Annales Academiæ Scientiarum Fennicæ
Series A. I. Mathematica
Volumen 5, 1980, 79—95

# ON THE BOUNDARY BEHAVIOR OF LOCALLY *K*-QUASICONFORMAL MAPPINGS IN SPACE

MATTI VUORINEN

## 1. Introduction

Let $B^n$, $n \geqq 2$, be the $n$-dimensional unit ball in $R^n$, let $b \in \partial B^n$, and let $f\colon B^n \to G'$ be a quasiconformal mapping. Suppose that $b \in \bar{E}_\varepsilon$ for all $\varepsilon > 0$, where $E_\varepsilon = \{x \in B^n\colon |f(x)| < \varepsilon\}$. This condition means that 0 belongs to the cluster set $C(f, b)$ of $f$ at $b$. Write $\delta_\varepsilon = \operatorname{cap}\underline{\operatorname{dens}}(E_\varepsilon, b)$, where cap $\underline{\text{dens}}$ refers to the lower (conformal) capacity density (for definitions, cf. Section 2). The conformal capacity density has been studied e.g. in [6] and [11]. In [20, 5.5] we proved that $f$ has angular limit 0 at $b$ if $\delta_\varepsilon(\log(1/\varepsilon))^{n-1} \to \infty$ when $\varepsilon \to 0$, i.e. if the numbers $\delta_\varepsilon$ do not tend "too" rapidly to 0. An alternative proof was presented in [21]. As it was shown in [20, Section 5], this result is a quasiconformal counterpart of a theorem of J. L. Doob [2, Theorem 4] about bounded analytic functions.

The purpose of the present paper is to prove related theorems for *locally* $K$-quasiconformal mappings. The main result, proved in Section 3, reads as follows. Let $f\colon B^n \to R^n$ be *locally* $K$-quasiconformal, let $b \in \partial B^n$, and let $D$ be an open cone in $B^n$ with vertex $b$. Write $\tilde{\delta}_\varepsilon = \operatorname{cap}\underline{\operatorname{dens}}(D \cap f^{-1}B^n(\varepsilon), b)$. If $n \geqq 3$ and $C(f, b) \subset \partial f B^n$ and $\tilde{\delta}_\varepsilon^{\,n/(n-1)}(\log(1/\varepsilon))^{n-1} \to \infty$ when $\varepsilon \to 0$, then $f$ has angular limit 0 at $b$. The proof of this theorem is based on the method used in [21, 4.12] and on an injectivity theorem of Martio, Rickman, and Väisälä [10, 2.3], which yields an upper bound for the maximal multiplicity of a locally $K$-quasiconformal mapping of $B^n$ in a non-tangential domain, provided that the dimension $n \geqq 3$. Instead of a cone with a fixed angle, like $D$ above, one may consider in the definition of $\tilde{\delta}_\varepsilon$ cones with the central angle increasing towards $\pi/2$ in a tempered way as $\varepsilon \to 0$. For details we refer the reader to Theorem 3.1.

In Section 4 we consider the situation of the above result if the condition $C(f, b) \subset \partial f B^n$ is removed. Employing now a different method we prove the following theorem. Let $f\colon B^n \to R^n$ be locally $K$-quasiconformal, let $b \in \partial B^n$, and let $D$ be an open cone in $B^n$ with vertex $b$. Suppose that $E \subset D$ and $\operatorname{cap}\underline{\operatorname{dens}}(E, b) > 0$. If $n \geqq 3$ and $f(x)$ tends to 0 when $x$ approaches $b$ through the set $E$, then $f$ has angular limit 0 at $b$. By an example we show that this result is, in a sense, the best possible.

doi:10.5186/aasfm.1980.0532

Finally, in Section 5, we consider a subclass of quasiregular mappings of $B^n$, $n \geqq 2$, characterized by the property that the maximal multiplicity is uniformly bounded in each hyperbolic ball with a fixed radius (cf. (5.1)). From the injectivity theorem [10, 2.3] it follows that locally $K$-quasiconformal mappings of $B^n$, $n \geqq 3$, have the same property. It is pointed out that the results in Sections 3 and 4 hold for mappings in this larger class as well. A normality criterion, related to a problem of W. K. Hayman, is given for functions in the mentioned class.

## 2. Preliminary results

The notation and terminology will be, in general, as in [20], [21], and [8]. For definitions and basic properties of quasiconformal and quasiregular mappings we refer the reader to Väisälä's book [19] and to the papers of Martio, Rickman, and Väisälä [8], [9], [10]. A mapping $f\colon G \to R^n$ is *locally K-quasiconformal* if there exists a number $K \in [1, \infty)$ such that $f$ is $K$-quasiconformal in a neighborhood of each point of $G$. Here $G \subset R^n$ is a domain. A sense-preserving mapping is locally $K$-quasiconformal if and only if it is a $K$-quasiregular local homeomorphism (cf. [8, p. 14]).

2.1. *Notation.* If $x \in R^n$, $n \geqq 2$, and $r > 0$, then $B^n(x, r) = \{y \in R^n\colon |x-y| < r\}$, $S^{n-1}(x, r) = \partial B^n(x, r)$, $B^n(r) = B^n(0, r)$, $S^{n-1}(r) = S^{n-1}(0, r)$, $B^n = B^n(1)$, and $S^{n-1} = S^{n-1}(1)$. For $x \in R^n$ and $r > s > 0$ we write $R(x, r, s) = B^n(x, r) \setminus \bar{B}^n(x, s)$ and $R(r, s) = R(0, r, s)$. The standard unit coordinate vectors are $e_1, \ldots, e_n$.

2.2. *Path families and their modulus.* A *path* is a continuous nonconstant mapping $\gamma\colon \Delta \to A$, $A \subset \bar{R}^n$, where $\Delta$ is an interval on the real axis. The point set $\gamma\Delta$ will be denoted by $|\gamma|$. Given $E$, $F$, and $G$ in $\bar{R}^n$, we let $\Delta(E, F; G)$ be the family of all paths $\gamma\colon [0, 1] \to G$ with $\gamma(0) \in E$ and $\gamma(1) \in F$ (cf. [19, p. 21]). For the definition and basic properties of the *(n-)modulus* $M(\Gamma)$ of a path family $\Gamma$ we refer the reader to Väisälä's book [19, Section 6]. If $u \in R^n$ and $t > r > 0$ and $\Gamma$ is a path family such that $|\gamma|$ intersects both boundary components of $R(u, t, r)$ for each $\gamma \in \Gamma$, then the following estimate holds ([19, 7.5]):

$$M(\Gamma) \leqq \omega_{n-1} \left( \log \frac{t}{r} \right)^{1-n}. \tag{2.3}$$

Here $\omega_{n-1}$ is the surface area of $S^{n-1}$. For $E \subset R^n$, $x \in R^n$, and $t > r > 0$ we abbreviate

$$M_t(E, r, x) = M\big(\Delta(S^{n-1}(x, t), \bar{B}^n(x, r) \cap E; R^n)\big),$$

$$M(E, r, x) = M_{2r}(E, r, x).$$

The *lower* and *upper capacity densities* of $E$ at $x$ are defined by (cf. [20] and Martio—Sarvas [11])

$$\operatorname{cap}\underline{\operatorname{dens}}(E, x) = \liminf_{r\to 0} M(E, r, x),$$

$$\operatorname{cap}\overline{\operatorname{dens}}(E, x) = \limsup_{r\to 0} M(E, r, x).$$

If $E$ is compact, this definition is equivalent to the one employed in [11], which is based on the use of $n$-capacities of condensers (cf. Ziemer [22]). Some sufficient conditions for $\operatorname{cap}\underline{\operatorname{dens}}(E, x)>0$ were given in [20, Section 2]. See also Martio [6, 3.1]. From a result of Wallin it follows that there are sets $E$ with $\operatorname{cap}\underline{\operatorname{dens}}(E, 0)>0$ which have Hausdorff dimension zero [20, 2.5 (3)]. For $t>s>r>0$

$$M_t(E, r, x) \leqq M_s(E, r, x) \leqq \left(\frac{\log(t/r)}{\log(s/r)}\right)^{n-1} M_t(E, r, x). \tag{2.4}$$

One can prove (2.4) by making use of a radial quasiconformal mapping which is identity in $\bar{B}^n(r)$ and maps $R(s, r)$ onto $R(t, r)$ (cf. [11, 2.7]). Using (2.4) we prove the following lemma.

