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ON THE POISSON REPRESENTATION OF DISTRIBUTIONS

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1. Introduction

1. Let $\mathscr{A}$ be a test function space on $\boldsymbol{R}$ and $\mathscr{A}'$ the corresponding space of distributions. The elements of $\mathscr{A}$ and $\mathscr{A}'$ are supposed to be complex valued. A complex valued function is called harmonic if its real and imaginary parts are harmonic functions. A distribution $T\in\mathscr{A}'$ has *a harmonic representation* in $\mathscr{B}$ if $\mathscr{B}\subset\mathscr{A}$ and there exists a complex valued function h harmonic in $\boldsymbol{C}\setminus\boldsymbol{R}$ such that

$$\lim_{\eta\to 0}\int_{-\infty}^{\infty} h(x+i\eta)\varphi(x)\,dx = \langle T,\varphi\rangle \tag{1.1}$$

for every $\varphi\in\mathscr{B}$. Similarly $T\in\mathscr{A}'$ has *an analytic representation* in $\mathscr{B}$ if $\mathscr{B}\subset\mathscr{A}$ and there exists an analytic function f in $\boldsymbol{C}\setminus\boldsymbol{R}$ such that (1.1) is valid for every $\varphi\in\mathscr{B}$ when $\eta\to 0+$ and $h(z)=f(z)-f(\bar z)$.

Let $\mathcal{O}_\alpha(\boldsymbol{R})$ be the test function space of functions $\varphi\colon \boldsymbol{R}\to\boldsymbol{C}$ such that $\varphi(t)$ together with its derivatives is asymptotically bounded by $|t|^\alpha$ as $|t|\to\infty$. In [2] Bremermann has shown, among other results, that $T\in\mathscr{E}'(\boldsymbol{R})$ has a harmonic representation in $\mathcal{O}_0(\boldsymbol{R})$ and that $T\in\mathcal{O}'_\alpha(\boldsymbol{R})$, $\alpha\geqq -1$, has an analytic representation in $\mathscr{D}(\boldsymbol{R})$.

In this paper we consider the harmonic representation of $T\in\mathcal{O}'_\alpha(\boldsymbol{R})$ in $\mathcal{O}_\alpha(\boldsymbol{R})$ following Bremermann. We mainly use the notations and terminology of [2]. For the basic properties of distributions we also refer to Schwartz [5]. Let us denote the Poisson kernel by $p(t,z)$, i.e.,

$$p(t,z)=\frac{1}{\pi}|y|\cdot|t-z|^{-2},\quad z=x+iy. \tag{1.2}$$

The function $p(\cdot,z)$ belongs to $\mathcal{O}_\alpha(\boldsymbol{R})$ for every $z\in\boldsymbol{C}\setminus\boldsymbol{R}$ if $\alpha\geqq -2$. For those values of α we can define for every $T\in\mathcal{O}'_\alpha(\boldsymbol{R})$ a complex valued function $z\to\langle T,p(\cdot,z)\rangle$ which is harmonic in $\boldsymbol{C}\setminus\boldsymbol{R}$. We call this function *the Poisson representation* of T, and show that it gives a harmonic representation of $T\in\mathcal{O}'_\alpha(\boldsymbol{R})$ in $\mathcal{O}_\alpha(\boldsymbol{R})$ if $-2\leqq\alpha<1$. As an immediate corollary we get an analytic representation of $T\in\mathcal{O}'_\alpha(\boldsymbol{R})$ in $\mathcal{O}_\alpha(\boldsymbol{R})$ if $-1\leqq\alpha<1$.

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2. Asymptotically bounded functions

2. Let $\alpha\in\boldsymbol{R}$. A complex valued function $f\colon t\to f(t)$, $t\in\boldsymbol{R}$, is called *asymptotically bounded by* $|t|^\alpha$ and denoted by $f(t)=\mathcal{O}(|t|^\alpha)$ if there exist non-negative constants c and r, *asymptotic constants*, such that $|f(t)|\leqq c|t|^\alpha$ if $|t|\geqq r$. We call α *the asymptotic degree of* f. The set of non-negative integers is denoted by $\boldsymbol{Z}_+$.

