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ON TWO-DIMENSIONAL QUASICONFORMAL GROUPS

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A *quasiconformal group* is a group G of homeomorphisms of some open set U in the n-ball S^n such that each element of G is K-quasiconformal for some fixed $K \geqq 1$. If we wish to specify K and U, we say that G is a *K-quasiconformal group acting* in U (or of U).

One can obtain quasiconformal groups as follows. Let $V \subset S^n$ be open and let G be a group of conformal homeomorphisms of V. (If $n \geqq 3$ and V is connected, G is a group of Möbius transformations.) Let $f\colon V \to U$ be quasiconformal. Then fGf^{-1} is a quasiconformal group. It has been proposed by F. Gehring that all quasiconformal groups are of this form. We offer here a proof of this conjecture for quasiconformal groups acting in open subsets of the Riemann sphere.

Kuusalo [3, Theorem 3 p. 21] has proved the following theorem, which is related to ours. Let S be a quasiconformal 2-manifold. Then S has a conformal structure which is compatible with the quasiconformal structure of S. Quasiconformal groups have been considered also in Gehring—Palka [2]. The theory of quasiconformal mappings we need can be found in Lehto—Virtanen [4].

I wish to thank Kari Hag who read the manuscript and made many valuable remarks. In particular, she informed me of Maskit's work [5, 6], and the Corollary was suggested by her.

Theorem. *Let G be a K-quasiconformal group acting in an open subset $U \subset S^2$. Then there is a K-quasiconformal homeomorphism $h\colon U \to V \subset S^2$ such that hGh^{-1} is a group of conformal self-maps of V and that h is the restriction of a K-quasiconformal homeomorphism f of S^2 whose complex dilatation μ_f vanishes a.e. outside U.*

A consequence of the above theorem is that if the fundamental group of a plane domain U is non-cyclic and G is a quasiconformal group of U, G is discrete, i.e. there are no sequences $f_i \in G \setminus \{\mathrm{id}\}$, $i \geqq 0$, such that $\lim_{i\to\infty} f_i = \mathrm{id}$. This follows since the group of conformal self-maps of a plane domain with non-cyclic fundamental group is discrete. This is in turn a consequence of the theory of Fuchsian groups, since in this case the universal cover of U is the open unit disk and the limit set of the cover translation group contains more than two points.

If $V \subset S^2$ is open and connected, there is a conformal homeomorphism $g\colon V \to V' \subset S^2$ such that $g\alpha g^{-1}$ is a Möbius transformation of V' whenever α is

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a conformal self-map of V, cf. Maskit [5, 6]; if V is simply connected, this follows by Riemann's mapping theorem. Therefore we have the following

Corollary. *Let G and U be as in the above theorem and assume that U is connected. Then there is a K-quasiconformal homeomorphism $h'\colon U\to U'\subset S^2$ such that $h'Gh'^{-1}$ is a group of Möbius transformations of U'.*

Note that in general we cannot extend h' to a quasiconformal homeomorphism of S^2.

Proof of the theorem. We find the conditions for μ guaranteeing that if a quasiconformal map f of S^2 has the complex dilatation μ, then f is a solution of our problem. Clearly, we must set

$$\mu|S^2\setminus U = 0. \tag{1}$$

The conformality of $f\circ g\circ (f^{-1}|f(U))$ for $g\in G$ is equivalent to the validity of

$$\mu_f(x) = \mu_{fg}(x) \tag{2}$$

for almost all $x\in U$ when $g\in G$ is fixed. Computing μ_{fg} in terms of μ_f and μ_g we must have

$$\mu_f(x) = \frac{\mu_g(x)+\mu_f\big(g(x)\big)e^{-2i\arg g_z(x)}}{1+\mu_g(x)\mu_f\big(g(x)\big)e^{-2i\arg g_{\bar z}(x)}} \tag{3}$$

a.e. in U. If we can find a measurable function $\mu\colon S^2\to C$ with $\|\mu\|_\infty \leqq (K-1)/(K+1)$ that satisfies (1) and (3) for $\mu=\mu_f$ a.e. in U if $g\in G$ is given, then any homeomorphic solution f of the equation

$$\mu_f = \mu \quad \text{a.e. in } S^2 \tag{4}$$

is also a solution of our problem.