2.5. Lemma. $\operatorname{cap}\underline{\operatorname{dens}}(E, 0)=\liminf_{r\to 0} M(E\cap B^n(r), r, 0)$.

*Proof.* Denote by $a$ and $b$ the left and right hand sides of the equality, respectively. Obviously $a\geqq b\geqq 0$. Hence it suffices to prove $a\leqq b$ and we may assume that $a>0$. Choose $a'\in(0, a)$ and $r_0\in(0, 1)$ in such a way that $M(E, r, 0)\geqq a'$ for all $r\in(0, r_0)$. Fix $r\in(0, r_0)$. For all $k=2, 3, \ldots$ we get by (2.4)

$$\begin{aligned} M(E\cap B^n(r), r, 0) &\geqq M_{2r}(E\cap B^n(r), r(1-1/k), 0) \\ &= M_{2r}(E, r(1-1/k), 0) \geqq d_k^{1-n} M(E, r(1-1/k), 0) \geqq d_k^{1-n} a', \end{aligned}$$

where $d_k=\log(2/(1-1/k))/\log 2$. Since $d_k\to 1$, this implies $M(E\cap B^n(r), r, 0)\geqq a'$. Hence $b\geqq a'$. Letting $a'\to a$ yields the desired conclusion.

The next lemma was proved by Näkki [15] (cf. also Martio, Rickman, and Väisälä [10, 3.11]). It will be called here, as in [15], *the comparison principle for the modulus*. Throughout the paper we let $c_n$ denote the positive constant in [19, 10.9], depending only on $n$.

2.6. Lemma. *Let $F_1$, $F_2$, and $F_3$ be three sets in $\bar{R}^n$ and let $\Gamma_{ij}=\Delta(F_i, F_j; \bar{R}^n)$, $1\leqq i, j\leqq 3$. If there exist $x\in R^n$ and $0<a<b$ such that $F_1, F_2\subset\bar{B}^n(x, a)$ and $F_3\subset\bar{R}^n\setminus B^n(x, b)$, then*

$$M(\Gamma_{12}) \geqq 3^{-n}\min\left\{M(\Gamma_{13}), M(\Gamma_{23}), c_n\log\frac{b}{a}\right\}.$$

2.7. Corollary. *Let $E_j\subset R^n$ with $M(E_j, s, 0)=\delta_j>0, j=1, 2,$ for some $s>0$ and let $t=3^{-n}\min\{\delta_1, \delta_2, c_n\log 2\}$. Choose $\lambda>1$ such that $\log\lambda=(t/6\omega_{n-1})^{1/(1-n)}$ and let $F_j=E_j\cap\bar{R}(s, s/\lambda), j=1, 2.$ Then $M(\Delta(F_1, F_2; R^n))\geqq 5t/6$.*

*Proof.* From the choice of $\lambda$ it follows by (2.3) that

$$M(\bar{B}^n(s/\lambda), s, 0) \leqq t/6,$$

and hence the proof follows from Lemma 2.6 and the estimates

$$M(F_j, s, 0) \geqq \delta_j - t/6 \geqq 3^n 5t/6, \quad j = 1, 2.$$

2.8. *The hyperbolic metric.* The *hyperbolic metric* $\varrho$ in $B^n$ is defined by the element of length $d\varrho = |dx|/(1-|x|^2)$. If $a$ and $b$ are points in $B^n$, then $\varrho(a, b)$ denotes the geodesic distance between $a$ and $b$ corresponding to this element of length. For $b \in B^n$ and $M \in (0, \infty)$ we let $D(b, M)$ denote the *hyperbolic ball* $\{x \in B^n: \varrho(b, x) < M\}$. Let $r_b = \min\{|z-b|: z \in \partial D(b, M)\}$. By integrating we get

$$r_b = \frac{(1-|b|^2)\tanh M}{1+|b|\tanh M}. \tag{2.9}$$

This implies that $B^n(b, \tanh M(1-|b|)) \subset D(b, M)$.

In what follows we shall need some properties of normal mappings. We recall that a mapping $f\colon B^n \to R^n$ is said to be *normal* if for each sequence $(h_k)$ of conformal self-mappings of $B^n$ there is a subsequence of $(f \circ h_k^{-1})$ converging uniformly on compact subsets of $B^n$ (or briefly $c$-uniformly) towards a limit mapping $g\colon B^n \to \bar{R}^n$ (cf. [19, p. 68], [20, Section 3]). The *cluster set* of $f$ at $b \in \partial B^n$ is the set $C(f, b)$ of all points $b' \in \bar{R}^n$ for which there exists a sequence $(x_k)$ in $B^n$ with $x_k \to b$ and $f(x_k) \to b'$. The next lemma makes use of some ideas of Bagemihl's and Seidel's [1, p. 5].

2.10. Lemma. *Let $f\colon B^n \to R^n$ be a quasiregular mapping, let $(b_k)$ be a sequence in $B^n$ with $b_k \to b \in \partial B^n$ and $f(b_k) \to \alpha$, and let $M \in (0, \infty)$ and $E = \cup D(b_k, M)$. Suppose that $\alpha \in \partial f B^n$. If $f$ is normal or if $C(f, b) \subset \partial f B^n$, then $f(x) \to \alpha$ as $x \to b$ through the set $E$.*

*Proof.* It is well known that $C(f, b)$ is a non-empty compact connected set (cf. [19, 17.1, 17.5 (1)]). If $C(f, b)$ consists of one point, there is nothing to prove. Otherwise $C(f, b)$ is a non-degenerate continuum. If $C(f, b) \subset \partial f B^n$, it follows that $\operatorname{cap}(\bar{R}^n \setminus f B^n) > 0$ in the terminology of [9], and hence $f$ is normal by [9, 3.17]. Hence $f$ is quasiregular and normal, and it follows from [17, p. 497] that the condition in [20, 6.3] is satisfied. The proof follows from [20, 6.3].

The assumption $\alpha \in \partial f B^n$ in the above lemma can be replaced by the requirement that $f^{-1}(\alpha)$ be finite (cf. [1, p. 5], [20, 6.4]). By considering the behavior of the function $f\colon B^2 \to B^2, f(z) = \exp((z+1)/(z-1))$, near $z=1$ we see that this assumption cannot be dropped. The next example shows that corresponding functions exist when the dimension $n=3$.

2.11. Example. We shall slightly modify the example constructed by Martio and Srebro in [12, 4.1]. Let $g\colon R^3_+ \to T$ be the locally $K$-quasiconformal automorphic mapping constructed in [12, 4.1], and let $h\colon B^3 \to R^3_+$ be the Möbius transformation $h(x)=2(x+e_1)|x+e_1|^{-2}-e_1$. Here $T$ is an open solid torus and the mapping $f=g\circ h$: $B^3\to T$ has a continuous extension $\bar{f}$: $\bar{B}^3\setminus\{e_1, -e_1\}\to\bar{T}$ with $\bar{f}(\partial B^3\setminus\{e_1, -e_1\})\subset\partial T$. By the construction of $g, f$ maps the segment $\{e_1 t\colon -1<t<1\}$ onto a closed curve $S\subset T$. Fix $\alpha, \beta\in S$, $\alpha\neq\beta$. By the construction of $f$ there are increasing sequences $(s_k)$, $(t_k)$ in $(0, 1)$ with $s_k<t_k<s_{k+1}$ for all $k$ such that $\lim s_k=1$, $f(s_k e_1)=\alpha$, $f(t_k e_1)=\beta$, $k=1, 2, \ldots$ and such that $\varrho(s_k e_1, t_k e_1)<M$ for all $k$ and some $M\in(0, \infty)$. Thus the conclusion of Lemma 2.10 does not hold for this function $f$. Hence the assumption $\alpha\in\partial fB^n$ cannot be dropped.

Let $f$: $B^n\to R^n$ be a mapping, $y\in R^n$, and $D\subset\bar{R}^n$. Then we denote by $N(y, f, D)$ the number of the points in $f^{-1}(y)\cap D$. The *maximal multiplicity* of $f$ in $D$ is

$$N(f, D) = \sup\{N(y, f, D)\colon y\in R^n\}.$$

The next lemma follows from [10, 2.3].