If $a, b\in\boldsymbol{R}\cup\{\pm\infty\}$, $z\in\boldsymbol{C}$, and the function f is defined on (a, b), we denote

$$PI(f, z; a, b) = \int_a^b p(t, z)f(t)\,dt, \tag{2.1}$$

provided that the integral exists. Here $p(t, z)$ is the Poisson kernel defined in (1.2). We also denote

$$PI(f, z) = PI(f, z; -\infty, \infty). \tag{2.2}$$

Using the Poisson integral we can represent asymptotically bounded functions as boundary values of harmonic functions.

Proposition 2.1. *Let* $f\in C^m(\boldsymbol{R})$ *and* $D^k f(t)=\mathcal{O}(|t|^\alpha)$ *for every* $k\in\boldsymbol{Z}_+$, $0\leqq k\leqq m$. *If* $\alpha<1$, *then*

$$\lim_{\eta\to 0} D^k PI(f, x+i\eta) = D^k f(x), \quad 0\leqq k\leqq m,$$

uniformly in compact subsets of $\boldsymbol{R}$.

Propositions of this kind are usually proved on the assumption that $\alpha=0$ (cf. [2] and [4]). The proof of this slightly more general form is essentially the same. One only has to be more careful with estimates because of the growth of f at infinity.

We say that $P(f, z)$ is *the Poisson representation of* f. It should be noted that

$$PI(1, z) = 1 \tag{2.3}$$

and that

$$D_x^k PI(f, z) = PI(D^k f, z). \tag{2.4}$$

3. Let $\beta<1$ and let $f\in C(\boldsymbol{R})$, $f(t)=\mathcal{O}(|t|^\beta)$. Then $PI(f, x+i\eta)$, the Poisson representation of f, exists and is a continuous function of x for fixed values of $\eta\in\boldsymbol{R}\setminus\{0\}$. We intend to study the asymptotical boundedness of $PI(f, x+i\eta)$ as a function of x. To be more exact, we want to know the asymptotic degree and, moreover, in which cases the asymptotic constants can be chosen to be independent of η. This information is needed in the next chapter.

Proposition 2.2. *Let* $f\in C(\boldsymbol{R})$, $f(t)=\mathcal{O}(|t|^\beta)$, $\beta<1$, *and let* $\alpha\geqq\max(\beta, -2)$. *If* $\eta_0>0$ *is given and* $0<|\eta|\leqq\eta_0$, *then* $PI(f, x+i\eta)=\mathcal{O}(|x|^\alpha)$, *and the asymptotic constants can be chosen to be independent of* η.

Proof. Let $\eta_0>0$ and $0<|\eta|\leqq\eta_0$. Let c and r be asymptotic constants of f. We divide the Poisson representation of f into three parts, $PI(f,\cdot)=PI(f,\cdot;-r,r)+PI(f,\cdot;r,\infty)+PI(f,\cdot;-\infty,-r)$, and estimate each part separately.

For the first term we get directly an upper bound

$$(2.5)\qquad |PI(f,x+i\eta;-r,r)|\leqq\frac{8r\eta_0}{\pi}\sup_{-r\leqq t\leqq r}|f(t)|\cdot|x|^{-2}\quad\text{if}\quad|x|\geqq 2r.$$

This has the required asymptotic behaviour.

The remaining two terms appear to have similar upper bounds

$$(2.6)\qquad \begin{aligned}|PI(f,x+i\eta;r,\infty)|&\leqq cPI(t^\beta,x+i\eta;r,\infty)\quad\text{if}\quad|x|\geqq r,\\ |PI(f,x+i\eta;-\infty,-r)|&\leqq cPI(t^\beta,-x+i\eta;r,\infty)\quad\text{if}\quad|x|\geqq r.\end{aligned}$$

Thus the proposition is proved if we show that there exist non-negative constants c_0 and r_0 such that

$$(2.7)\qquad PI(t^\beta,x+i\eta;r,\infty)\leqq c_0|x|^\alpha,\quad\text{if}\quad|x|\geqq r_0,$$

for every $\eta\in\boldsymbol{R}$, $0<|\eta|\leqq\eta_0$.

