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ON THE CONSTRUCTION OF THE IRREDUCIBLE REPRESENTATIONS OF THE HYPERALGEBRA OF A UNIVERSAL CHEVALLEY GROUP

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1. Introduction. Let K be an algebraically closed field of characteristic $p>3$. Let $\mathfrak{g}$ be a finite dimensional complex semisimple Lie algebra and G the universal Chevalley group of type $\mathfrak{g}$ over K. If U_Z denotes Kostant's $\boldsymbol{Z}$-form of the universal enveloping algebra of $\mathfrak{g}$, then the infinite dimensional associative K-algebra $U_K=U_Z\otimes_Z K$ is known as the *hyperalgebra* of G ([2], [6]).

Each finite dimensional irreducible rational G-module is known to admit the structure of a U_K-module and vice versa ([2], [5]). Moreover, for each positive integer r the algebra U_K contains a finite dimensional subalgebra $\boldsymbol{u}_r$ in such a way that finite dimensional irreducible $\boldsymbol{u}_r$-modules correspond one-to-one to finite dimensional irreducible G_r-modules over K, where G_r is the group of points in G rational over the finite field of p^r elements.

In [10] it was observed that the algebras U_K and $\boldsymbol{u}_r$ all admit a s.c. good triangular decomposition. Using this we show how to explicitly construct the finite dimensional irreducible U_K- and $\boldsymbol{u}_r$-modules starting from certain one-dimensional representations of a suitable subalgebra. This provides a new direct way of realizing and classifying all irreducible U_K- and $\boldsymbol{u}_r$-modules, without relying on the classical highest weight theory of G-modules as in [6]. Moreover, the finite dimensional irreducible U_K-modules are shown to have a certain tensor product decomposition (Proposition 6). When interpreted as a result of G-modules this provides a new proof for the celebrated Steinberg's tensor product theorem.

2. Algebras with good triangular decomposition. Let A be an associative algebra with 1 over a field K. Let B be a subalgebra of A $(1\in B)$ and assume $B=H\oplus R$, a vector space direct sum, where H is a subalgebra of A $(1\in H)$ and R a two-sided ideal of B. Let us assume furthermore that R acts nilpotently on every irreducible finite dimensional left A-module. As in [10], we say that A admits a *triangular decomposition* over B if there exists a left B-, right H-module homomorphism $\gamma\colon A\to B$ such that $\gamma|_B=1_B$. In this case $A=H\oplus R\oplus\ker(\gamma)$.

If M is an arbitrary left A-module let $M^{\ker(\gamma)}=\{m\in M\,|\ker(\gamma)m=0\}$. Then $M^{\ker(\gamma)}$ is a left H-module. Supposing A admits a triangular decomposition over

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B and $M^{\ker(\gamma)} \neq 0$ for every finite dimensional left A-module $M \neq 0$, then A is said to admit a *good triangular decomposition* over B.

Suppose now that A admits a triangular decomposition over B via γ. Then any left H-module W becomes a left B-module by way of

$$(h+r)v = hv \quad \text{for every} \quad h \in H, \quad r \in R, \quad v \in W.$$

Furthermore, $P(W) = \mathrm{Hom}_B(A, W)$ is viewed as a left A-module in the usual fashion: $(a \cdot f)(a') = f(a'a)$ for every $a, a' \in A, f \in P(W)$. Finally the map $\omega\colon W \to P(W)$, given by $\omega(v)(a) = \gamma(a)v$, $v \in W$, $a \in A$, is an injective left H-module map. We denote by $W' = A\omega(W)$ the left A-submodule of $P(W)$ generated by $\omega(W)$.

Remark. If $\dim A = \infty$, the module W' may sometimes be infinite dimensional even though $\dim W < \infty$. In the sequel all modules and representations are automatically assumed to be finite dimensional.

An argument, similar to those in [10] and [13], yields the following result.

Proposition 1. *Assume A admits a good triangular decomposition over B via γ and that* $\ker(\gamma)\ker(\gamma) \subseteq \ker(\gamma)$. *Then the following are true:*

(i) $W_1 \cong_H W_2$ *if and only if* $W_1' \cong_A W_2'$.

(ii) $(W')^{\ker(\gamma)} = \omega(W) \cong_H W$ *for each left H-module W.*

(iii) *If W is an irreducible left H-module and* $\dim W' < \infty$, *then W' is an irreducible left A-module.*

(iv) *If M is an irreducible left A-module, then $M^{\ker(\gamma)}$ is an irreducible left H-module and* $M \cong_A (M^{\ker(\gamma)})'$.

Let $\mathrm{Irr}(A)$ (resp. $\mathrm{Irr}(H)$) denote the set of isomorphism classes of irreducible left A-modules (resp. H-modules). Proposition 1 established a bijective correspondence between the sets $\mathrm{Irr}(A)$ and $\{[W] \in \mathrm{Irr}(H) \mid \dim W' < \infty\}$ where $[W]$ denotes the class of H-modules isomorphic to W. If $\dim A < \infty$ this latter set is of course equal to $\mathrm{Irr}(H)$.