2.12. Lemma. *Let $n\geqq 3$ and $K\geqq 1$. Then there is a constant $\psi(n, K)\in(0, 1)$ such that if $f$: $B^n\to R^n$ is a locally $K$-quasiconformal mapping, then $f$ is injective in $B^n(\psi(n, K))$. Moreover, for every $r\in(0, 1)$ there is a number $c(n, K, r)\in[1, \infty)$ depending only on $n$, $K$, and $r$, and a number $b(n)$ depending only on $n$ such that*

$$N(f, \bar{B}^n(r)) \leqq c(n, K, r) \leqq \left(\frac{b(n)}{\psi(n, K)(1-r)}\right)^n.$$

*Proof.* The first part of the lemma was proved by Martio, Rickman, and Väisälä [10, 2.3]. From the first part it follows that $f$ is injective in $B_x^t=B^n(x, \psi(n, K)t)$, $x\in B^n$, when $0<t\leqq 1-|x|$ and one may define $c(n, K, r)$ to be the smallest number of the balls $B_x^t$ needed to cover $\bar{B}^n(r)$. The estimate for $c(n, K, r)$ follows from known properties of coverings by families of balls (cf. [14, Lemma 3] and [4, p. 197, Lemma 3.2]).

Note that Lemma 2.12 is false for $n=2$ (cf. [10, 2.11]).

2.13. Remark. One can improve the upper bound for $c(n, K, r)$ by making use of ideas presented in [7, 5.27]. In this way one obtains an estimate of the type $c(n, K, r)\leqq A(1-r)^{1-n}\log(2/(1-r))$, where $A>0$ depends only on $n$ and $K$, but we shall not need such an estimate here.

Using Lemma 2.12 we shall now prove an upper bound for the maximal multiplicity of a locally $K$-quasiconformal mapping in a non-tangential domain of a particular shape. For $b\in\partial B^n$ and $\varphi\in(0, \pi/2)$ we let $K(b, \varphi)$ denote the cone $\{z\in R^n\colon (b|b-z)>|b-z|\cos\varphi\}$. Here $(x|y)$ is the inner product $\sum_{i=1}^n x_i y_i$.

2.14. Lemma. *If $n\geqq 3$, $K\geqq 1$, and $\varphi\in(0, \pi/2)$, then there are constants $a(n, K)>0$ and $d(n, K, \varphi)>0$, depending only on the numbers indicated, with the following properties. Let $f$: $B^n\to R^n$ be a locally $K$-quasiconformal mapping, $b\in\partial B^n$,* *let* $t\in(0, \cos\varphi)$, *and let* $A_\lambda^\varphi(t)=K(b,\varphi)\cap R(b,t,t/\lambda)$ *for* $\lambda>1$. *Then for* $\lambda\geqq 2$ *the following estimates hold:*

$$N(f, \bar{A}_\lambda^\varphi(t)) \leqq d(n, K, \varphi)\log\lambda,$$

*where*

$$d(n, K, \varphi) \leqq a(n, K)\cos^{-2n}\varphi.$$

*Proof.* Fix $b\in\partial B^n$, $\varphi\in(0,\pi/2)$, and $t\in(0,\cos\varphi)$. We first consider the case $\lambda=2$. By elementary geometry $\bar{A}_2^\varphi(t)\subset\bar{B}^n(x,r)$, where

$$\begin{cases} x = \left(1-\dfrac{3t}{4\cos\varphi}\right)b \\ r = \dfrac{t}{4}(9\tan^2\varphi+1)^{1/2}. \end{cases}$$

Then $N(f,\bar{A}_2^\varphi(t))\leqq N(f,\bar{B}^n(x,r))$ and by Lemma 2.12

$$N(f, \bar{B}^n(x,r)) \leqq c(n, K, v_\varphi),$$

$$v_\varphi = r/|b-x| = (\sin^2\varphi+(1/9)\cos^2\varphi)^{1/2}.$$

Let us now consider the case $\lambda>2$. Fix $\lambda>2$. Define

$$m = \min\{k\in N\colon\ 2^{-k}t \leqq t/\lambda\} \geqq 2.$$

Thus $2^{-m}\leqq 1/\lambda\leqq 2^{-m+1}\leqq 2^{-m/2}$ and hence $m\leqq\log\lambda/\log\sqrt{2}$. Using the estimate obtained in the case $\lambda=2$ we get

$$N(f, \bar{A}_\lambda^\varphi(t)) \leqq \sum_{j=1}^{m} N(f, \bar{A}_2^\varphi(2^{-j+1}t)) \leqq c(n, K, v_\varphi)\log\lambda/\log\sqrt{2}.$$

These estimates hold for $\lambda=2$ as well. Hence for all $\lambda\geqq 2$ we may choose $d(n,K,\varphi)=c(n,K,v_\varphi)/\log\sqrt{2}$. Since $1-v_\varphi\geqq(4/9)\cos^2\varphi$, the desired estimate with

$$a(n,K) = (9b(n)/4\psi(n,K))^n/\log\sqrt{2}$$

follows from Lemma 2.12.

2.15. Remark. We shall now show by investigating the mapping $f$ of Example 2.11 that the upper bound of Lemma 2.14 is of the correct order of magnitude for this mapping. By the construction of the automorphic mapping $f\colon B^3\to T$ there exist $a\in T$ and a sequence $(u_k)$ in $(0,1)$ with $\lim u_k=1$ such that $f(u_k e_1)=a$ for all $k=1,2,\ldots$ and a number $M\in(0,\infty)$ such that $\varrho(u_k e_1, u_{k+1}e_1)<M$ for all $k=1,2,\ldots$. Fix $\varphi\in(0,\pi/2)$. After relabeling if necessary we may assume that $1-u_1<\cos\varphi$ and $1-u_1<1/2$. For $\lambda\geqq 2$ let $A_\lambda^\varphi=K(e_1,\varphi)\cap R(e_1, 1-u_1, (1-u_1)/\lambda)$. Define

$$p = \min\left\{k\in N\colon\ M(k+1) > \varrho\left(e_1u_1, e_1\left(1-\frac{1-u_1}{\lambda}\right)\right)\right\}.$$

Then $N(f, \bar{A}_\lambda^\varphi) \geqq N(a, f, \bar{A}_\lambda^\varphi) \geqq p$. Since for $0 \leqq v < w < 1$

$$\varrho(e_1 v, e_1 w) = \frac{1}{2} \log \frac{1+w}{1-w} \cdot \frac{1-v}{1+v},$$

we get the estimates

$$\varrho\left(e_1 u_1, e_1\left(1-\frac{1-u_1}{\lambda}\right)\right) \geqq \frac{1}{2} \log\left(\lambda - \frac{1}{3}\right) \geqq \frac{1}{4} \log \lambda$$

for $\lambda \geqq 2$, where we have used the fact $1-u_1 < 1/2$. Hence if $\lambda \geqq 2$ is large enough, then $p \geqq 1$, and hence $M(p+1) \leqq 2Mp$, which together with the above estimates yields

$$N(f, \bar{A}_\lambda^\varphi) \geqq \frac{1}{8M} \log \lambda.$$

We have thus shown that the dependence on $\lambda$ in the upper bound of Lemma 2.14 is the best possible when $\varphi$ is fixed.

For $\theta \in (0, \pi/2)$ let $C(\theta) = \{x \in R^n \colon (x|e_n) > |x| \cos \theta\}$. If $A \subset R^n$ we write $A_+ = \{x \in R^n \colon x_n > 0\}$. In the next lemma we construct a quasiconformal mapping $f\colon \bar{R}^3 \to \bar{R}^3$ such that $f$ maps the truncated cone $C(\theta) \cap R(1, s)$ onto $B^3$ for given $\theta \in (0, \pi/2)$ and $s \in (0, 1)$ and such that we get an appropriate upper bound for $K(f)$. The numerical value of this upper bound is probably not the best possible.