If β is not an integer, (2.7) can be obtained by means of residue calculus. So we suppose first that $\beta<1$, $\beta\notin\boldsymbol{Z}$, and consider a value of x for which $|x|\geqq 2r+4\eta_0$. We imbed the t-axis into the complex plane $\boldsymbol{C}_w$, $w=t+iu$, choose a constant R such that $R\geqq 2|x|+4\eta_0$, and define paths γ_j: $[0,1]\to\boldsymbol{C}_w$ as follows: γ_1: $s\to Re^{i2\pi s}$, γ_2: $s\to(R+s(r-R))e^{i2\pi}$, γ_3: $s\to re^{i2\pi(1-s)}$, γ_4: $s\to r+s(R-r)$. We also define $\gamma_0=\gamma_1+\gamma_2+\gamma_3+\gamma_4$. Along these paths we define integrals

$$(2.8)\qquad I_j=\frac{|\eta|}{\pi}\int_{\gamma_j}\frac{w^\beta}{(w-x)^2+\eta^2}\,dw,\quad j=0,1,2,3,4.$$

Here the argument of w^β is taken between 0 and $2\pi\beta$. Because I_2 and I_4 are of the type of (2.1) we get a representation

$$(2.9)\qquad PI(t^\beta,x+i\eta;r,R)=(1-e^{2\pi i\beta})^{-1}(I_0-I_1-I_3).$$

In order to get (2.7) we estimate each integral of the right side separately.

The value of I_0 is equal to the sum of the residues of the integrand multiplied by $2\pi i$ (cf. e.g. [1]). Because the integrand has only two simple poles at $x\pm i\eta$, $I_0=\operatorname{sgn}(\eta)[(x+i\eta)^\beta-(x-i\eta)^\beta]$. This gives easily an estimate of the right kind

$$(2.10)\qquad |I_0|\leqq 4|x|^\beta\quad\text{if}\quad|x|\geqq 2r+4\eta_0.$$

A straightforward calculation gives an estimate of I_1 in R,

$$(2.11)\qquad |I_1|\leqq 16\eta_0R^{\beta-1}\quad\text{if}\quad R\geqq 2|x|+4\eta_0.$$

Because $\beta<1$, I_1 tends to zero as R tends to infinity. In the same way we can estimate I_3 in x,

$$(2.12)\qquad |I_3|\leqq 16\eta_0r^{\beta+1}|x|^{-2}\quad\text{if}\quad|x|\geqq 2r+4\eta_0.$$

If we take absolute values of both sides of (2.9), use the estimates (2.10), (2.11), (2.12), and let R tend to infinity, we get an estimate of the type of (2.7) with values $c_0=|1-e^{2\pi i\beta}|^{-1}(4+16\eta_0 r^{\beta+1})$ and $r_0=2r+4\eta_0$. These values are clearly independent of η. So (2.7) is proved in the case $\beta\notin \boldsymbol{Z}$.

For integer values of β (2.7) can be obtained by direct integration and estimation. If $\beta=0$, then $\alpha\geqq 0$ and the statement follows from (2.3). In the cases $\beta=-1$ and $\beta=-2$ we get the same final estimate

$$PI(t^\beta, x+i\eta; r, \infty) \leqq (\eta_0+1)|x|^\beta \quad \text{if} \quad |x| \geqq 2r \geqq 2. \tag{2.13}$$

This shows (2.7) for values $\beta=-1, -2$. If finally $\beta\in\boldsymbol{Z}$, $\beta<-2$, then $\alpha\geqq -2$ and $PI(t^\beta, x+i\eta;\ r, \infty)\leqq PI(t^{-2}, x+i\eta;\ r, \infty)$ when $r\geqq 1$. This implies (2.7) for the remaining values of β by (2.13). The proposition is proved.