We write (3) in the form

$$\mu_f(x) = T_g(x)\big(\mu_f(g(x))\big) = \mu_{fg}(x), \tag{5}$$

where

$$T_g(x)(z) = \frac{\mu_g(x)+e^{-2i\arg g_z(x)}z}{1+\mu_g(x)e^{-2i\arg g_{\bar z}(x)}z} = \frac{a+\bar b z}{b+\bar a z}$$

with $a=g_{\bar z}(x)$, $b=g_z(x)$. Thus, whenever defined (i.e. a.e. in U for fixed g), $T_g(x)$ is a conformal self-map of the open unit disk D and is an isometry in the hyperbolic metric.

Consider the sets $M_x=\{\mu_g(x)\colon g\in G\}$, $x\in U$. By (5) we have

$$\begin{aligned} T_g(x)(M_{g(x)}) &= \{T_g(x)\big(\mu_{g'}(g(x))\big)\colon g'\in G\} \\ &= \{\mu_{g'g}(x)\colon g'\in G\} = \{\mu_{g'}(x)\colon g'\in G\} \\ &= M_x \end{aligned}$$

for almost all $x\in U$ and every $g\in G$ if G is countable. Let us assume that there is a map $X\mapsto P(X)\in D$ that assigns a point to every non-empty subset $X\subset D$ which is bounded in the hyperbolic metric in such a way that $P\big(g(X)\big)=g\big(P(X)\big)$ for every isometry g of D. Then, for countable G, the map $\mu(x)=P(M_x)$ satisfies (3) (with $\mu_f=\mu$) a.e. in U for all $g\in G$.

Now we construct such a map P. Let $X\subset D$ be bounded, $X\neq\emptyset$. Then there is a unique closed hyperbolic disk $D(x, r)$ with center x and radius $r\geqq 0$ with the properties

(i) $D(x, r)\supset X$, and

(ii) if $D(y, r')\supset X$, $y\neq x$, then $r'>r$.

To see the existence of $D(x, r)$ we can reason as follows. In any case there is a smallest $r\geqq 0$ such that if $r'>r$, there is $y\in D$ with $D(y, r')\supset X$. Next it is easy to see that there is at least one $x\in D$ such that $D(x, r)\supset X$. Assume that there is another point $y\in D$ with $D(y, r)\supset X$. Let w be one of the two points of $\partial D(x, r)\cap\partial D(y, r)$ and let z be the orthogonal projection (in hyperbolic geometry) of w onto the hyperbolic line through x and y. Consider the hyperbolic triangle with vertices x, z and w. It has a right angle at z and therefore it is geometrically evident that $d(x, w)=r>d(z, w)$. This follows also from the relation $\cosh r=\cosh d(x, z)\cosh d(z, w)$ (cf. e.g. Coxeter [1]). But then, if $r'=d(z, w)$, $r'<r$ and $D(z, r')\supset D(x, r)\cap D(y, r)\supset X$. This proves the uniqueness of x. Therefore, if we let $P(X)$ be the center of the smallest closed hyperbolic disk containing X, we have a well-defined map P. Clearly, $P(g(X))=g(P(X))$ for any isometry g of D. It has also the following property.

(A) *If* $X\subset D(y, s)$, $X\neq\emptyset$, *then* $P(X)\in D(y, s)$.

To see the validity of (A), note first that $r\leqq s$ if r is the radius of the smallest disk containing X. Then, if $d(y, P(X))>s$, we can reason as above and find $D(z, r')\supset X$ with $r'<r$. Therefore $d(y, P(X))\leqq s$.