3. The algebras U_K and u_r. Let $\mathfrak{h}$ be a maximal torus in the semisimple Lie algebra $\mathfrak{g}$, and Φ the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Let $\Delta = \{\alpha_1, \ldots, \alpha_l\}$ denote a basis of Φ and $\Phi^+ = \{\alpha_1, \ldots, \alpha_m\}$, $\Phi^- = -\Phi^+$, the sets of positive (resp. negative) roots with respect to Δ. We fix a Chevalley basis $\{x_\alpha, \alpha \in \Phi;\ h_i = h_{\alpha_i}\ (i = 1, \ldots, l)\}$ for $\mathfrak{g}$ and let $U_{\mathbf{Z}}$ denote Kostant's $\mathbf{Z}$-form of $U(\mathfrak{g})$, i.e. the subring generated by all $x_\alpha^t/t!$, $\alpha \in \Phi$, $t \in \mathbf{Z}^+$.

From now on let K be an algebraically closed field of characteristic $p > 3$. Set $X_{\alpha,n} = x_\alpha^n/n! \otimes 1$ ($\alpha \in \Phi$, $n \in \mathbf{Z}^+$) and

$$H_{i,b} = \binom{h_i}{b} \otimes 1 \quad (i = 1, \ldots, l,\ b \in \mathbf{Z}^+).$$

The products

$$\left\{ \prod_{i=1}^{m} X_{-\alpha_i, a_i} \prod_{i=1}^{l} H_{i, b_i} \prod_{i=1}^{m} X_{\alpha_i, c_i} \,\middle|\, a_i, b_i, c_i \in \mathbf{Z}^+ \right\} \tag{1}$$

are known to form a K-basis for the K-algebra $U_K = U_Z \otimes_Z K$, which is called the *hyperalgebra.*

If S is any subset of U_K let $\langle S \rangle$ denote the subalgebra of U_K generated by S. For any $r \in N$ let $\boldsymbol{u}_r = \langle \{X_{\alpha,t} | \alpha \in \Phi,\ 0 \leqq t < p^r\} \rangle$. It was shown in [6] that the products

$$\left\{ \prod_{i=1}^{m} X_{-\alpha_i, a_i} \prod_{i=1}^{l} H_{i, b_i} \prod_{i=1}^{m} X_{\alpha_i, c_i} \middle| 0 \leqq a_i,\ b_i,\ c_i < p^r \right\} \tag{2}$$

form a K-basis for $\boldsymbol{u}_r$. Moreover $X_{\alpha,t}^p = 0$ and $H_{i,b}^p = H_{i,b}$ whenever $\alpha \in \Phi$, $t \in N$, $i = 1, \ldots, l$ and $b \in \boldsymbol{Z}^+$. The following subalgebras are also needed

$$\begin{aligned} \boldsymbol{h}_r &= \langle \{H_{i,b} | 1 \leqq i \leqq l,\ 0 \leqq b < p^r\} \rangle, \\ \boldsymbol{x}_r &= \langle \{X_{\alpha,c} | \alpha \in \Phi^+,\ 0 < c < p^r\} \rangle, \\ \boldsymbol{y}_r &= \langle \{X_{-\alpha,a} | \alpha \in \Phi^+,\ 0 < a < p^r\} \rangle, \\ \boldsymbol{b}_r &= \langle \boldsymbol{h}_r \cup \boldsymbol{y}_r \rangle. \end{aligned} \tag{3}$$

Lemma 1. *The algebras $\boldsymbol{h}_r$, $\boldsymbol{x}_r$ and $\boldsymbol{y}_r$ are already generated as algebras by the sets $\{1, H_{i,p^j} | 1 \leqq i \leqq l,\ 0 \leqq j \leqq r-1\}$, $\{X_{\alpha,p^j} | \alpha \in \Delta,\ 0 \leqq j \leqq r-1\}$ and $\{X_{-\alpha,p^j} | \alpha \in \Delta,\ 0 \leqq j \leqq r-1\}$ respectively.*

Proof. For $\boldsymbol{h}_r$ this follows directly from the proof of Proposition 2.1 in [6]. In case of $\boldsymbol{x}_r$ let $\alpha \in \Phi^+$ and $0 < c < p^r$. Using induction with respect to c, it can easily be shown that $X_{\alpha,c} \in \langle \{X_{\alpha,p^j} | 0 \leqq j \leqq r-1\} \rangle$. Since $p > 3$ the proof of Proposition (71) in [8] implies that $X_{\alpha,p^j} \in \langle \{X_{\alpha,p^j} | \alpha \in \Delta,\ 0 \leqq j \leqq r-1\} \rangle$ for all $\alpha \in \Phi^+$, $0 \leqq j \leqq r-1$. This proves the assertion about $\boldsymbol{x}_r$, and $\boldsymbol{y}_r$ can be handled analogously.