3. The Poisson representation of distributions

4. We define the space of asymptotically bounded test functions as in [2]: If $\alpha\in\boldsymbol{R}$, then

$$\mathcal{O}_\alpha(\boldsymbol{R}) = \{\varphi\in C^\infty(\boldsymbol{R}) | D^k\varphi(t) = \mathcal{O}(|t|^\alpha) \text{ for every } k\in\boldsymbol{Z}_+\}. \tag{3.1}$$

If $\alpha\leqq\beta$, then $\mathcal{D}(\boldsymbol{R})\subset\mathcal{O}_\alpha(\boldsymbol{R})\subset\mathcal{O}_\beta(\boldsymbol{R})\subset\mathcal{E}(\boldsymbol{R})$. Using Propositions 2.1 and 2.2 together with (2.4) we get the following result:

Proposition 3.1. *Let* $\varphi\in\mathcal{O}_\beta(\boldsymbol{R})$, $\beta<1$ *and let* $\alpha\geqq\max(\beta, -2)$. *If* $\eta\in\boldsymbol{R}\setminus\{0\}$, *then* $PI(\varphi, x+i\eta)\in\mathcal{O}_\alpha(\boldsymbol{R})$ *as a function of x and*

$$\lim_{\eta\to 0} PI(\varphi, x+i\eta) = \varphi(x) \tag{3.2}$$

in $\mathcal{O}_\alpha(\boldsymbol{R})$.

5. The distribution space corresponding to $\mathcal{O}_\alpha(\boldsymbol{R})$ is denoted by $\mathcal{O}'_\alpha(\boldsymbol{R})$. It consists of continuous linear functionals $T\colon \mathcal{O}_\alpha(\boldsymbol{R})\to\boldsymbol{C}$. The value of a distribution T applied to a test function φ is denoted by $\langle T, \varphi\rangle$. It is easily seen that the Poisson kernel $p(\cdot, z)$ belongs to $\mathcal{O}_\alpha(\boldsymbol{R})$ for every $z\in\boldsymbol{C}\setminus\boldsymbol{R}$ if $\alpha\geqq -2$. Thus we can define a function

$$PI(T, z) = \langle T, p(\cdot, z)\rangle \tag{3.3}$$

for every $T\in\mathcal{O}'_\alpha(\boldsymbol{R})$, $\alpha\geqq -2$. We call this function *the Poisson representation of T*. The harmonicity of the Poisson kernel implies that also the Poisson representation is a harmonic function outside the real axis.

Proposition 3.2. *If* $T\in\mathcal{O}'_\alpha(\boldsymbol{R})$, $\alpha\geqq -2$, *then its Poisson representation* $PI(T, \cdot)$ *is harmonic in* $\boldsymbol{C}\setminus\boldsymbol{R}$.

Proof. We begin by showing that

$$D_x PI(T, z) = \langle T, D_x p(\cdot, z)\rangle. \tag{3.4}$$

Suppose that $z \in \boldsymbol{C}\setminus\boldsymbol{R}$, $t\in\boldsymbol{R}$ and $h\in\boldsymbol{R}\setminus\{0\}$. Consider the difference

$$A_x(t, h) = \frac{1}{h}(p(t, z+h)-p(t, z))-D_x p(t, z). \tag{3.5}$$

If $n\in\boldsymbol{Z}_+$, we can show by induction that

$$D_t^n A_x(t, h) = hB_n(t, z, h)(|t-z-h|^2|t-z|^4)^{-n-1}, \tag{3.6}$$

where B_n is a polynomial of t, z and h. As a polynomial of t the degree of B_n is at most $5n+2$. Thus (3.6) implies that $D_t^n A_x(t, h)$ converges to zero uniformly on compact sets as $h\to 0$. Moreover, if $h_0>0$ is small enough, then $D_t^n A_x(t, h)=\mathcal{O}(|t|^{-2})$ and the asymptotic constants are independent of h when $0<|h|\leqq h_0$. Therefore

$$\lim_{h\to 0} A_x(t, h) = 0 \tag{3.7}$$

in $\mathcal{O}_\alpha(\boldsymbol{R})$, $\alpha\geqq -2$. This proves (3.4).