2.16. Lemma. *Let* $n=3$, $\theta_0 \in (0, \pi/2)$, *and* $s \in (0, 1)$. *Then there exists a constant* $Q(3, \theta_0, s) \geqq 1$ *and a* $Q(3, \theta_0, s)$*-quasiconformal mapping* $f\colon \bar{R}^3 \to \bar{R}^3$ *which maps* $C(\theta_0) \cap R(1, s)$ *onto* $B^3$. *Moreover,* $Q(3, \theta_0, s) \leqq Q(3, \theta_0, r)$ *when* $\theta_0 \in (0, \pi/2)$ *and* $0 < s \leqq r < 1$, *and* $Q(3, \theta_0, s) \geqq Q(3, \theta, s)$ *when* $0 < \theta_0 \leqq \theta < \pi/2$ *and* $s \in (0, 1)$.

*Proof.* The proof makes use of some ideas of Gehring and Väisälä [3] (cf. Lemma 8.2 in [3] and the proof of Lemma 3.4 in Martio—Srebro [13]).

Let $(R, \varphi, \theta)$ be the spherical coordinates in $R^3$, where $\varphi \in [0, 2\pi)$ is measured from the direction of $e_1$ to the direction of $e_2$ and $\theta \in [0, \pi]$ is measured from the direction of $e_3$ (cf. [19, 16.4]). Let $f_1\colon \bar{R}^3 \to \bar{R}^3$ be the mapping defined by $f_1(\infty) = \infty$ and

$$\begin{cases} f_1(R, \varphi, \theta) = \left(R, \varphi, \dfrac{\pi}{2\theta_0}\theta\right), & \text{if } 0 \leqq \theta < \theta_0, \\ f_1(R, \varphi, \theta) = \left(R, \varphi, \dfrac{\pi\theta}{2(\pi-\theta_0)} + \dfrac{\pi(\pi-2\theta_0)}{2(\pi-\theta_0)}\right), & \text{if } \theta_0 \leqq \theta \leqq \pi. \end{cases}$$

Then $f_1$ is quasiconformal and maps $C(\theta_0)$ onto $R^3_+$ and $C(\theta_0) \cap R(1, s)$ onto $R(1, s)_+$. It follows from [19, 16.4, 35.1] that

$$K(f_1) \leqq \left(\frac{\pi}{\theta_0} - 1\right)^2 \sin^{-2}\theta_0. \tag{2.17}$$

Let $f_2$: $\bar{R}^3 \to \bar{R}^3$ be the Möbius transformation defined by $f_2(-e_1)=\infty, f_2(\infty)=-e_1$, and

$$f_2(x) = 2\frac{x+e_1}{|x+e_1|^2} - e_1, \quad \text{if} \quad x \in R^3 \setminus \{-e_1\}.$$

Then $f_2$ maps $R(1, s)_+$ onto $\{x \in R^3_+ : x_1 > 0\} \setminus \bar{B}$, where $B = B^3((1+s^2)e_1/(1-s^2), 2s/(1-s^2))$ and $f_2 B^3(s) = B$. Let $T$ be the tangent plane of $B$ which contains the $x_2$-axis and passes through the point $e_1 + (2s/(1-s^2))e_3 \in \partial B$. Denote by $\alpha_s$ the acute angle between $T$ and $e_3$. Then $\alpha_s = \overline{\text{arc}} \tan((1-s^2)/2s)$. Let $(r, \varphi, x_2)$ be the cylindrical coordinates in $R^3$ with the $x_2$-axis as the symmetry axis, where $\varphi$ is measured from the direction of $e_3$. Let $f_3$: $\bar{R}^3 \to \bar{R}^3$ be a quasiconformal folding defined by $f_3(\infty) = \infty$ and

$$\begin{cases} f_3(r, \varphi, x_2) = (r, \varphi, x_2), & \text{if } 0 \leqq \varphi < \frac{\pi}{2} - \alpha_s, \\ f_3(r, \varphi, x_2) = \left(r, \frac{\pi + 2\alpha_s}{2\alpha_s} + \frac{\pi}{2}\frac{2\alpha_s - \pi}{2\alpha_s}, x_2\right), & \text{if } \frac{\pi}{2} - \alpha_s \leqq \varphi < \frac{\pi}{2}, \\ f_3(r, \varphi, x_2) = \left(r, \frac{2}{3}\varphi + \frac{2\pi}{3}, x_2\right), & \text{if } \frac{\pi}{2} \leqq \varphi < 2\pi. \end{cases}$$

Then $f_3$ maps $\{x \in R^3_+ : x_1 > 0\} \setminus \bar{B}$ onto $R^3_+ \setminus \bar{B}$ and it follows from [19, 16.3, 35.1] that

$$K(f_3) = \max\left\{\left(\frac{3}{2}\right)^2, \left(\frac{\pi}{2\alpha_s} + 1\right)^2\right\}; \quad \alpha_s = \overline{\text{arc}} \tan \frac{1-s^2}{2s}. \tag{2.18}$$

Let $f_4$: $\bar{R}^3 \to \bar{R}^3$ be the Möbius transformation defined by $f_4(a) = \infty$, $a = ((1+s)/(1-s))e_1$, $f_4(\infty) = a$, and

$$f_4(x) = c^2 \frac{x-a}{|x-a|^2} + a, \quad \text{if} \quad x \in R^3 \setminus \{a\}; \; c^2 = \frac{4s}{(s-1)^2}.$$

Then $f_4$ maps $B$ onto $\{x \in R^3 : x_1 > 0\}$ in such a way that $B_+$ is mapped onto $\{x \in R^3_+ : x_1 > 0\}$. Let $(r, \varphi, x_2)$ be the same cylindrical coordinate system as above and define $f_5$: $\bar{R}^3 \to \bar{R}^3$ by $f_5(\infty) = \infty$ and

$$\begin{cases} f_5(r, \varphi, x_2) = (r, 2\varphi, x_2), & \text{if } 0 \leqq \varphi < \frac{\pi}{2}, \\ f_5(r, \varphi, x_2) = \left(r, \frac{\varphi}{2} + \frac{3\pi}{4}, x_2\right), & \text{if } \frac{\pi}{2} \leqq \varphi < \frac{3\pi}{2}, \\ f_5(r, \varphi, x_2) = (r, \varphi, x_2), & \text{if } \frac{3\pi}{2} \leqq \varphi < 2\pi. \end{cases}$$

Then $f_5$ maps $f_4(\{x \in R^3_+ : x_1 > 0\} \setminus \bar{B}) = \{x \in R^3_+ : x_1 > 0\}$ onto $R^3_+$ and $f_5$ is $2^2$-quasiconformal (cf. [19, 16.3, 35.1]). The mapping $f_6 = f_5 \circ f_4 \circ f_3 \circ f_2 \circ f_1$: $\bar{R}^3 \to \bar{R}^3$ is quasiconformal with $f_6(C(\theta_0) \cap R(1, s)) = R^3_+$ and the assertion follows from this in view of (2.17) and (2.18).

2.19. Lemma. *Let* $n \geqq 3$ *and* $\theta_0 \in (0, \pi/2)$. *Then there exists a constant* $Q(n, \theta_0) \geqq 1$ *depending only on n and* $\theta_0$ *such that if* $\lambda \geqq 2$ *and* $\theta \in [\theta_0, \pi/2)$, *there exists a* $Q(n, \theta_0)$*-quasiconformal mapping* $f\colon \bar{R}^n \to \bar{R}^n$ *with* $f(C(\theta) \cap R(1, 1/\lambda)) = B^n$.

*Proof.* Since the constant $Q(3, \theta, s)$ of Lemma 2.16 is increasing as a function of $s$ and decreasing as a function of $\theta$ we may choose $Q(3, \theta_0) = Q(3, \theta_0, 1/2)$. The proof of the general case $n \geqq 3$ can be carried out by generalization of Lemma 2.16 to the $n$-dimensional case.

2.20. Remark. Martio and Srebro have studied in [13] the problem of mapping strictly star shaped domains of $R^n$ onto $B^n$ by means of bi-lipschitzian quasiconformal mappings of $R^n$. Note that the domains in 2.19 need not be star shaped.

The next lemma is one variant of a symmetry principle for the modulus (cf. Gehring—Väisälä [3, Lemma 3.3] and also [20, 4.3]).