Lemma 2. *Let $\alpha \in \Phi$, $i \in \{1, \ldots, l\}$, $a, c, k, s \in \boldsymbol{Z}^+$ and $r \in N$. Let $\beta \in \Phi$ be such that $\alpha \neq -\beta$ and $\alpha + \beta \notin \Phi$. Then the following commutation rules hold:*

$$X_{\alpha,c} X_{-\alpha,a} = \sum_{k=0}^{\min(a,c)} X_{-\alpha,a-k} \left\{ \binom{h_\alpha - a - c + 2k}{k} \otimes 1 \right\} X_{\alpha,c-k}, \tag{4}$$

$$H_{i,a} X_{\alpha,c} = X_{\alpha,c} \left\{ \binom{h_i + c\alpha(h_i)}{a} \otimes 1 \right\}, \tag{5}$$

$$X_{\alpha,c} H_{i,a} = \left\{ \binom{h_i - c\alpha(h_i)}{a} \otimes 1 \right\} X_{\alpha,c}, \tag{6}$$

$$X_{\alpha,k} X_{\beta,s} = X_{\beta,s} X_{\alpha,k} \tag{7}$$

$$X\boldsymbol{x}_r = \boldsymbol{x}_r X,\ Y\boldsymbol{y}_r = \boldsymbol{y}_r Y, \quad \textit{where} \quad X = \bigcup_{r=1}^{\infty} \boldsymbol{x}_r,\ Y = \bigcup_{r=1}^{\infty} \boldsymbol{y}_r. \tag{8}$$

Proof. The identities (4)—(6) are well known ([4]). Because of our assumptions, $[x_\alpha, x_\beta] = 0$ and from this (7) follows immediately. To prove the commutation rule $X\boldsymbol{x}_r = \boldsymbol{x}_r X$ we first prove $X\boldsymbol{x}_r \subseteq \boldsymbol{x}_r X$. It suffices to show that $X_{\alpha,k} X_{\beta,s} \in \boldsymbol{x}_r X$ for all $\alpha, \beta \in \Phi^+$, $k \in N$ and $0 < s < p^r$. If $\alpha + \beta \notin \Phi$ then by (7) $X_{\alpha,k} X_{\beta,s} = X_{\beta,s} X_{\alpha,k} \in \boldsymbol{x}_r X$. We suppose now that $\alpha + \beta \in \Phi$. Let $U_K[[t, u]]$ be the ring of formal power series over

U_K, where t and u are independent variables. Let $x_\gamma(t)=\sum_{n\geqq 0} t^n X_{\gamma,n}\in U_K[[t,u]]$, where $\gamma\in\Phi$. Then there are integers c_{ij} such that

$$x_\alpha(t)x_\beta(u) = \prod_{i,j\geqq 1} x_{i\alpha+j\beta}(c_{ij}t^i u^j)x_\beta(u)x_\alpha(t)$$

([11], p. 22). Comparing the coefficients of the term $t^k u^s$ in the above equation we see that $X_{\alpha,k}X_{\beta,s}\in \boldsymbol{x}_r X$. The inclusion $\boldsymbol{x}_r X\subseteq X\boldsymbol{x}_r$ can be proved analogously. The commutation rule $Y\boldsymbol{y}_r=\boldsymbol{y}_r Y$ is handled similarly.

Set $\boldsymbol{r}_r=\boldsymbol{y}_r\boldsymbol{h}_r$. Lemma 2 implies $\boldsymbol{y}_r\boldsymbol{h}_r=\boldsymbol{h}_r\boldsymbol{y}_r$, whence $\boldsymbol{r}_r$ is a two-sided ideal of $\boldsymbol{b}_r$. It is not difficult to see that the algebras $\boldsymbol{x}_r$, $\boldsymbol{y}_r$ and $\boldsymbol{r}_r$ are all nilpotent. Write

$$\boldsymbol{u}_r = \boldsymbol{b}_r\oplus\boldsymbol{b}_r\boldsymbol{x}_r = \boldsymbol{h}_r\oplus\boldsymbol{r}_r\oplus\boldsymbol{b}_r\boldsymbol{x}_r$$

and let γ_r: $\boldsymbol{u}_r\to\boldsymbol{b}_r$ be the projection onto the first factor. The algebra $\boldsymbol{u}_r$ now admits a good triangular decomposition over $\boldsymbol{b}_r$ with respect to γ_r (cf. [10]). Moreover, $\ker(\gamma_r)\ker(\gamma_r)\subseteq\ker(\gamma_r)$.

Similarly, if we set $H=\bigcup_{r=1}^{\infty}\boldsymbol{h}_r$, $B=\bigcup_{r=1}^{\infty}\boldsymbol{b}_r$ and $R=\bigcup_{r=1}^{\infty}\boldsymbol{r}_r$ then $R=YH=$ $=HY$ and

$$U_K = B\oplus BX = H\oplus R\oplus BX.$$

The hyperalgebra U_K now admits a good triangular decomposition over B with respect to the projection γ: $U_K\to B$ and $\ker(\gamma)\ker(\gamma)\subseteq\ker(\gamma)$.