The rest of the proof runs along the same line. If we replace p by $D_x p$ in (3.4), we see in a similar way that (3.7) is still valid in $\mathcal{O}_\alpha(\boldsymbol{R})$, $\alpha\geqq -2$. Thus

$$D_x^2 PI(T, z) = \langle T, D_x^2 p(\cdot, z)\rangle. \tag{3.8}$$

Proceeding to the y-direction in the same way we see that (3.8) remains valid if D_x^2 is replaced by D_y^2. Because $p(t, z)$ is harmonic in $\boldsymbol{C}\setminus\boldsymbol{R}$ these facts imply that $(D_x^2+D_y^2)PI(T, z)=\langle T, (D_x^2+D_y^2)p(\cdot, z)\rangle=0$ if $z\in\boldsymbol{C}\setminus\boldsymbol{R}$. The proposition is proved.

6. The Poisson representation of a distribution T can be used for the characterization of T as follows:

Proposition 3.3. *Let $\beta<1$ and $\alpha\geqq\max(\beta, -2)$. If $T\in\mathcal{O}'_\alpha(\boldsymbol{R})$ and $\varphi\in\mathcal{O}_\beta(\boldsymbol{R})$, then*

$$\lim_{\eta\to 0}\int_{-\infty}^{\infty} PI(T, x+i\eta)\varphi(x)\,dx = \langle T, \varphi\rangle. \tag{3.9}$$

Proof. Because $\beta\leqq\alpha$, we have $\varphi\in\mathcal{O}_\alpha(\boldsymbol{R})$ and the right side of (3.9) is well defined. We consider the integral on the left side. Let $a<0<b$ and let $\{x_j\}_{-n}^n$ be a division of the interval (a, b) into equal subintervals. We show that if $\eta\in\boldsymbol{R}\setminus\{0\}$, then

$$\lim_{n\to\infty}\sum_{j=-n}^{n}\varphi(x_j)p(t, x_j+i\eta)\Delta x_j = PI(\varphi, t+i\eta; a, b) \tag{3.10}$$

in $\mathcal{O}_\alpha(\boldsymbol{R})$. Denote the sum of the left side by σ_n. If $m\in\boldsymbol{Z}_+$, it is not difficult to see that

$$\lim_{n\to\infty} D_t^m\sigma_n = D_t^m PI(\varphi, t+i\eta; a, b)$$

uniformly in compact sets. If c and r, $r \geqq 2\max(-a, b)$, are asymptotic constants of $D_t^m p(t, x_j + i\eta)$, we can suppose that they are independent of the division of the interval (a, b). Thus the estimate

$$|D_t^m \sigma_n| \leqq (b-a) \cdot \sup_{x \in (a,b)} |\varphi(x)| \cdot c \cdot |t|^\alpha, \quad \text{if} \quad |t| \geqq r,$$

gives the right asymptotic boundedness independent of n. So (3.10) is valid in $\mathcal{O}_\alpha(\boldsymbol{R})$. This means that

$$(3.11) \qquad \int_a^b PI(T, x+i\eta)\varphi(x)\,dx = \lim_{n\to\infty} \sum_{j=-n}^{n} PI(T, x_j+i\eta)\varphi(x_j)\Delta x_j$$

$$= \lim_{n\to\infty} \sum_{j=-n}^{n} \langle T_t, p(t, x_j+i\eta)\varphi(x_j)\Delta x_j\rangle = \langle T_t, PI(\varphi, t+i\eta; a, b)\rangle.$$