Now we assume for a moment that G is countable, $G=\{g_0, g_1, \ldots\}$. We define a map μ by setting

$$\mu|S^2\setminus U = 0, \quad \text{and}$$
$$\mu(x) = P(M_x) \quad \text{if} \quad x\in U,$$

which defines μ a.e. in S^2. Since $M_x\subset D(0, r)$, where $r=d(0, (K-1)/(K+1))$, for almost all $x\in U$, $\|\mu\|_\infty\leqq(K-1)/(K+1)$ by (A). We have already observed that μ satisfies (3) (with $\mu_f=\mu$) a.e. in U for all $g\in G$. It is also measurable. To see this, let

$$\mu_n(x) = P(\{\mu_{g_i}(x)\colon\ i \leqq n\})$$

if $x\in U$ and $n\geqq 0$. Then μ_n is a.e. defined and it is certainly measurable. Since $\mu(x)=\lim_{n\to\infty}\mu_n(x)$ a.e. in U, also μ is measurable. Therefore, if G is countable, there is a map f satisfying the conditions of the theorem.

If G is not countable, there is always a countable subgroup $G'\subset G$ which is dense in the topology of uniform convergence in compact sets. This follows from the separability of the set of all continuous maps $U\to S^2$ in this topology (they can be approximated by PL maps). Then if f satisfies the conditions of the theorem with respect to G', it satisfies them also with respect to G.

Remarks. An earlier version of this note proved the preceding theorem under the assumptions that G was discrete and that the limit set of G had zero measure. Unable to find a solution of (3) with $\mu=\mu_f$ for general G, I had to make these assumptions on which one can find a measurable fundamental set S for G, and set $\mu|S=0$, determining μ completely. After it was written, F. W. Gehring called my attention to the paper [7] by Sullivan. This paper contains a sketch of the proof of the above theorem. Sullivan's proof differs from ours in the definition of μ; we have defined $\mu(x)=P(M_x)$ whereas Sullivan sets $\mu(x)=B(M_x)$, where $B(X)$ is the barycenter of the convex hull of X, both in hyperbolic geometry. Since Sullivan gives only the barest outline of the proof and since he makes also some unnecessary assumptions (e.g. G was assumed to be discrete), the publication of this little note is perhaps justified.

The use of the map P seems also to have some slight advantages over the use of the map B. No doubt one can take barycenters also in hyperbolic geometry, but to prove the existence of $B(X)$ is non-trivial, whereas this proof is very simple for $P(X)$. Secondly, $X\mapsto P(X)$ is continuous but $X\mapsto B(X)$ is not. That is, if $\varepsilon>0$ is given, there is $\delta=\delta(\varepsilon)>0$ such that if $X, Y\subset D$ are non-empty and bounded and if

$$\varrho(X, Y) = \sup\{d(x, Y),\ d(X, y):\ x\in X,\ y\in Y\} < \delta,$$

then $d(P(X), P(Y))<\varepsilon$; cf. (8) below. To see the discontinuity of B, let $A=\{0, 1\}$ and $A_n=\{0, 1, 1+i/n\}\subset C$, $n>0$. Then if we take the barycenter of the convex hull in the euclidean geometry of C, we have $B(A)=1/2$ but $\lim_{n\to\infty} B(A_n)=2/3$.

