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A NON-NORMAL FUNCTION WHOSE DERIVATIVE HAS FINITE AREA INTEGRAL OF ORDER $0<p<2$

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1. Introduction

Let f be a function holomorphic in $D=\{|z|<1\}$, and let

$$I_p(f) = \iint_D |f'(z)|^p \, dx\, dy \quad (z = x+iy,\ 0 < p < \infty).$$

It is known that if $I_2(f)<\infty$, then f is normal in D in the sense of O. Lehto and K. I. Virtanen [4], or equivalently, $\sup_{z\in D}(1-|z|)|f'(z)|/(1+|f(z)|^2)<\infty$. We shall show that there exists a non-normal f such that $I_p(f)<\infty$ for each p, $0<p<2$. H. Allen and C. Belna [1, Theorem 1] proved that there exists a non-normal f such that $I_1(f)<\infty$. Our example therefore fills up the remaining gap between 1 and 2. Note that if $I_p(f)<\infty$, then $I_q(f)<\infty$ for all q, $0<q<p$.

Theorem. *Let the zeros* $\{z_n\}$ *of the Blaschke product*

$$B(z) = \prod_{n=1}^{\infty} \frac{|z_n|}{z_n} \frac{z_n - z}{1-\bar{z}_n z} \tag{1.1}$$

satisfy the inequality

$$1-|z_{n+1}| \leqq \beta(1-|z_n|) < \beta, \quad n \geqq 1, \tag{1.2}$$

where β, $0<\beta<1$, *is a constant. Assume that* 1 *is the limit of a subsequence of* $\{z_n\}$. *Then the function*

$$f(z) = B(z) \log \frac{1}{1-z} \quad (\log 1 = 0) \tag{1.3}$$

is not normal in D, yet it satisfies $I_p(f)<\infty$ *for all* p, $0<p<2$.

A typical example of $\{z_n\}$ is $\{1-\beta^n\}$.

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2. Proof of Theorem

Lemma 1. *Consider B of* (1.1) *with* (1.2). *Assume that a holomorphic function g in D satisfies*

$$\lim_{j\to\infty} g(z_{n_j}) = \infty$$

for a certain subsequence $\{z_{n_j}\}$ of $\{z_n\}$. Then the product Bg is not normal in D.

Since

$$|B'(z_n)| = \frac{1}{1-|z_n|^2} \prod_{\substack{k=1\\k\neq n}}^{\infty} \left| \frac{z_n - z_k}{1-\bar{z}_k z_n} \right|,$$

it follows from the proof of [5, Theorem 2] that

$$(1-|z_n|)\,|B'(z_n)| \geqq \frac{1}{2} \prod_{\substack{k=1\\k\neq n}}^{\infty} \left| \frac{z_n - z_k}{1-\bar{z}_k z_n} \right| \geqq \frac{1}{2}\left[\prod_{k=1}^{\infty} \frac{1-\beta^k}{1+\beta^k}\right]^2 = a > 0.$$

Since

$$(1-|z_{n_j}|)\frac{|(B(z_{n_j})g(z_{n_j}))'|}{1+|B(z_{n_j})g(z_{n_j})|^2} = (1-|z_{n_j}|)|B'(z_{n_j})g(z_{n_j})| \geqq a\,|g(z_{n_j})| \to \infty$$

as $j\to\infty$, the function Bg is not normal.

Lemma 2. *Assume that $p>1/2$ and $1/2<\gamma<1$. Then, for each r, $1/(2\gamma)\leqq r<1$, and for each real constant τ, the following estimate*

$$J(\gamma, r, p) \equiv \int_0^{2\pi} |1-\gamma r e^{i(\theta-\tau)}|^{-2p} d\theta \leqq c_p(1-\gamma r)^{-2p+1} \tag{2.1}$$

holds, where

$$c_p = \int_{-\infty}^{+\infty} [1+2\pi^{-2}t^2]^{-p}\, dt < \infty.$$

Since $1/2\leqq \gamma r<\gamma<1$, it follows from the known estimate [2, p. 66, line 4 from above] that

$$J(\gamma, r, p) = \int_{-\pi}^{\pi} \frac{d\theta}{(1-2\gamma r\cos\theta+\gamma^2 r^2)^p} \leqq c_p(1-\gamma r)^{-2p+1}.$$

Proof of Theorem. Set $g(z)=\log[1/(1-z)]$ in D. It follows from Lemma 1 that $f=Bg$ is not normal.