2.21. Lemma. *Let D be a domain in* $R^n$, $n \geqq 2$, *and suppose that there exists a quasiconformal mapping* $f\colon \bar{R}^n \to \bar{R}^n$ *with* $fD = B^n$. *If E and F are two subsets of D, then*

$$M(\Delta(E, F; D)) \geqq M(\Delta(E, F; R^n))/2K(f)^2.$$

*Proof.* By quasiconformality and [20, 4.3] we obtain

$$\begin{aligned} M(\Delta(E, F; R^n)) &\leqq M(\Delta(fE, fF; R^n))K(f) \leqq M(\Delta(fE, fF; fD))2K(f) \\ &\leqq M(\Delta(E, F; D))2K(f)K(f^{-1}) = M(\Delta(E, F; D))2K(f)^2. \end{aligned}$$

Hereafter we shall use Lemma 2.21 when $D$ is a truncated cone as in Lemma 2.19.

## 3. The main result

In [20, 5.5, 5.6] we proved theorems about quasiconformal mappings of $B^n$, $n \geqq 2$, which are analogous to a theorem of J. L. Doob [2, Theorem 4] regarding angular limits of bounded analytic functions. Using the results of Section 2, we shall now prove a related theorem for *locally K-quasiconformal* mappings of $B^n$, $n \geqq 3$. As will be pointed out in Section 5, the same proof works in the case of somewhat more general mappings as well.

3.1. Theorem. *Let* $f\colon B^n \to R^n$ *be locally K-quasiconformal, let* $b \in \partial B^n$, *and let* $C(f, b) \subset \partial f B^n$. *For* $\varepsilon > 0$ *let* $\varphi_\varepsilon \in (0, \pi/2)$, $E_\varepsilon = K(b, \varphi_\varepsilon) \cap f^{-1} B^n(\varepsilon)$, *and* $\delta_\varepsilon =$ cap $\underline{\text{dens}}\,(E_\varepsilon, b)$. *Moreover, let* $\varphi_\varepsilon$ *be decreasing and* $\delta_\varepsilon$ *increasing. If* $n \geqq 3$ *and*

$$\limsup_{\varepsilon \to 0} \cos^{2n} \varphi_\varepsilon \delta_\varepsilon^{n/(n-1)} \left(\log \frac{1}{\varepsilon}\right)^{n-1} = \infty,$$

*then f has angular limit* 0 *at b.*

*Proof.* Suppose that this is not the case. Then there is $\varphi_0\in(0, \pi/2)$ and a sequence $(b_k)$ in $K(b, \varphi_0)\cap B^n$ with $b_k\to b$, $f(b_k)\to\beta\neq 0$. Fix $r_0\in(0, 1)$ such that $\beta\in\bar{R}^n\setminus B^n(2r_0)$. Since $b_k\in K(b, \varphi_0)$ and $b_k\to b$ there is an integer $k_1$ such that $1-|b_k|>|b_k-b|(\cos\varphi_0)/2$ for $k\geqq k_1$. Since $C(f, b)\subset\partial fB^n$ there is by Lemma 2.10 an integer $k_0\geqq k_1$ such that $fD(b_k, 1)\subset R^n\setminus\bar{B}^n(r_0)$ for $k\geqq k_0$. If $A$ is a proper subset of $B^n$ and $r>0$, we abbreviate $A(r)=B^n(b, r)\cap A$. Let $E=K(b, \varphi_0)\cap(\bigcup_{k\geqq k_0} D(b_k, 1))$. By (2.9) $B^n(b_k, (\tanh 1)(1-|b_k|))\subset D(b_k, 1)$ for all $k$ and by [20, 1.10] or [19, 10.12] we get the estimate

$$M(E(|b_k-b|), |b_k-b|, b)\geqq c(n, \varphi_0)=c_n\log(1+(\tanh 1\cos\varphi_0)/2)$$

for $k\geqq k_0$. From Lemma 2.5 it follows that for $\varepsilon\in(0, r_0)$ there is an integer $k_\varepsilon\geqq k_0$ such that $\varrho_\varepsilon=|b_{k_\varepsilon}-b|\leqq\min\{\cos\varphi_0, \cos\varphi_\varepsilon\}$ and $M(E_\varepsilon(\varrho_\varepsilon), \varrho_\varepsilon, b)\geqq\delta_\varepsilon/2$, where $\delta_\varepsilon=\text{cap}\,\underline{\text{dens}}\,(E_\varepsilon, b)$. Write

$$t_\varepsilon=3^{-n}\min\{\delta_\varepsilon/2, c(n, \varphi_0), c_n\log 2\}>0 \tag{3.2}$$

for $\varepsilon\in(0, r_0)$ and let $\lambda_\varepsilon\geqq 2$ be defined by $\lambda_\varepsilon=\max\{2, \tilde{\lambda}_\varepsilon\}$, where $\tilde{\lambda}_\varepsilon>1$ satisfies

$$\log\tilde{\lambda}_\varepsilon=(t_\varepsilon/6\omega_{n-1})^{1/(1-n)}. \tag{3.3}$$

Let $F_\varepsilon=E_\varepsilon(\varrho_\varepsilon)\setminus\bar{B}^n(b, \varrho_\varepsilon/\lambda_\varepsilon)$, $F=E(\varrho_\varepsilon)\setminus\bar{B}^n(b, \varrho_\varepsilon/\lambda_\varepsilon)$, and $\tilde{\Gamma}_\varepsilon=\Delta(F, F_\varepsilon; R^n)$, when $\varepsilon\in(0, r_0)$. It follows from Corollary 2.7 that $M(\tilde{\Gamma}_\varepsilon)\geqq 5t_\varepsilon/6$. For $\varepsilon\in(0, r_0)$ let $D_\varepsilon=K(b, \varphi_\varepsilon^*)\cap R(b, \varrho_\varepsilon, \varrho_\varepsilon/\lambda_\varepsilon)$, where $\varphi_\varepsilon^*=\max\{\varphi_0, \varphi_\varepsilon\}$, and let $\Gamma_\varepsilon=\Delta(F, F_\varepsilon; D_\varepsilon)$. Observe that $F, F_\varepsilon\subset D_\varepsilon$. Since $\lambda_\varepsilon\geqq 2$ we get then by Lemmas 2.19 and 2.21

$$M(\Gamma_\varepsilon)\geqq t_\varepsilon/3Q(n, \varphi_\varepsilon^*)^2\geqq t_\varepsilon/3Q(n, \varphi_0)^2$$

for $\varepsilon\in(0, r_0)$. By (2.3) we obtain

$$M(f\Gamma_\varepsilon)\leqq\omega_{n-1}\left(\log\frac{r_0}{\varepsilon}\right)^{1-n}$$

for $\varepsilon\in(0, r_0)$. From the modulus inequality [8, 3.2] it follows that

$$M(\Gamma_\varepsilon)\leqq KN(f, D_\varepsilon)M(f\Gamma_\varepsilon).$$

Since $\varrho_\varepsilon<\min\{\cos\varphi_0, \cos\varphi_\varepsilon\}$ and $\lambda_\varepsilon\geqq 2$ we get by Lemma 2.14 and by the above inequalities

$$\begin{aligned} t_\varepsilon/3Q(n, \varphi_0)^2 &\leqq Kd(n, K, \varphi_\varepsilon^*)\log\lambda_\varepsilon\omega_{n-1}\left(\log\frac{r_0}{\varepsilon}\right)^{1-n} \\ &\leqq Ka(n, K)\cos^{-2n}\varphi_\varepsilon^*\log\lambda_\varepsilon\omega_{n-1}\left(\log\frac{r_0}{\varepsilon}\right)^{1-n} \end{aligned} \tag{3.4}$$

for $\varepsilon\in(0, r_0)$. Since $\varphi_\varepsilon$ is decreasing, the limit $\lim_{\varepsilon\to 0+}\varphi_\varepsilon=\theta$ exists. Below we shall assume that $\theta=\pi/2$: the slightly easier case $\theta<\pi/2$ can be dealt with by means of a similar reasoning. Then there exists $r_1\in(0, r_0)$ such that $\varphi_\varepsilon\in(\varphi_0, \pi/2)$ for $\varepsilon\in(0, r_1)$ and so $\varphi_\varepsilon^*=\varphi_\varepsilon$ for $\varepsilon\in(0, r_1)$. Since $\delta_\varepsilon$ is increasing, the limit $\lim_{\varepsilon\to 0+}\delta_\varepsilon=d$