Appendix 1. It is easy to derive an estimate for $d(P(X), P(Y))$ in terms of $\varrho(X, Y)$, and since we will need it in a future paper, we do it here. A consequence of this estimate is that if the family $\{\mu_g:\ g\in G\}$ is equicontinuous, $\mu_f|U$ is continuous. Let $X, Y\subset D$ be non-empty and bounded. Let $d=\varrho(X, Y)$, $x=P(X)$, $y=P(Y)$, and let $D(x, r)\supset X$ and $D(y, r')\supset Y$ be the smallest disks containing X and Y, respectively. Then $D(x, r+d)\supset Y$, implying $r'\leqq r+d$. Similarly, $D(y, r'+d)\supset X$, and therefore $D(y, r+2d)\supset X$. We consider the disks $D(x, r)\supset X$ and $D(y, r+2d)\supset X$. If $2d\leqq d(x, y)\leqq 2r+2d$, $\partial D(x, r)\cap\partial D(y, r+2d)\neq\emptyset$. We assume now that $d(x, y)>2d$. Since $D(x, r+d)\supset Y$, by (A) $y\in D(x, r+d)$, implying $d(x, y)\leqq r+d$. Therefore there is a point $w\in\partial D(x, r)\cap\partial D(y, r+2d)$. Consider the hyperbolic triangle T with vertices x, y and w. Let z be the orthogonal projection (in hyperbolic geometry) of w onto the hyperbolic line through x and y. If $z\in T$, $z\neq x$, then $r''=d(z, w)<r$, and $D(z, r'')\supset D(x, r)\cap D(y, r+2d)\supset X$, contradicting the definition of r. Therefore $z\notin T\setminus\{x\}$, i.e., $\varphi\geqq\pi/2$ when φ is the angle of T at x. Now, keep r and $r+2d$ fixed and decrease φ from π to $\pi/2$. Then $d(x, y)$ increases from $2d$ to a value d' with $\cosh(r+2d)=\cosh r\cosh d'$. This is geometrically evident and follows also from the relation $\cosh(r+2d)=\cosh r\cosh d(x, y)-\sinh r\sinh d(x, y)\cos\varphi$; cf. [1]. It follows

$$\cosh d(x, y) \leqq \cosh(r+2d)/\cosh r. \tag{6}$$

This is also valid if $d(x, y) \leqq 2d$, which case we have excluded from the above discussion.

We have $\cosh(r+2d)/\cosh r = e^{2d}(1+e^{-2(r+2d)})/(1+e^{-2r}) \leqq e^{2d}$. Thus, substituting back into (6) $d=\varrho(X, Y)$, $x=P(X)$ and $y=P(Y)$, we get

$$(7) \qquad d(P(X), P(Y)) \leqq \operatorname{ar\,cosh} e^{2\varrho(X,Y)} = \log\left(e^{2\varrho(X,Y)}+(e^{4\varrho(X,Y)}-1)^{1/2}\right) < 2\varrho(X, Y)+\log 2.$$

If $\varrho(X, Y)$ is small, we get a Hölder-type inequality. Let $c=e^{2\varrho(X,Y)}-1$. Then $\log\left(e^{2\varrho(X,Y)}+(e^{4\varrho(X,Y)}-1)^{1/2}\right)=\log\left(1+c+(2c+c^2)^{1/2}\right)=\log\left(1+c^{1/2}(c^{1/2}+(2+c)^{1/2})\right) \leqq c^{1/2}(c^{1/2}+(2+c)^{1/2})$. If $\varrho(X, Y) \leqq R$, $c \leqq 2e^{2R}\varrho(X, Y)$. We have then by (7)

$$(8) \qquad d(P(X), P(Y)) \leqq C(R)\,\varrho(X, Y)^{1/2} \quad \text{if} \quad \varrho(X, Y) \leqq R,$$

where $C(R)=2e^R\left((Re^{2R})^{1/2}+(1+Re^{2R})^{1/2}\right)$.

Note that $P(X)$ exists and that (6), (7), (8) and (A) are valid also if X and Y are non-empty bounded subsets of the n-dimensional hyperbolic space.

Appendix 2. (Added December 1979.) It is possible to give a sharper estimate for the dilatation of the map f of the preceding Theorem. In fact,

f is K'-quasiconformal where $K' = \left(\sqrt{K+1/K}+\sqrt{K}-1/\sqrt{K}\right)/\sqrt{2} \leqq \min(K^{1/\sqrt{2}}, \sqrt{2K})$.

This is an immediate consequence of the following lemma. Note that always $0=\mu_{\mathrm{id}}(z) \in M_z$ and that $d(0, (K-1)/(K+1))=\log K$ when the hyperbolic metric of D is given by $2|dz|/(1-|z|^2)$ in which the formulae of hyperbolic trigonometry are valid.