4. The irreducible representations of U_K and $\boldsymbol{u}_r$. We have seen above that the hyperalgebra U_K and each of its finite dimensional subalgebras $\boldsymbol{u}_r$ satisfy the hypotheses in Proposition 1. Hence the irreducible U_K- and $\boldsymbol{u}_r$-modules can all be obtained from irreducible H- and $\boldsymbol{h}_r$-modules respectively using the lifting process of § 2. The question now arises: which irreducible H-modules yield finite dimensional U_K-modules?

The algebras H and $\boldsymbol{h}_r$ are commutative. The set $m_r=\{\prod_{i=1}^{l} H_{i,b_i}|0\leqq b_i<p^r\}$ is a K-basis for $\boldsymbol{h}_r$ and $M=\bigcup_{r=1}^{\infty} m_r$ a K-basis for H. Since $H_{i,b}^p=H_{i,b}$ for all i and b the same argument as in [7], p. 193, shows that all H- and $\boldsymbol{h}_r$-modules are completely reducible. Irreducible representations of H and $\boldsymbol{h}_r$ are naturally one-dimensional, let φ: $H\to K$, φ_r: $\boldsymbol{h}_r\to K$ be any such. Finally, let $W(\varphi)=Kw_\varphi$ and $W(\varphi_r)=Kw_{\varphi_r}$ denote the one-dimensional modules corresponding to these algebra homomorphisms.

If $\lambda_1,\dots,\lambda_l$ are the fundamental dominant weights, $P=\boldsymbol{Z}\lambda_1\oplus\dots\oplus\boldsymbol{Z}\lambda_l$ the set of all weights, then the weights in $P_{p^r}=\{\sum_{i=1}^{l} m_i\lambda_i|0\leqq m_i<p^r\}$ are called restricted and those in $P^+=\bigcup_{r=1}^{\infty}P_{p^r}$ dominant. Each $\lambda\in P$ gives rise to an algebra homomorphism

$$\varphi_\lambda:\ H\to K,\ \varphi_\lambda(H_{i,b}) = \binom{\lambda(h_i)}{b}\quad\text{for all}\quad i=1,\dots,l,\ b\in\boldsymbol{Z}^+$$

and one knows that $\varphi_\lambda\neq\varphi_\mu$ whenever $\lambda\neq\mu$. For each $\lambda\in P_{p^r}$ let $\varphi_\lambda|\boldsymbol{h}_r=\varphi_{r,\lambda}$, a one-dimensional representation of $\boldsymbol{h}_r$. Let us also use the following shorter notations $W(r, \lambda)=W(\varphi_{r,\lambda})=Kw_{r,\lambda}$ and $V(r, \lambda)=W(r, \lambda)'$ whenever $\lambda\in P_{p^r}$. It was shown in [6] that the representations $\varphi_{r,\lambda}$ are pairwise non-equivalent and constitute all algebra homomorphisms $\boldsymbol{h}_r\to K$. These observations together with Proposition 1 prove

Proposition 2. *If* $\lambda, \mu\in P_{p^r}$, $\lambda\neq\mu$, *then the irreducible* $\boldsymbol{u}_r$*-modules* $V(r, \lambda)$ *and* $V(r, \mu)$ *are mutually non-isomorphic. In fact* $\operatorname{Irr}(\boldsymbol{u}_r)=\{[V(r, \lambda)]|\lambda\in P_{p^r}\}$.

According to Proposition 1 $\operatorname{Irr}(U_K)=\{[W(\varphi)']|\dim W(\varphi)'<\infty\}$. We shall show next that $\{\varphi|\dim W(\varphi)'<\infty\}=\{\varphi_\lambda|\lambda\in P^+\}$, and thus that $\operatorname{Irr}(U_K)=\{[W(\lambda)']|\lambda\in P^+\}$, where $W(\lambda)=W(\varphi_\lambda)=Kw_\lambda$. The equality $\operatorname{Irr}(U_K)=\bigcup_{r=1}^{\infty}\operatorname{Irr}(\boldsymbol{u}_r)$ then also follows.

Lemma 3. $\{\varphi|\dim W(\varphi)'<\infty\}\subseteqq\{\varphi_\lambda|\lambda\in P^+\}$.

Proof. Assuming $\dim W(\varphi)'<\infty$ we first establish the existence of a $b\in\boldsymbol{N}$ such that $\varphi(H_{i,a})=0$ whenever $a>b$, $i=1, \ldots, l$. Suppose this were not the case. There would then exist an index $i\in\{1, \ldots, l\}$ and an infinite sequence of integers $b_1<b_2<\ldots$ such that $\varphi(H_{i,b_j})\neq0$ for all $j\in\boldsymbol{N}$. Let ω: $W(\varphi)\to P(W(\varphi))$ be as in § 2. Using the commutation rule (4) one shows readily that the elements $X_{-\alpha_i,b_j}\omega(w_\varphi)\in W(\varphi)'$, $j\in\boldsymbol{N}$, are all linearly independent. This, however, contradicts the assumption $\dim W(\varphi)'<\infty$.