It suffices to prove that $I_p(f)<\infty$ for each p, $1<p<2$. From $f'=B'g+Bg'$ it follows that

$$|f'|^p \leqq 2^p(|B'g|^p+|Bg'|^p)$$

(see [2, p. 2]). Since

$$\iint_D |B(z)g'(z)|^p\, dx\, dy \leqq \iint_D |g'(z)|^p\, dx\, dy = \iint_D |1-z|^{-p}\, dx\, dy < \infty,$$

because $p<2$, we have only to prove that

$$P \equiv \iint_{3/4<|z|<1} |B'(z)g(z)|^p \, dx\, dy < \infty. \tag{2.2}$$

From $M(r, g)=\max_{|z|=r}|g(z)| \leqq -\log(1-r)+(\pi/2)$, $0<r<1$, we conclude that, for each $s>0$,

$$\int_0^1 M(r, g)^s \, dr < \infty. \tag{2.3}$$

Let N be a natural number such that $1/(2|z_n|)<3/4$ for all $n>N$. Since

$$|B'(z)| = \left| \sum_{n=1}^{\infty} \frac{|z_n|}{z_n} \frac{|z_n|^2-1}{(1-\bar{z}_n z)^2} \prod_{\substack{k=1\\k\neq n}}^{\infty} \frac{|z_k|}{z_k} \frac{z_k-z}{1-\bar{z}_k z} \right| \leqq 2 \sum_{n=1}^{\infty} n^{-1} \cdot n \frac{1-|z_n|}{|1-\bar{z}_n z|^2},$$

it follows from the Hölder inequality with $p^{-1}+q^{-1}=1$ that

$$\begin{aligned} |B'(z)|^p &\leqq 2^p \left(\sum_{n=1}^{\infty} n^{-q} \right)^{p/q} \sum_{n=1}^{\infty} n^p \left(\frac{1-|z_n|}{|1-\bar{z}_n z|^2} \right)^p \\ &\leqq K_1 \left[\sum_{n=1}^{N} n^p (1-|z_n|)^{-p} + \sum_{n=N+1}^{\infty} n^p \left(\frac{1-|z_n|}{|1-\bar{z}_n z|^2} \right)^p \right] \\ &\leqq K_2 + K_1 \sum_{n=N+1}^{\infty} n^p (1-|z_n|)^p |1-\bar{z}_n z|^{-2p}, \quad z \in D. \end{aligned}$$

Here and hereafter, K_j, $j=1, \dots, 8$, are positive constants. It now follows from Lemma 2 that, for each r, $3/4<r<1$,

$$\begin{aligned} &\int_0^{2\pi} |B'(re^{i\theta})g(re^{i\theta})|^p \, d\theta \\ &\leqq M(r, g)^p \left[K_3 + K_1 \sum_{n=N+1}^{\infty} n^p (1-|z_n|)^p J(|z_n|, r, p) \right] \\ &\leqq K_3 M(r, g)^p + K_4 \sum_{n=N+1}^{\infty} n^p (1-|z_n|)^p M(r, g)^p (1-|z_n| r)^{-2p+1} \\ &\equiv Q(r). \end{aligned} \tag{2.4}$$

Therefore, on considering (2.3), one obtains the bound

$$P \leqq \int_{3/4}^{1} Q(r)\, dr \leqq K_5 + K_4 \sum_{n=N+1}^{\infty} n^p (1-|z_n|)^p \int_0^1 M(r, g)^p (1-|z_n| r)^{-2p+1} dr. \tag{2.5}$$

Choose $\lambda>1$ and $\mu>1$ such that

$$\mu < (p-1)^{-1} \quad \text{and} \quad \lambda^{-1}+\mu^{-1} = 1.$$

It then follows from the Hölder inequality, together with (2.3), that the last integral in (2.5) is not greater than

$$\left[\int_0^1 M(r,g)^{\lambda p}\,dr\right]^{1/\lambda}\left[\int_0^1 (1-|z_n|r)^{\mu(-2p+1)}\,dr\right]^{1/\mu}$$
$$\leqq K_6(3/2)^{1/\mu}[\mu(2p-1)-1]^{-1/\mu}(1-|z_n|)^t,$$

where

$$t=-2p+1+\mu^{-1}<0.$$

It then follows from (2.5) that

$$P\leqq K_5+K_7\sum_{n=N+1}^{\infty} n^p(1-|z_n|)^{t+p}\leqq K_5+K_8\sum_{n=1}^{\infty} n^2\beta^{(t+p)n}<\infty$$

because

$$1-|z_n|\leqq \beta^{n-1}(1-|z_1|),\quad n\geqq 1,$$

and $t+p>0$.

3. Concluding remarks

Since the function $g(z)=\log[1/(1-z)]$ is a member of

$$H\equiv \bigcap_{0<\lambda<\infty} H^\lambda,$$

where H^λ is the Hardy class (see [7, p. 58]), it follows from $|f|\leqq|g|$, that f of Theorem also belongs to H. On the other hand, if h is holomorphic in D such that $I_2(h)<\infty$, then $h\in H$ (see [2, p. 106]). It might be of interest to note that this follows from [3, Theorem] because the area of the image $h(D)$ (not the Riemannian image F) of D by h is less than that of F, being $I_2(h)$.

