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ON THE INNER RADIUS OF UNIVALENCY FOR NON-CIRCULAR DOMAINS

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Let A be a domain in the extended complex plane, conformally equivalent to a disc. We denote by $\varrho_A|dz|$ the Poincaré metric in A, a conformal invariant so normalized that the density ϱ_A satisfies $\varrho_H(z)=(2\,\mathrm{Im}\,z)^{-1}$ for the upper half-plane H. The norm of the Schwarzian derivative S_f of a locally injective meromorphic function f in A is the number

$$\|S_f\|_A = \sup_{z\in A} |S_f(z)|/\varrho_A(z)^2.$$

The inner radius of univalency $\sigma(A)$ of A, introduced by Lehto ([3], [4]), is the supremum — or maximum — of numbers a such that $\|S_f\|_A \leqq a$ is a sufficient condition for f to be univalent in A. A Möbius transformation of a domain does not change its inner radius of univalency. By a result of Gehring [1], $\sigma(A)$ is positive only if A is bounded by a quasicircle. Also, if $\|S_f\|_A<\sigma(A)$, then $f(A)$ is bounded by a quasicircle, and hence is a Jordan domain.

The classical results of Nehari [5] and Hille [2] show that if A is a disc, then $\sigma(A)=2$. If the monotonicity of $\varrho_A(z)$ with respect to A is taken into account, it is not difficult to conclude that $\sigma(A)\leqq 2$ for any A. Our aim is to sharpen this inequality by proving

Theorem. *For any domain A, not Möbius equivalent to a disc, $\sigma(A)<2$.*

The proof of the Theorem rests on two lemmas, one analytical and the other geometrical:

Lemma 1. *Let $B_k=\{z|0<\arg z<k\pi\}$, $1<k<2$. Then*

$$\sigma(B_k) \leqq 4k-2k^2. \tag{1}$$

Proof. We shall establish the existence of a conformal map $f\colon B_k\to E$ such that $\|S_f\|_{B_k}=4k-2k^2$ and E is not a Jordan domain. In fact, set

$$E = \{z||\arg z| < k\pi/2\}\cap\{z||\arg(1-z)| < k\pi/2\}.$$

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The Schwarzian derivative of a conformal map $g\colon H\to E\cap H$ with $g(0)=1, g(\infty)=0$, $g(1)=\infty$ is given by (see e.g. [6, p. 203])

$$S_g(z) = \frac{4-k^2}{2z^2}+\frac{2k-k^2}{2(z-1)^2}+\frac{k^2-2k}{2z(z-1)}.$$

A conformal map $h\colon H\to E$ is obtained if the map $z\mapsto g(z^2)$, defined in the first quadrant of the plane, is reflected over the imaginary axis. The composition rule for the Schwarzian yields

$$S_h(z) = \frac{1-k^2}{2z^2}+\frac{4k-2k^2}{(z^2-1)^2}.$$

Now $S_\varphi(z)=(1-k^2)/(2z)^2$ for $\varphi(z)=z^k$. Set $f=h\circ\varphi^{-1}$. Then, by $\|S_f\|_{B_k}=\|S_h-S_\varphi\|_H$ and an elementary computation,

$$\|S_f\|_{B_k} = \sup_{y\neq 0}\frac{4y^2(4k-2k^2)}{(x^2-y^2-1)^2+4x^2y^2} = 4k-2k^2.$$

Remark. Lemma 1 complements a result by Lehto [4], who proved an inequality opposite to (1). We have thus actually established the equality

$$\sigma(B_k) = 4k-2k^2$$

for $1\leqq k\leqq 2$.

Lemma 2. *Assume $A\subset H$ is a Jordan domain having two finite boundary points a and b on the real axis such that the open interval (a, b) is in the complement of ∂A. Then A lies in the opening of an obtuse angle whose both sides contain a point of ∂A at an equal distance from the vertex.*

Proof. Choose an s, $0<s<(b-a)/4$. A positive r exists such that the rectangle R with vertices $a+s$, $b-s$, $b-s+ir$, $a+s+ir$ is in the complement of A. For each point p in $\bar{R}\cap H$ one may consider the smallest obtuse angle $B(p)$ with vertex p containing A in its opening. Let $q_1(p)$ and $q_2(p)$ be the points of ∂A closest to p on the left and right side of $B(p)$, respectively. Set

$$\psi(p) = \frac{|p-q_1(p)|}{|p-q_2(p)|}.$$

For any fixed point $a+s+it$, $0<t<r$, let p move on the right side of $B(a+s+it)$ from $a+s+it$ towards $q_2(a+s+it)$. Evidently $\psi(p)$ then increases monotonically from 0 to values >1. There is a unique $p=q_0(t)$ at which either $\psi(p)=1$ or ψ has a finite jump from a value <1 to a value >1. Now consider $q_1(q_0(t))$. Its real part is a monotonically increasing function of t, and a jump in $\psi(p)$ at $q_0(t)$ means a jump in Re $q_1(q_0(t))$. Since the latter ranges between a and $a+s$, there can be only a denumerable number of t's associated with a jump, and a t_0 can be chosen such that $B(q_0(t_0))$ is the desired angle.

The proof of the Theorem is now evident: Any Jordan domain A not a disc is Möbius equivalent to a domain A' of the type described in Lemma 2, and hence also to a domain A'' in some B_k, $1<k<2$, with $\{1, e^{ik\pi}\}\subset\partial A''$. The restriction to A'' of the map f, discussed in the proof of Lemma 1, carries A'' onto a domain not bounded by a Jordan curve, and the monotonicity of the density of the Poincaré metric implies

$$\|S_{f|A''}\|_{A''} \leqq \|S_f\|_{B_k} = 4k-2k^2 < 2.$$

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