Lemma. *Let $X \subset D(0, r)$, $r \geqq 0$, and assume that $0 \in X$. Then the center of the smallest hyperbolic disk containing X satisfies*

$$d(0, P(X)) \leqq \beta(r) = \operatorname{ar\,cosh}(\cosh r)^{1/2}.$$

We have the following relations for the function β: $r/2 < \beta(r) < r/\sqrt{2}$ if $r>0$, $\beta(r) < r/2+\log\sqrt{2}$ and $\lim_{r\to\infty}(\beta(r)-r/2)=\log\sqrt{2}$.

Proof. Let $D(x, \varrho)$ be the smallest hyperbolic disk containing X. We can assume that $x \in R$, $x \geqq 0$. It also suffices to consider the case $d(0, x) > r/2$; by (A) always $x \in D(0, r)$. Then

$$(9) \qquad r/2 < d(0, x) \leqq \varrho \leqq r,$$

since $0 \in D(x, \varrho)$ and in any case $\varrho \leqq r$. Thus $\partial D(x, \varrho) \cap \partial D(0, r)$ consists of two points; let z be one of them. Let w be the orthogonal projection (in hyperbolic geometry) of z onto $R \cap D$ (=the hyperbolic line joining 0 and x). We consider the following three cases

$$(\alpha) \quad w \leqq 0; \qquad (\beta) \quad 0 < w < x; \qquad (\gamma) \quad w \geqq x.$$

Let T be the hyperbolic triangle with vertices 0, x and z. In case (α) the angle of T at $0 \geqq \pi/2$. Therefore [1, eq. 12.94] $\cosh\varrho = \cosh d(x, z) \geqq \cosh d(0, x)\cosh d(0, z) = \cosh d(0, x)\cosh r$. This implies $\varrho > r$ which is impossible by (9). Thus (α) is impossible. Case (β) cannot occur either, since now $D(w, d(w, z)) \supset D(0, r) \cap D(x, \varrho) \supset X$. This is impossible since the triangle with vertices x, z and w has a right angle at w and thus $d(w, z) < d(x, z) = \varrho$.

Thus, if $d(0, x) > r/2$, (γ) is the only possibility. Now the angle of T at $x \geqq \pi/2$. This, together with (9), implies $\cosh r = \cosh d(0, z) \geqq \cosh d(0, x) \cosh d(x, z) = \cosh d(0, x) \cosh \varrho \geqq \cosh^2 d(0, x)$. Thus $d(0, x) \leqq \operatorname{ar} \cosh (\cosh r)^{1/2}$, proving the inequality for $d(0, P(X)) = d(0, x)$.

We then examine the properties of $\beta(r)$. Differentiating $\beta(r)$, we get

$$\beta'(r) = \frac{\cosh r/2}{\sqrt{2} \cosh r} = \sqrt{\frac{e^r + e^{-r} + 2}{4(e^r + e^{-r})}} \in (1/2, 1/\sqrt{2})$$

if $r > 0$, proving the first inequalities for β. We get the next, since

$$\begin{aligned} \beta(r) &= \log\left(\sqrt{(e^r + e^{-r})/2} + \sqrt{(e^r + e^{-r})/2 - 1}\right) \\ &= \log\left(\sqrt{e^r + e^{-r}} + e^{r/2} - e^{-r/2}\right) - \log \sqrt{2} \\ &< \log\left(e^{r/2} + e^{-r/2} + e^{r/2} - e^{-r/2}\right) - \log \sqrt{2} = r/2 + \log \sqrt{2}. \end{aligned}$$

Finally, the above expression for $\beta(r)$ gives immediately $\lim_{r \to \infty} (\beta(r) - r/2) = \log \sqrt{2}$.

We remark that the function β is best possible in the above lemma. In fact, let T be the triangle with vertices 0, x and y where $x, y \in \partial D(0, r)$ and T has equal angles at x and y. Choose these angles in such a way that if t is the orthogonal projection of 0 onto the opposite side, we have $d(t, 0) = d(t, x) = d(t, y)$. Then $d(0, t) = d(0, P(T)) = \beta(r)$.

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