We have now seen that $\varphi(M\setminus m_r)=0$ for some $r\in\boldsymbol{N}$. On the other hand φ induces a representation for the subalgebra $\boldsymbol{h}_r$ so that $\varphi|\boldsymbol{h}_r=\varphi_\lambda|\boldsymbol{h}_r$ for some $\lambda\in P_{p^r}$. Then $\varphi_\lambda(M\setminus m_r)=0$ and $\varphi|_{m_r}=\varphi_\lambda|_{m_r}$ hence $\varphi=\varphi_\lambda$ since they agree on the basis M.

Lemma 4. *If* $i\in\{1, \ldots, l\}$ *and* $\lambda\in P^+$ *then* $X_{-\alpha_i,a}\omega(w_\lambda)=0$ *for all* $a>\lambda(h_i)$.

Proof. According to Proposition 1 $\left(W(\lambda)'\right)^{\ker(\gamma)}=\omega(W(\lambda))=\omega(Kw_\lambda)=K\omega(w_\lambda)$. Now $K\omega(w_\lambda)\cap Y\omega(w_\lambda)=0$ since $\omega(w_\lambda)(1)=\gamma(1)w_\lambda=w_\lambda$ and $(y\omega(w_\lambda))(1)=\gamma(y)w_\lambda=yw_\lambda=0$ for all $y\in Y$. Because $X_{-\alpha_i,a}\omega(w_\lambda)\in Y\omega(w_\lambda)$ it suffices to prove that $X_{-\alpha_i,a}\omega(w_\lambda)\in\left(W(\lambda)'\right)^{\ker(\gamma)}$ whenever $a>\lambda(h_i)$. Furthermore, since $\ker(\gamma)=BX$ and $X=\langle\{X_{\alpha_j,c}|j=1, \ldots, l, c\in\boldsymbol{N}\}\rangle$ (Lemma 1), it suffices to prove that $X_{\alpha_j,c}X_{-\alpha_i,a}\omega(w_\lambda)=0$ for all $j\in\{1, \ldots, l\}$, $c\in\boldsymbol{N}$, $a>\lambda(h_i)$. Using the commutation rules of Lemma 2 this follows by induction on a.

Proposition 3. $\{\varphi|\dim W(\varphi)'<\infty\}=\{\varphi_\lambda|\lambda\in P^+\}$.

Proof. It suffices to show $\dim W(\lambda)'<\infty$ for all $\lambda\in P^+$ (Lemma 3). To this end fix $\lambda\in P^+=\bigcup_{s=1}^{\infty}P_{p^s}$ and assume $\lambda\in P_{p^r}$. If we can establish

$$W(\lambda)'=\boldsymbol{u}_r\,\omega(W(\lambda))=\boldsymbol{u}_r\,\omega(w_\lambda) \tag{9}$$

we are done. The proof of this can be reduced to showing that $M=\boldsymbol{u}_r\omega(w_\lambda)$ is a U_K-submodule of $W(\lambda)'$ and this follows if $XM\subseteqq M$ and $YM\subseteqq M$.

Now $M=K\omega(w_\lambda)\oplus \boldsymbol{y}_r\omega(w_\lambda)$. First of all $X\omega(w_\lambda)=0$. Using Lemmas 1 and 2 one sees easily that $X\boldsymbol{y}_r\subseteqq \boldsymbol{u}_r+\boldsymbol{u}_rX$. Hence

$$XM = X\boldsymbol{y}_r\omega(w_\lambda) \subseteqq \boldsymbol{u}_r\omega(w_\lambda) = M.$$

Since $Y=\langle\{X_{-\alpha_i,a}|i=1, \dots, l,\ a\in \boldsymbol{N}\}\rangle$ (Lemma 1) Lemma 4 says that $Y\omega(w_\lambda)\subseteqq \boldsymbol{y}_r\omega(w_\lambda)+Y\boldsymbol{y}_r\omega(w_\lambda)$. The algebra $\boldsymbol{y}_r$ is nilpotent and $Y\boldsymbol{y}_r=\boldsymbol{y}_rY$ by Lemma 2, hence $Y\omega(w_\lambda)\subseteqq \boldsymbol{y}_r\omega(w_\lambda)$. This implies

$$YM = Y\big(K\omega(w_\lambda)+\boldsymbol{y}_r\omega(w_\lambda)\big) \subseteqq \boldsymbol{y}_r\omega(w_\lambda) = M.$$

Now M is a finite dimensional U_K-submodule of $W(\lambda)'$ and U_K admits a good triangular decomposition over B via γ. Therefore $M^{\ker(\gamma)}\neq 0$. According to Proposition 1

$$0 \neq M^{\ker(\gamma)} \subseteqq \big(W(\lambda)'\big)^{\ker(\gamma)} = \omega\big(W(\lambda)\big) = K\omega(w_\lambda).$$

Thus $M^{\ker(\gamma)}=K\omega(w_\lambda)$ and it follows that $W(\lambda)'=U_K\omega(w_\lambda)=U_KM^{\ker(\gamma)}\subseteqq M$. This means $W(\lambda)'=M=\boldsymbol{u}_r\omega(w_\lambda)$ and the proof is complete.

