . The previous results show that T is maximal symmetric/self-adjoint iff $n_+ = 0$ or $n_- = 0$ / $n_+ = n_- = 0$.

A mapping $U: \mathfrak{H} \rightarrow \mathfrak{H}$ is called a *conjugation* if $U^2 = I$ and $[Uf | Ug] = [g | f]$ for all $f, g \in \mathfrak{H}$. A linear relation T in \mathfrak{H} is said to be *real* with respect to the conjugation U , if $(f, g) \in T$ implies $(Uf, Ug) \in T$. A useful criteria for the equality $n_+ = n_-$ is given by the following result:

Theorem 4.11. *Let T and \mathfrak{H} be as in Theorem 4.10. If T is real with respect to a conjugation, then the defect numbers of T are equal.*

As the spaces $\mathfrak{R}_{\pm ia}$ with $a > c_T$ are Hilbert spaces, we can use the proof given for the Hilbert space operators; see e.g. [12], Theorem XII.4.18.

As in the Hilbert space case, the sums in (iv) of Theorem 4.10 are orthogonal for $z \in \{i, -i\}$, but it can very well happen here that $\pm i \in \mathbb{C}_T$. In order to get orthogonal sums also in this case, we need to modify the inner product:

Theorem 4.12. *Let T and \mathfrak{H} be as in Theorem 4.10 and let $a > c_T$. Define*

$$[(f, g) | (h, k)]_+ := a^2 [f | h] + [g | k]$$

for all $(f, g), (h, k) \in T^+$. Then T^+ is a π_x -space with the inner product $[\cdot | \cdot]_+$ and

$$T^+ = T[\dot{+}]_+ M_{ia}[\dot{+}]_+ M_{-ia}$$

The proof follows the same lines as in the operator case; see [20].

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