exists. Suppose that $d>0$. Choose $r_2\in(0, r_1)$ in such a way that $\delta_\varepsilon \geqq d/2$ for $\varepsilon\in(0, r_2)$. Then (3.4) yields

$$\cos^{2n}\varphi_\varepsilon\left(\log\frac{1}{\varepsilon}\right)^{n-1} \leqq C_1$$

for $\varepsilon\in(0, r_2)$, where $C_1\in(0, \infty)$ is independent of $\varepsilon$ in view of (3.2) and (3.3). Letting $\varepsilon\to 0+$ yields a contradiction. Hence $d=0$, i.e. $\delta_\varepsilon\to 0$ when $\varepsilon\to 0+$ and by (3.2) and (3.3) there is a number $r_3\in(0, r_1)$ such that $t_\varepsilon\geqq 3^{-n-1}\delta_\varepsilon$ and $\tilde{\lambda}_\varepsilon=\lambda_\varepsilon$ for $\varepsilon\in(0, r_3)$. Then (3.2) and (3.4) yield

$$\cos^{2n}\varphi_\varepsilon\delta_\varepsilon^{n/(n-1)}\left(\log\frac{1}{\varepsilon}\right)^{n-1} \leqq C_2$$

for $\varepsilon\in(0, r_3)$, where $C_2\in(0, \infty)$ does not depend on $\varepsilon$. Letting $\varepsilon\to 0+$ yields a contradiction.

3.5. Corollary. *Let $f\colon B^n\to R^n$ be locally K-quasiconformal, let $b\in\partial B^n$, let $\varphi_0\in(0, \pi/2)$, and let $C(f, b)\subset\partial fB^n$. For $\varepsilon>0$ let $E_\varepsilon=K(b, \varphi_0)\cap f^{-1}B^n(\varepsilon)$ and $\delta_\varepsilon=$*cap $\underline{\text{dens}}\,(E_\varepsilon, b)$. *If $n\geqq 3$ and*

$$\limsup_{\varepsilon\to 0}\delta_\varepsilon^{n/(n-1)}\left(\log\frac{1}{\varepsilon}\right)^{n-1} = \infty,$$

*then $f$ has angular limit* 0 *at $b$.*

3.6. Corollary. *Let $f\colon B^n\to R^n$ be locally K-quasiconformal, let $b\in\partial B^n$, let $\varphi_0\in(0, \pi/2)$, and let $C(f, b)\subset\partial fB^n$. Suppose that there is a set $E\subset K(b, \varphi_0)\cap B^n$ such that* $\lim_{x\to b, x\in E}f(x)=0$. *If $n\geqq 3$ and* cap $\underline{\text{dens}}\,(E, b)=\delta>0$, *then $f$ has angular limit* 0 *at $b$.*

*Proof.* The proof follows from Corollary 3.5 since here $\delta_\varepsilon\geqq\delta>0$. for all $\varepsilon>0$.

## 4. Further results

In this section we shall study the situation of Theorem 3.1 if the assumption $C(f, b)\subset\partial fB^n$ is dropped. Now one cannot use Lemma 2.10, on which a central part of the proof of Theorem 3.1 was based, and we shall employ here a different method. For this purpose we shall prove the following lemma, where an appropriate upper bound for the absolute value of a quasiregular mapping is found. More specifically, we consider a quasiregular mapping $f\colon B^n\to R^n$, wishing to find an upper bound for $|f(x)|$ when $x$ belongs to a ball $B^n(r)$, $r\in(0, 1)$, containing a sufficiently large portion of the set where $|f|$ is small. This method enables us to prove that Corollary 3.6 holds without the assumption $C(f, b)\subset\partial fB^n$.

4.1. Lemma. *Let* $f\colon B^n \to R^n$ *be quasiregular, let* $E_\varepsilon = f^{-1}B^n(\varepsilon)$ *for* $\varepsilon \in (0, 1)$, *let* $r \in (0, 1)$, *and let* $\theta > 1$ *with* $\theta r < 1$. *If* $\delta^r_\varepsilon = M(E_\varepsilon, r, 0) > 0$ *and* $N^{\theta r} = N(f, \bar{B}^n(\theta r))$, *then for* $x \in \bar{B}^n(r)$

$$|f(x)| \leqq \varepsilon \exp (c\delta^r_\varepsilon / N^{\theta r})^{1/(1-n)},$$

*where c is a positive constant depending only on n,* $K_O(f)$, *and* $\theta$.

*Proof.* Fix $x \in \bar{B}^n(r)$. If $|f(x)| \leqq \varepsilon$, there is nothing to prove and we may assume $|f(x)| > \varepsilon$. Let $\beta\colon [0, \infty) \to R^n$ be the path $\beta(t) = f(x)(1+t)$, $t \in [0, \infty)$, and let $\gamma\colon [0, c) \to B^n$ be a maximal lifting of $\beta$, starting at $x$. Then $\gamma(t) \to \partial B^n$ when $t \to c$ and, in particular, $|\gamma| \cap \partial B^n(\theta r) \neq \emptyset$ (cf. [10, 3.12, 3.11]). Let $\Gamma = \Delta(E_\varepsilon, |\gamma|; B^n(\theta r))$. If we write $F_1 = E_\varepsilon \cap \bar{B}^n(r)$, $F_2 = |\gamma| \cap B^n(\theta r)$, $F_3 = S^{n-1}(2\theta r)$, and $\Gamma_{ij} = \Delta(F_i, F_j; R^n)$ $1 \leqq i, j \leqq 3$, we get by the comparison principle of Lemma 2.6 and by Lemma 2.21

$$M(\Gamma) \geqq 2^{-1} 3^{-n} \min \{M(\Gamma_{13}), M(\Gamma_{23}), c_n \log 2\}.$$

Since $\bar{F}_2 \cap S^{n-1}(\theta r) \neq \emptyset \neq F_2 \cap S^{n-1}(r)$ it follows from [20, 1.10] or [19, 10.12] that $M(\Gamma_{23}) \geqq c_n \log (2 - \theta^{-1})$. Let $A = (\log 2 / \log 2\theta)^{n-1}$. By (2.4) we obtain

$$M(\Gamma) \geqq 3^{-n-1} \min \{c_n \log (2-\theta^{-1}), A\delta^r_\varepsilon\} \geqq a\delta^r_\varepsilon,$$

where $a = A \cdot 3^{-n-1} \min \{1, c_n (\log (2-\theta^{-1}))/(\omega_{n-1}(\log 2)^{1-n})\}$ and the upper bound (2.3) for $\delta^r_\varepsilon$ has been used. Since $f|\gamma| \subset R^n \setminus B^n(|f(x)|)$ we obtain by (2.3)

$$M(f\Gamma) \leqq \omega_{n-1} \left( \log \frac{|f(x)|}{\varepsilon} \right)^{1-n}.$$

The modulus inequality in [8, 3.2] yields

$$M(\Gamma) \leqq K_O(f) N^{\theta r} M(f\Gamma).$$

The asserted inequality follows from the above estimates with the constant $c = a/(K_O(f)\omega_{n-1}) > 0$.