Proposition 4. *If $\lambda, \mu\in P^+$, $\lambda\neq\mu$, then the irreducible U_K-modules $W(\lambda)'$ and $W(\mu)'$ are mutually non-isomorphic. In fact* $\mathrm{Irr}\,(U_K)=\{[W(\lambda)']|\lambda\in P^+\}$.

Proof. This is an immediate consequence of Propositions 1 and 3.

For $\lambda\in P^+$ let $v_\lambda=\omega(w_\lambda)$ and $V(\lambda)=W(\lambda)'=Kv_\lambda\oplus Yv_\lambda$. The vectors

$$X_{-\alpha_1,a_1}\dots X_{-\alpha_m,a_m}v_\lambda \quad (a_i\in \boldsymbol{Z}^+,\ i=1, \dots, m) \tag{10}$$

span the vector space $V(\lambda)$. Because of (5) we get

$$hX_{-\alpha_1,a_1}\dots X_{-\alpha_m,a_m}v_\lambda = \varphi_\mu(h)X_{-\alpha_1,a_1}\dots X_{-\alpha_m,a_m}v_\lambda, \tag{11}$$

$\mu=\lambda-\sum_{j=1}^m a_j\alpha_j$, for all $a_i\in \boldsymbol{Z}^+$ $(i=1, \dots, m)$, $h\in H$. This implies that $V(\lambda)$ has a weight space decomposition

$$V(\lambda) = \oplus \sum_{\mu\in P(\lambda)} V(\lambda)_\mu$$

where $V(\lambda)_\mu=\{v\in V(\lambda)|hv=\varphi_\mu(h)v$ for all $h\in H\}$ and $P(\lambda)=\{\mu\in P|V(\lambda)_\mu\neq 0\}$ is the set of weights of $V(\lambda)$ with respect to H. Formula (11) means that $P(\lambda)\subseteqq \{\lambda-\sum_{j=1}^m a_j\alpha_j|a_j\in \boldsymbol{Z}^+\}$. If W is any H-submodule of $V(\lambda)$ then (cf. [8], p. 4)

$$W = \oplus \sum_{\mu\in P(\lambda)} (V(\lambda)_\mu\cap W). \tag{12}$$

Lemma 5. *If $\lambda\in P_{p^r}$, $\alpha\in\Delta$ and $\lambda-i\alpha\in P(\lambda)$ then $i\leqq p^r-1$.*

Proof. Among the weight vectors (10) only $X_{-\alpha,i}v_\lambda$ is of weight $\lambda-i\alpha$. Hence Lemma 5 follows from Lemma 4.

Proposition 5. *Considered as $\boldsymbol{u}_r$-modules $V(\lambda)$ and $V(r, \lambda)$, $\lambda\in P_{p^r}$, are isomorphic.*

Proof. We have seen above in (9) that $V(\lambda)=\boldsymbol{u}_rv_\lambda$. Therefore it suffices to prove $V(\lambda)^{\ker(\gamma_r)}=Kv_\lambda$. Write $N=V(\lambda)^{\ker(\gamma_r)}$. Because of $\boldsymbol{x}_rH\subseteqq H\boldsymbol{x}_r$ and (12), $N=\oplus\sum_{\mu\in P(\lambda)}\big(V(\lambda)_\mu\cap N\big)$. Using the fact that $X=\langle\{X_{\alpha_i,c}|i=1, \dots, l,\ c\in \boldsymbol{N}\}\rangle$, Lemma 5 and the commutation rule $Xx_r = x_r X$ we can see, as in [1], pp. 43 and 44, that $V(\lambda)_\mu \cap N = 0$ for all $\mu \neq \lambda$. Hence $N = V(\lambda)_\lambda = Kv_\lambda$.

Corollary. (i) $\mathrm{Irr}(\boldsymbol{u}_1) \subset \mathrm{Irr}(\boldsymbol{u}_2) \subset \dots$

$$\text{(ii) } \mathrm{Irr}(U_K) = \bigcup_{r=1}^{\infty} \mathrm{Irr}(\boldsymbol{u}_r).$$

Finally, we show how the irreducible U_K-module $V(\lambda)$ can be constructed as a tensor product of modules of the form $V(p^k\mu)$, $k \in \boldsymbol{Z}^+$, $\mu \in P_p$.