Next we show that

$$(3.12) \qquad \lim_{-a, b\to\infty} PI(\varphi, t+i\eta; a, b) = PI(\varphi, t+i\eta)$$

in $\mathcal{O}_\alpha(\boldsymbol{R})$. If $m \in \boldsymbol{Z}_+$, it is not difficult to see that

$$\lim_{-a, b\to\infty} D_t^m PI(\varphi, t+i\eta; a, b) = D_t^m PI(\varphi, t+i\eta)$$

uniformly in compact sets. It remains to show the asymptotic boundedness. The derivative $D_t^m p(x, t+i\eta)$ can be written in the form $B_m(t, x) \cdot p(x, t+i\eta)$, where the function $B_m\colon \boldsymbol{R}\times\boldsymbol{R} \to \boldsymbol{R}$ is bounded. If d is an upper bound of $|B_m|$, if c and r are asymptotic constants of φ and if $-a \geqq r$, $b \geqq r$, then we get an estimate

$$\big|D_t^m\big(PI(\varphi, t+i\eta) - PI(\varphi, t+i\eta; a, b)\big)\big|$$
$$\leqq cd\big(PI(|x|^\beta, t+i\eta; r, \infty) + PI(|x|^\beta, -t+i\eta; r, \infty)\big).$$

By Proposition 2.2 there exist constants c_0 and r_0 such that

$$PI(|x|^\beta, t+i\eta; r, \infty) \leqq c_0 |t|^\alpha \quad \text{if} \quad |t| \geqq r_0$$

for every $r \geqq r_0$. This fact together with the preceding estimate gives the right asymptotic boundedness. So (3.12) is valid in $\mathcal{O}_\alpha(\boldsymbol{R})$. Together with (3.11) it shows that

$$(3.13) \qquad \int_{-\infty}^{\infty} PI(T, x+i\eta)\varphi(x)\,dx = \langle T_t, PI(\varphi, t+i\eta)\rangle.$$

The statement (3.9) follows now from (3.13) by Proposition 3.1. The proposition is proved.

7. Now we can formulate our main result. We recall that a distribution $T \in \mathcal{O}_\alpha'(\boldsymbol{R})$ has a harmonic representation in $\mathcal{O}_\alpha(\boldsymbol{R})$ if there exists a harmonic function $h\colon \boldsymbol{C}\setminus\boldsymbol{R} \to \boldsymbol{C}$ such that

$$(3.14) \qquad \lim_{\eta\to 0} \int_{-\infty}^{\infty} h(x+i\eta)\varphi(x)\,dx = \langle T, \varphi\rangle$$

for every $\varphi \in \mathcal{O}_\alpha(\mathbf{R})$. The Poisson representation provides us a simple means of obtaining a harmonic representation in some cases. In fact, Propositions 3.2 and 3.3 give us immediately:

Theorem 1. *Let* $T \in \mathcal{O}'_\alpha(\mathbf{R})$, $-2 \leqq \alpha < 1$. *Then* $PI(T, \cdot)$ *is a harmonic representation of* T *in* $\mathcal{O}_\alpha(\mathbf{R})$.

4. The Cauchy representation of distributions

8. Let $\alpha \geqq -1$ and $T \in \mathcal{O}'_\alpha(\mathbf{R})$. The function

$$\hat{T}(z) = \frac{1}{2\pi i}\left\langle T_t, \frac{1}{t-z}\right\rangle \tag{3.15}$$

is called *the Cauchy representation of* T. It is an analytic function of z in $\mathbf{C} \setminus \operatorname{spt} T$ (cf. [2]). Because $\hat{T}(z) - \hat{T}(\bar{z}) = \operatorname{sgn}(\operatorname{Im} z) PI(T, z)$ we have by Proposition 3.3

$$\lim_{\eta \to 0_+} \int_{-\infty}^{\infty} \left(\hat{T}(x+i\eta) - \hat{T}(x-i\eta)\right)\varphi(x)\,dx = \langle T, \varphi\rangle \tag{3.16}$$

for every $\varphi \in \mathcal{O}_\alpha(\mathbf{R})$ if $-1 \leqq \alpha < 1$. In other words, the following result is valid:

Theorem 2. *Let* $T \in \mathcal{O}'_\alpha(\mathbf{R})$, $-1 \leqq \alpha < 1$. *Then* $\hat{T}$ *is an analytic representation of* T *in* $\mathcal{O}_\alpha(\mathbf{R})$.

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