4.2. Theorem. *Let* $f\colon B^n \to R^n$ *be locally K-quasiconformal, let* $b \in \partial B^n$, $\varphi_0 \in (0, \pi/2)$, *and let* $E \subset K(b, \varphi_0) \cap B^n$ *be a set with* $\operatorname{cap} \underline{\operatorname{dens}} (E, b) = \delta > 0$. *If* $n \geqq 3$ *and the limit* $\lim_{x \to b,\, x \in E} f(x) = 0$ *exists, then f has angular limit* 0 *at b.*

*Proof.* Fix $\varphi \in (\varphi_0, \pi/2)$. Let $\varepsilon \in (0, 1)$. Choose $t_\varepsilon \in (0, \cos \varphi)$ such that $E \cap \bar{B}^n(b, t_\varepsilon) \subset E_\varepsilon = f^{-1}B^n(\varepsilon)$ and $M(E_\varepsilon, s, b) \geqq 2\delta/3$ for all $s \in (0, t_\varepsilon]$. Let $\lambda \geqq 3$ be such that (cf. (2.3))

$$M(\bar{B}^n(b, s/\lambda), s, b) \leqq \delta/3$$

for all $s > 0$ and let $B^n(x_s, r_s)$ be the smallest ball containing $A^\varphi_\lambda(s) = K(b, \varphi) \cap R(b, s, s/\lambda)$ when $s \in (0, t_\varepsilon]$. Then

$$\begin{cases} x_s = \left(1 - \dfrac{s(1+1/\lambda)}{2 \cos \varphi}\right) b \\ r_s = \dfrac{s}{2} ((1+1/\lambda)^2 \tan^2 \varphi + (1-1/\lambda)^2)^{1/2}. \end{cases}$$

If we use the notation $F_1=E_\varepsilon\cap\bar{A}^\varphi_\lambda(s)$, $F_2=S^{n-1}(x_s, |x_s-b|)\cap B^n(b, s)$, $F_3=S^{n-1}(b, 2s)$, and $\Gamma_{ij}=\Delta(F_i, F_j; R^n)$, $1\leqq i, j\leqq 3$, we get by the comparison principle of Lemma 2.6, in view of the choice of $\lambda$ and (2.3),

$$M_{|x_s-b|}\big(E_\varepsilon\cap\bar{A}^\varphi_\lambda(s), r_s, x_s\big)\geqq M(\Gamma_{12})$$

$$\geqq 3^{-n}\min\{M(\Gamma_{13}), M(\Gamma_{23}), c_n\log 2\}\geqq 3^{-n}\min\{\delta/3, c_n\log 2\}\geqq a\delta$$

for all $s\in(0, t_\varepsilon]$; here $a=3^{-n-1}\min\{1, c_n(\log 2)^n/\omega_{n-1}\}$ and the upper bound (2.3) for $\delta$ has been used as in the proof of Lemma 4.1. We have also used here the lower bound $M(\Gamma_{23})\geqq c_n\log 2$, which follows from [20, 1.10] or [19, 10.12] because $S^{n-1}(b, u)\cap F_2\neq\emptyset$ for $u\in(0, s)$. Since $\lambda\geqq 3$ it follows that $2r_s>|x_s-b|>r_s$ and hence we get in view of (2.4)

$$M\big(E_\varepsilon\cap\bar{A}^\varphi_\lambda(s), r_s, x_s\big)\geqq\big(\log(|x_s-b|/r_s)/\log 2)^{n-1}a\delta = d_1$$

for all $s\in(0, t_\varepsilon]$; here, as in what follows, $d_j$, $j=1, 2, \dots$, denotes a positive constant depending only on some of the numbers $n$, $K$, $\varphi$, and $\delta$. Let $\theta=(|x_s-b|+r_s)/2r_s=d_2>1$. Since $n\geqq 3$ it follows from Lemma 2.12 that $N^{\theta r_s}=N\big(f, \bar{B}^n(x_s, \theta r_s)\big)=d_3$. If we now apply Lemma 4.1 to the mapping $f|B^n(x_s, |x_s-b|)$, we get for $u\in B^n(x_s, r_s)$ the estimate

$$|f(u)|\leqq\varepsilon d_4$$

for all $s\in(0, t_\varepsilon]$. Since

$$K(b, \varphi)\cap B^n(b, t_\varepsilon)\subset\bigcup_{s\in(0, t_\varepsilon]}B^n(x_s, r_s)$$

and since $\varepsilon\in(0, 1)$ was arbitrary, the proof follows now from the definition of an angular limit.

4.3. Corollary. *Let $f$: $B^n\to R^n$ be locally $K$-quasiconformal and let $b\in\partial B^n$. If $n\geqq 3$ and the radial limit $\lim_{t\to 1}f(tb)=0$ exists, then $f$ has angular limit* 0 *at* $b$.

*Proof.* By [19, 10.12] the conditions in Theorem 4.2 are satisfied by $\delta=c_n\log 2$.

The next example shows that the condition on the set $E$ in Theorem 4.2 is in a sense the best possible.

4.4. Examples. (1) The locally $K$-quasiconformal mapping in Example 2.11 shows that the conditions $\operatorname{cap}\underline{\operatorname{dens}}(E, b)>0$ and $\lim_{x\to b, x\in E}f(x)=0$ cannot be replaced by the requirement that there exist a sequence $(b_k)$ in $K(b, \varphi_0)\cap B^n$ with $b_k\to b$, $f(b_k)\to 0$, and $\limsup\varrho(b_k, b_{k+1})<\infty$. Note that the situation is different for a quasiconformal mapping $g$: $B^n\to G'$, $n\geqq 2$, since then the inclusion $C(g, b)\subset\partial fB^n=\partial G'$ in Lemma 2.10 holds (cf. [20, 6.5, 6.7]).

(2) Theorem 4.2 and Corollary 4.3 fail to hold for the dimension $n=2$. A counter-example is provided by the analytic function $h$: $B^2\to R^2\setminus\{0\}$, $h(z)=\exp(-(1-z)^{-4})$, $z\in B^2$, which is a local homeomorphism and has a radial limit 0 at $z=1$. On the other hand the function $h$ does not have any angular limit at 1, since $h(z)\to\infty$ when

$z \to 1$ through the line $y = -x+1$. An essential feature of this function $h$ is that, in view of Lemma 4.7 and Corollary 5.5, it does not satisfy condition (5.1) below.

(3) From the example given in [20, 6.6] it follows that the condition $\operatorname{cap} \underline{\operatorname{dens}}(E, b) > 0$ in Theorem 4.2 cannot be replaced by $\operatorname{cap} \overline{\operatorname{dens}}(E, b) > 0$.

According to Example 4.4 (1) the set $E$ in Theorem 4.2 cannot be replaced, in general, by a sequence with the property in 4.4 (1). Following an idea of Bagemihl and Seidel [1, Lemma 1] we shall now show that $E$ can be replaced by a sufficiently dense sequence of the same type, provided that the mapping in question is normal. The next lemma completes the result in [20, 6.3] and Lemma 2.10 above, both proved under the assumption $\alpha \in \partial f B^n$, which is not needed here.

4.5. Lemma. *Suppose that* $f\colon B^n \to R^n$ *is normal,* $b \in \partial B^n$, *and* $(b_k)$ *is a sequence in* $B^n$ *with* $b_k \to b$ *and* $f(b_k) \to \alpha$. *If* $(r_k)$ *is a sequence of positive numbers with* $\lim r_k = 0$, *then* $f(x) \to \alpha$ *as* $x$ *approaches* $b$ *through the set* $E = \cup D(b_k, r_k)$.

*Proof.* Suppose that this is not the case. Then there is a sequence $(a_k)$ in $E$ with $a_k \to b$, $f(a_k) \to \beta \neq \alpha$. After relabeling, if necessary, we may assume that $\varrho(a_k, b_k) < r_k$, $k = 1, 2, \ldots$. Let $h_k\colon B^n \to B^n$ be a conformal self-mapping of $B^n$ with $h_k(b_k) = 0$. Choose a subsequence of $(f \circ h_k^{-1})$, denoted again by $(f \circ h_k^{-1})$, converging $c$-uniformly towards $g\colon B^n \to \overline{R}^n$. Since $f(b_k) = f(h_k^{-1}(0)) \to \alpha$, $g(0) = \alpha$. On the other hand $h_k(a_k) \to 0$ and $f(a_k) = f\big(h_k^{-1}(h_k(a_k))\big) \to \beta \neq \alpha$, which contradicts the continuity of $g$ at 0. The proof is complete.

4.6. Theorem. *Let* $f\colon B^n \to R^n$ *be a locally* $K$*-quasiconformal mapping omitting one finite point, let* $b \in \partial B^n$, $\varphi \in (0, \pi/2)$, *and let* $(b_k)$ *be a sequence in* $B^n \cap K(b, \varphi)$ *with* $b_k \to b$ *and* $f(b_k) \to \alpha$. *If* $n \geqq 3$ *and* $\lim \varrho(b_k, b_{k+1}) = 0$, *then* $f$ *has angular limit* $\alpha$ *at* $b$.