Lemma 6. *Let $r \in \boldsymbol{N}$ and $\lambda \in P_p$. Then $\boldsymbol{y}_r V(p^r\lambda) = 0$ and $\boldsymbol{x}_r V(p^r\lambda) = 0$.*

Proof. Let $v_r = v_{p^r\lambda}$ in which case $V(p^r\lambda) = Kv_r \oplus Yv_r$. Since $\boldsymbol{y}_r Y = Y\boldsymbol{y}_r$ and $\boldsymbol{y}_r$ is generated by elements of the form $X_{-\alpha_i, a}$, $1 \leqq i \leqq l$, $0 < a < p^r$, the assertion $\boldsymbol{y}_r V(p^r\lambda) = 0$ follows if $X_{-\alpha_i, a} v_r = 0$ for this i, a. But using induction on a, it can easily be proved that

$$X_{\beta, c} X_{-\alpha_i, a} v_r = 0 \quad \text{for all} \quad \beta \in \Delta,\, c \in \boldsymbol{N},\, i \in \{1, \dots, l\}, \quad 0 < a < p^r.$$

Then $X_{-\alpha_i, a} v_r \in V(p^r\lambda)^{\ker(\gamma)} \cap Yv_r = 0$, which proves the first claim.

According to (9) we may write $V(p^r\lambda) = Kv_r \oplus \boldsymbol{y}_{r+1} v_r$. Since the elements $X_{-\alpha_i, p^j}$, $1 \leqq i \leqq l$, $0 \leqq j \leqq r$, generate $\boldsymbol{y}_{r+1}$ it is now clear that vectors

$$v_r,\ X_{-\beta_1, p^r} \dots X_{-\beta_s, p^r} v_r, \quad s \in \boldsymbol{N},\, \beta_i \in \Delta, \tag{13}$$

span $V(p^r\lambda)$. Hence

$$\boldsymbol{x}_r V(p^r\lambda) \subseteqq \boldsymbol{u}_{r+1} \boldsymbol{x}_r v_r + \boldsymbol{u}_{r+1} \boldsymbol{y}_r \boldsymbol{u}_{r+1} v_r = 0,$$

because $\boldsymbol{x}_r v_r = 0$ and $\boldsymbol{y}_r V(p^r\lambda) = 0$.

Now U_K has a natural Hopf algebra structure arising from the Hopf algebra ttructure of the universal enveloping algebra $U(\mathfrak{g})$. Let Δ denote the diagonalization map and set $\Delta_1 = \Delta$, $\Delta_n = (\Delta \otimes I^{n-1})\Delta_{n-1}$, $n \geqq 2$. This is determined explicitly by

$$\Delta_{n-1}(X_{\alpha, a}) = \sum_{a_1 + \dots + a_n = a} X_{\alpha, a_1} \otimes \dots \otimes X_{\alpha, a_n}$$

$$\Delta_{n-1}(H_{i, b}) = \sum_{b_1 + \dots + b_n = b} H_{i, b_1} \otimes \dots \otimes H_{i, b_n}$$

for all $n \geqq 2$, $\alpha \in \Phi$, $1 \leqq i \leqq l$, $a, b \in \boldsymbol{Z}^+$. If $V_1, \dots, V_n$ are left U_K-modules then so is $V_1 \otimes \dots \otimes V_n$, the action being given by Δ_{n-1}.

Proposition 6. *Write $\lambda \in P^+$ in the form $\lambda = \lambda_0 + p\lambda_1 + \dots + p^k\lambda_k$ where $\lambda_i \in P_p$. Then*

$$V(\lambda) \cong_{U_K} V(\lambda_0) \otimes V(p\lambda_1) \otimes \dots \otimes V(p^k\lambda_k).$$

Proof. Let $M = V(\lambda_0) \otimes \dots \otimes V(p^k\lambda_k)$ and $v_i = v_{p^i\lambda_i}$, $i = 0, \dots, k$. The assertion follows if we can show

$$M^{\ker(\gamma_{k+1})} = K(v_0 \otimes \dots \otimes v_k),$$

$$M = \boldsymbol{u}_{k+1}(v_0 \otimes \dots \otimes v_k)$$

and

$$K(v_0 \otimes \dots \otimes v_k) \cong_{h_{k+1}} Kv_\lambda.$$

Let $m=\sum_{i=1}^{n} w_i \otimes x_i \in M^{\ker(\gamma_{k+1})}$ where each $w_i \in V(\lambda_0)$ and $x_i \in V(p\lambda_1)\otimes\ldots\otimes V(p^k\lambda_k)$. We may assume that the vectors $x_1, \ldots, x_n$ are linearly independent. If $\alpha\in\Phi^+$ Lemma 6 implies

$$0 = X_{\alpha,1} m = \sum_{i=1}^{n} X_{\alpha,1} w_i \otimes x_i .$$

The linear independence of the vectors x_i then forces $X_{\alpha,1} w_i = 0$ for all $i=1, \ldots, n$. Thus each $w_i \in V(\lambda_0)^{\ker(\gamma_1)} = Kv_0$ and we get

$$M^{\ker(\gamma_{k+1})} \subseteqq Kv_0 \otimes \big(V(p\lambda_1)\otimes\ldots\otimes V(p^k\lambda_k)\big)^{\ker(\gamma_{k+1})}.$$