It is obvious that f in Theorem is not Bloch [6], that is,

$$\sup_{z\in D}(1-|z|)|f'(z)|=\infty.$$

However, *f is "near Bloch" in the sense that*

$$\sup_{z\in D}(1-|z|)^\alpha|f'(z)|<\infty \tag{3.1}$$

for every $\alpha>1$. In fact, for each $h\in H$, and for each $\alpha>1$, (3.1) is valid if f is replaced by h. For the proof we set $p=(\alpha-1)^{-1}$. Since $h\in H^p$, it follows from the inequality [2, Lemma, p. 36] (see also [2, p. 144, line 2 from above]) that

$$\sup_{z\in D}(1-|z|)^{\alpha-1}|h(z)|<\infty.$$

This, together with [2, Theorem 5.5 in the case $p=\infty$, p. 80], shows that (3.1) is true for h.

Consider next the weighted integral $I_2(h)$,

$$A_\alpha(h)=\iint_D (1-|z|)^\alpha|h'(z)|^2\,dx\,dy\quad(\alpha\geqq 0)$$

of arbitrary h holomorphic in D. Obviously, $I_2(h)=A_0(h)$, and $A_\alpha(h)\leqq A_\beta(h)$ if $\alpha\geqq\beta$. We shall show that *our function f in Theorem also satisfies $A_\alpha(f)<\infty$ for all $\alpha>0$.*

For the proof we may assume that $0<\alpha<1$. Since

$$\iint_D (1-|z|)^\alpha |B(z)g'(z)|^2 dx\,dy \leqq \iint_D |1-z|^{\alpha-2} dx\,dy <\infty,$$

it suffices to show that

$$P_1 \equiv \iint_{3/4<|z|<1} (1-|z|)^\alpha |B'(z)g(z)|^2 dx\,dy <\infty.$$

We follow the proof of (2.2) up to the estimate (2.4), where, in the present case, we may set $p=2$. Then,

$$\int_0^{2\pi} |B'(re^{i\theta})g(re^{i\theta})|^2 d\theta$$

$$\leqq C_1 M(r,g)^2 + C_2 \sum_{n=N+1}^{\infty} n^2(1-|z_n|)^2 M(r,g)^2(1-|z_n|r)^{-3}$$

$$\equiv Q_1(r), \quad 3/4<r<1.$$

Here and hereafter, C_j, $j=1, \dots, 5$, are positive constants. On considering

$$(1-r)^\alpha M(r,g)^2 \leqq M(r,g)^2$$

and

$$(1-r)^\alpha(1-|z_n|r)^{-3} \leqq (1-|z_n|r)^{-3+\alpha},$$

one obtains from (2.3) the following estimate:

$$P_1 \leqq \int_{3/4}^{1} (1-r)^\alpha Q_1(r)\,dr \tag{3.2}$$

$$\leqq C_3 + C_2 \sum_{n=N+1}^{\infty} n^2(1-|z_n|)^2 \int_0^1 M(r,g)^2(1-|z_n|r)^{-3+\alpha} dr.$$

With the choice $1<\mu<(1-\alpha)^{-1}$, $\lambda^{-1}+\mu^{-1}=1$, the last integral in (3.2) is not greater than

$$\left[\int_0^1 M(r,g)^{2\lambda} dr\right]^{1/\lambda} \left[\int_0^1 (1-|z_n|r)^{\mu(-3+\alpha)} dr\right]^{1/\mu} \leqq C_4(1-|z_n|)^{-3+\alpha+\mu^{-1}}.$$

It follows that

$$P_1 \leqq C_3 + C_5 \sum_{n=1}^{\infty} n^2(1-|z_n|)^{-1+\alpha+\mu^{-1}} <\infty,$$

because $-1+\alpha+\mu^{-1}>0$.

References

[1] ALLEN, H., and C. BELNA: Non-normal functions $f(z)$ with $\iint_{|z|<1}|f'(z)|\,dx\,dy<\infty$. - J. Math. Soc. Japan 24, 1972, 128—131.

[2] DUREN, P. L.: Theory of H^p spaces. - Academic Press, New York—London, 1970.

[3] HANSEN, L. J.: The Hardy class of a function with slowly-growing area. - Proc. Amer. Math. Soc. 45, 1974, 409—410.

[4] LEHTO, O., and K. I. VIRTANEN: Boundary behaviour and normal meromorphic functions. - Acta Math. 97, 1957, 47—65.

[5] NEWMAN, D. J.: Interpolation in H^∞. - Trans. Amer. Math. Soc. 92, 1959, 501—507.

[6] POMMERENKE, C.: On Bloch functions. - J. London Math. Soc. (2) 2, 1970, 689—695.

[7] PRIWALOW, I. I.: Randeigenschaften analytischer Funktionen. - VEB Deutscher Verlag der Wissenschaften, Berlin, 1956.

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