*Proof.* From [10, 2.9] and [19, 20.4] it follows that $f$ is normal, and hence by Lemma 4.5 $f(x)$ tends to $\alpha$ as $x$ approaches $b$ through a curve $C \subset K(b, \varphi) \cap B^n$ terminating at $b$ and consisting of the geodesics of the hyperbolic metric, joining $b_k$ to $b_{k+1}$, for each $k = 1, 2, \ldots$. The proof now follows from Theorem 4.2.

The next result follows from the proof of [7, 5.8].

4.7. Lemma. *Let* $f\colon B^n \to R^n$ *be a normal quasiregular mapping having a radial limit* $\alpha$ *at* $b \in \partial B^n$. *Then* $f$ *has angular limit* $\alpha$ *at* $b$.

4.8. Remark. Lemma 4.7 was pointed out to the author by Prof. O. Martio. From 4.7 and [10, 2.10] we get an alternative proof for Corollary 4.3 under the additional assumption $R^n \setminus f B^n \neq \emptyset$.

## 5. Concluding remarks

As was pointed out in Example 4.4 (2), Theorem 4.2 fails to hold when the dimension $n=2$. In this final section we shall show how Theorems 4.2 and 3.1 can be generalized to cover the case $n=2$ as well and how the condition on the mapping can be weakened. As a byproduct we obtain a normality criterion for quasiregular mappings, which is related to a problem of W. K. Hayman concerning meromorphic functions.

Let $f\colon B^n \to R^n$, $n \geqq 2$, be a quasiregular mapping with the following property: there exist numbers $p \in [1, \infty)$ and $t \in (0, \infty)$ such that

$$N(f, D(x, t)) \leqq p \quad \text{for all} \quad x \in B^n. \tag{5.1}$$

Then one can easily prove for $f$ estimates of the same type as those in Lemmas 2.12 and 2.14, where only locally $K$-quasiconformal mappings of $B^n$, $n \geqq 3$, were considered. This observation, together with the fact that Lemma 2.19 holds for $n=2$ as well, is all that is needed to carry over the proofs of Theorems 3.1 and 4.2 to the case of quasiregular mappings satisfying (5.1). Thus we have

5.2. Theorem. *Theorems* 3.1 *and* 4.2 *hold for a quasiregular mapping* $f\colon B^n \to R^n$, $n \geqq 2$, *with property* (5.1).

Condition (5.1) has also some interest in the theory of normal functions. In [16, Problem 3.5] W. K. Hayman asked whether there exists a non-normal meromorphic function $f\colon B^2 \to \bar{R}^2$ with property (5.1). This question was answered in the affirmative by Lappan [5], who constructed a non-normal *analytic* function satisfying (5.1) for some $t>0$ and for $p=1$. From the next result, based on a theorem of Rickman [18], it follows that such a function cannot omit any point in $R^2$.

5.3. Theorem. *A quasiregular mapping* $f\colon B^n \to R^n \setminus \{d\}$, $n \geqq 2$, $d \in R^n$, *with property* (5.1) *is normal.*

*Proof.* Let $(h_j)$ be a sequence of conformal self-mappings of $B^n$. By Ascoli's theorem [19, Chapter 19] it will be enough to show that $(f_j)$, $f_j = f \circ h_j$, is equicontinuous at 0. Fix $s>0$ with $B^n(s) \subset D(0, t)$. Then by (5.1) $N(f_j, B^n(s)) \leqq p$ and $K_I(f_j) = K_I(f)$ in the notation of [18]. By [18, 4.4] there are constants $C_1>0$ and $C_2>0$ depending only on $K_I(f)$, $n$, and $d$ such that

$$q(f_j B^n(s/\lambda)) \leqq C_1\left(\exp\left(C_2 p^{n+1} (\log \lambda)^{1-n}\right) - 1\right)$$

for all $\lambda>1$, where $q$ is the spherical metric. Letting $\lambda \to \infty$ yields the desired conclusion.

For the next result the reader is referred to [10, 2.10].

5.4. Corollary. *A locally K-quasiconformal mapping* $f\colon B^n \to R^n \setminus \{0\}$ *is normal if* $n \geqq 3$.

*Proof.* Condition (5.1) is satisfied by [10, 2.3].

The following corollary seems also to be well-known.

5.5. Corollary. *An analytic function* $f\colon B^2 \to R^2 \setminus \{0\}$ *with property* (5.1) *is normal.*

5.6. Remarks. (1) For the remark that Theorem 5.3 is related to Hayman's problem, the author wishes to thank Prof. P. Lappan.

(2) Some results in [14] concerning locally $K$-quasiconformal mappings of $B^n$, $n \geqq 3$, can be generalized in the spirit of this section.

## References


[1] BAGEMIHL, F., and W. SEIDEL: Sequential and continuous limits of meromorphic functions. - Ann. Acad. Sci. Fenn. Ser. A I 280, 1960, 1—17.

[2] DOOB, J. L.: The boundary values of analytic functions. - Trans. Amer. Math. Soc. 34, 1932, 153—170.

[3] GEHRING, F. W., and J. VÄISÄLÄ: The coefficients of quasiconformality of domains in space. - Acta Math. 114, 1965, 1—70.

[4] LANDKOF, N. S.: Foundations of modern potential theory. - Die Grundlehren der mathematischen Wissenschaften 180, Springer-Verlag, Berlin—Heidelberg—New York, 1972.

[5] LAPPAN, P.: A non-normal locally uniformly univalent function. - Bull. London Math. Soc. 5, 1973, 291—294.

[6] MARTIO, O.: Capacity and measure densities. - Ann. Acad. Sci. Fenn. Ser. A I 4, 1978/1979, 109—118.

[7] MARTIO, O., and S. RICKMAN: Boundary behavior of quasiregular mappings. - Ibid. 507, 1972, 1—17.

[8] MARTIO, O., S. RICKMAN, and J. VÄISÄLÄ: Definitions for quasiregular mappings. - Ibid. 448, 1969, 1—40.

[9] MARTIO, O., S. RICKMAN, and J. VÄISÄLÄ: Distortion and singularities of quasiregular mappings. - Ibid. 465, 1970, 1—13.

[10] MARTIO, O., S. RICKMAN, and J. VÄISÄLÄ: Topological and metric properties of quasiregular mappings. - Ibid 488, 1971, 1—31.

[11] MARTIO, O., and J. SARVAS: Density conditions in the $n$-capacity. - Indiana Univ. Math. J. 26, 1977, 761—776.

[12] MARTIO, O., and U. SREBRO: Automorphic quasimeromorphic mappings in $R^n$. - Acta Math. 135, 1975, 221—247.

[13] MARTIO, O., and U. SREBRO: On the existence of automorphic quasimeromorphic mappings in $R^n$. - Ann. Acad. Sci. Fenn. Ser. A I 3, 1977, 123—130.

[14] MINIOWITZ, R.: Distortion theorems for quasiregular mappings. - Ibid. 4, 1978/1979, 63—74.

[15] NÄKKI, R.: Extension of Loewner's capacity theorem. - Trans. Amer. Math. Soc. 180, 1973, 229—236.

[16] POMMERENKE, CH.: Problems in complex function theory. - Bull. London Math. Soc. 4, 1972, 354—366.

[17] REŠETNJAK, JU. G.: Mappings with bounded deformation as extremals of Dirichlet type integrals. - Siberian Math. J. 9, 1968, 487—498.

[18] RICKMAN, S.: A path lifting construction for discrete open mappings with application to quasimeromorphic mappings. - Duke Math. J. 42, 1975, 797—809.

[19] VÄISÄLÄ, J.: Lectures on $n$-dimensional quasiconformal mappings. - Lecture Notes in Mathematics 229, Springer-Verlag, Berlin—Heidelberg—New York, 1971.

[20] VUORINEN, M.: On the existence of angular limits of $n$-dimensional quasiconformal mappings. - Ark. Mat. 1979. (To appear).

[21] VUORINEN, M.: Lower bounds for the moduli of path families with applications to non-tangential limits of quasiconformal mappings. - Ann. Acad. Sci. Fenn. Ser. A I 4, 1978/1979, 279—291.

[22] ZIEMER, W. P.: Extremal length and $p$-capacity. - Michigan Math. J. 16, 1969, 43—51.

University of Helsinki
Department of Mathematics
SF-00100 Helsinki 10
Finland

Received 6 April 1979
Revision received 21 August 1979