Replacing $X_{\alpha,1}$ by $X_{\alpha,p}$ in the above argument gives

$$\big(V(p\lambda_1)\otimes\ldots\otimes V(p^k\lambda_k)\big)^{\ker(\gamma_{k+1})} \subseteqq Kv_1 \otimes \big(V(p^2\lambda_2)\otimes\ldots\otimes V(p^k\lambda_k)\big)^{\ker(\gamma_{k+1})}.$$

Continuing this process leads to $M^{\ker(\gamma_{k+1})} = K(v_0\otimes\ldots\otimes v_k)$.

Let

$$N_j = \{1, X_{-\beta_1, p^j} \ldots X_{-\beta_s, p^j} | s\in \mathbf{N}, \beta_i \in \Delta, i = 1, \ldots, s\}$$

where $j=0, \ldots, k$. The set $\{x_0 v_0 \otimes\ldots\otimes x_k v_k | x_j \in N_j, j=0, \ldots, k\}$ is seen to span M as a vector space (cf. (13)). But Lemmas 4 and 6 imply $(x_0 \ldots x_k)(v_0\otimes\ldots\otimes v_k) = x_0 v_0 \otimes\ldots\otimes x_k v_k$, hence $M = \boldsymbol{u}_{k+1}(v_0\otimes\ldots\otimes v_k)$.

Finally, it can easily be confirmed that

$$H_{i,b}(v_0\otimes\ldots\otimes v_k) = \varphi_\lambda(H_{i,b})(v_0\otimes\ldots\otimes v_k) \quad \text{for all} \quad 1 \leqq i \leqq l,\ b\in \mathbf{Z}^+.$$

Therefore $K(v_0\otimes\ldots\otimes v_k) \cong_{h_{k+1}} Kv_\lambda$ and the proof is complete.

5. Irreducible representations of the universal Chevalley group. Let $G = \langle x_\alpha(t) | \alpha\in\Phi, t\in K\rangle$ be the universal Chevalley group of type $\mathfrak{g}$ over the field K ([4], p. 161). Let $K_{p^r} \subset K$ $(r\in \mathbf{N})$ be a finite field of order p^r and $G_r = \langle x_\alpha(t) | \alpha\in\Phi, t\in K_{p^r}\rangle$ the corresponding finite subgroup of G. The irreducible rational G-modules are known to correspond one-to-one to irreducible U_K-modules and similarly for irreducible G_r- and $\boldsymbol{u}_r$-modules. Thus

$$\operatorname{Irr}(G) = \operatorname{Irr}(U_K) = \{[V(\lambda)] | \lambda\in P^+\},$$
$$\operatorname{Irr}(G_r) = \operatorname{Irr}(\boldsymbol{u}_r) = \{[V(\lambda)] | \lambda\in P_{p^r}\}$$

and the groups G, G_r act on the modules $V(\lambda) = W(\lambda)'$ according to the rule

$$x_\alpha(t)v = \sum_{n\geqq 0} t^n X_{\alpha,n} v \quad \text{for all} \quad \alpha\in\Phi, t\in K, v\in V(\lambda). \tag{14}$$

This shows how the irreducible rational G-modules and irreducible G_r-modules over K can all be obtained starting from one-dimensional H-modules (resp. $\boldsymbol{h}_r$-modules) lifting them up to U_K-modules (resp. $\boldsymbol{u}_r$-modules) as in § 2 and then transforming them into G-modules (resp. G_r-modules) using (14).

Let $\lambda\in P_p$ and $k\in \mathbf{Z}^+$. Then $V(p^k\lambda) \cong_G V(\lambda)^{(p^k)}$ where $V(\lambda)^{(p^k)} = V(\lambda)$ with G-action given by $x_\alpha(t)v = x_\alpha(t^{p^k})v$, $\alpha\in\Phi$, $t\in K$, $v\in V(\lambda)$ ([11], p. 217). If the tensor product of U_K-modules in Proposition 6 is viewed as a G-module it becomes an ordinary tensor product of G-modules $V(\lambda_0), \ldots, V(p^k\lambda_k)$. Hence as a corollary to Proposition 6 one obtains a new proof of Steinberg's tensor product theorem:

Proposition 7. *Let* $\lambda = \lambda_0 + p\lambda_1 + \ldots + p^k\lambda_k$ *where* $\lambda_i \in P_p$. *Then* $V(\lambda) \cong_G V(\lambda_0) \otimes V(\lambda_1)^{(p)} \otimes \ldots \otimes V(\lambda_k)^{(p^k)}$.

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