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ON EXCEPTIONAL VALUES OF FUNCTIONS MEROMORPHIC OUTSIDE A LINEAR SET

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1. Introduction

1. Let E be a closed set in the complex plane and f a function mermorphic outside E omitting a set $F=F(f)$. We shall consider the following problem: If E is thin, under what conditions is F thin, too?

If the number of the points of E card (E) is finite and the set of the essential singularities of f sing (f) is non-empty, Picard's theorem says that card $(F)\leqq 2$. If the linear measure of E $l(E)$ is zero and f non-constant, then the Hausdorff dimension of F dim (F) is at most one. If the logarithmic capacity of E cap (E) is zero and f non-constant, then cap $(F)=0$, and Matsumoto [3] has proved that this result is sharp in the sense that given a closed set B with cap $(B)=0$, there exist a set E with cap $(E)=0$ and a function f with sing $(f)=E$ such that $B=F(f)$.

However, the thickness of the set E does not always guarantee the existence of a function f such that F is thick, too. Lehto [2] has proved that E may be chosen such that card $(E)=\infty$ and card $(F)\leqq 2$ for any f with sing $(f)\neq\emptyset$. Carleson [1] proved that there exists a set E with cap $(E)>0$ such that if sing $(f)\neq\emptyset$, then card $(F)\leqq 3$. Matsumoto [4] constructed a perfect set E such that card $(F)\leqq 2$ for any f with sing $(f)\neq\emptyset$. There exists a set E with cap $(E)>0$ (Matsumoto [5], Toppila [6]) such that if sing $(f)=E$, then card $(F)\leqq 2$. A set E with dim $(E)>0$ is constructed in [8] such that if sing $(f)=E$, then card $(F)\leqq 4$, and in the same paper it is proved that there exists a set E with dim $(E)>0$ such that cap $(F)=0$ for any f with sing $(f)\neq\emptyset$.

In this paper, we shall consider sets E with dim $(E)<1$. We shall prove that if E is a linear set with dim $(E)<1$ and f non-constant, then dim $(F)\leqq$ dim (E). In [7] another geometrical condition is given under which dim $(F)\leqq$ dim (E). For example usual Cantor sets satisfy the condition given in [7], but all linear sets do not satisfy it.

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2. Linear sets

2. We prove the following theorem.

Theorem. *Let E be a closed linear set with* $\dim(E)<1$. *If f is non-constant and meromorphic outside E and omits F, then* $\dim(F)\leqq\dim(E)$.

3. *Proof.* It does not mean any essential restriction to assume that $\infty\in F$, E is contained in the segment $[0, 1]$ on the real axis, and that f is non-rational. In order to prove the theorem it is sufficient to prove that for any λ, $\dim(E)<\lambda<1$, and any R, $0<R<\infty$, we have $\dim(B)\leqq\lambda$, where B is that part of F which lies on $|w|\leqq R$. Let these λ and R be chosen. We choose α such that $\dim(E)<\alpha<\lambda$.

We denote $U(a, r)=\{z\colon |z-a|<r\}$. Let n be a positive integer. We choose a covering

$$\bigcup_{v=1}^{N_n} U(x_v, r_v) \supset E$$

such that every x_v lies on the segment $[0, 1]$ and

$$\sum_{v=1}^{N_n} r_v^\alpha < \frac{1-(1/2)^\alpha}{16n}. \tag{1}$$

If $U(x_s, r_s)\cap U(x_k, r_k)\neq\emptyset$, then both of these discs are contained in a disc U with radius r_s+r_k satisfying $(r_s+r_k)^\alpha\leqq r_k^\alpha+r_s^\alpha$. Therefore we may assume that the discs $U(x_v, r_v)$ have no common points.

Let v be fixed. We choose $\delta_v>0$ such that the discs $U(x_v-r_v, \delta_v)$ and $U(x_v+r_v, \delta_v)$ do not contain any point of E. We divide the circle $C_v\colon |z-x_v|=r_v$ in the following manner. Let k be so large that $\pi r_v<2^k\delta_v$. We set

$$\gamma_s = \{z = x_v+r_v e^{i\varphi}\colon\ \pi/2^{s+1} \leqq \varphi \leqq \pi/2^s\}$$

for $s=1, 2, \ldots, k-1$, and $\gamma_k=\{z\in C_v\colon 0\leqq\arg(z-x_v)\leqq\pi/2^k\}$. The length of γ_s $l(\gamma_s)$ is less than $4r_v/2^s$ and

$$\sum_{s=1}^{k} (l(\gamma_s))^\alpha \leqq (4r_v)^\alpha \sum_{s=1}^{k} (1/2^s)^\alpha < \frac{4r_v^\alpha}{1-(1/2)^\alpha}.$$

The arcs $\{z\in C_v\colon p\pi/2\leqq\arg(z-x_v)\leqq(p+1)\pi/2\}$, $p=1, 2, 3$, are divided in a similar manner, and we see that the boundary of the complement of

$$\bigcup_{v=1}^{N_n} U(x_v, r_v)$$

consists of arcs t_v, $v=1, 2, \ldots, P_n$, such that

$$\sum_{v=1}^{P_n} (l(t_v))^\alpha \leqq \frac{16}{1-(1/2)^\alpha} \sum_{v=1}^{N_n} r_v^\alpha < 1/n, \tag{2}$$

and if $\zeta\in t_v$, then $U(\zeta, l(t_v)/2)$ does not contain any point of E.

We denote $\omega(t_\nu, a)=\min\{|f(z)-a|: z\in t_\nu\}$. Applying repeatedly Schottky's theorem in the discs $U(\zeta, l(t_\nu)/2)$ $(\zeta\in t_\nu)$, we get the following lemma.

Lemma. *There exists an absolute constant* $K>4$ *such that if* $\omega(t_\nu, a)\leqq\varrho$ $(\varrho>0)$ *for some* $a\in B$, *then* $\omega(t_\nu, b)>K\varrho$ *for any* $b\in B-U(a, 2K\varrho)$.

4. For any $a\in B$ we set

$$f_a(z)=(2\pi i)^{-1}\int_{|\zeta|=4}\frac{d\zeta}{(f(\zeta)-a)(\zeta-z)}.$$

The function f_a is regular in $|z|<4$ and therefore $f_a(z)\not\equiv 1/(f(z)-a)$. We set $G=\{z: 3<|z|<4\}$. Because $f_a(z)$ and $1/(f(z)-a)$ are continuous functions of a $(a\in B)$ for any fixed $z\in G$ and B is compact, there exists $\beta>0$ such that

$$\sup_{z\in G}|f_a(z)-1/(f(z)-a)|>\beta \tag{3}$$

for any $a\in B$. It follows from Cauchy's integral theorem that

$$\left|f_a(z)-1/(f(z)-a)\right|=\left|(2\pi i)^{-1}\sum_{\nu=1}^{P_n}\int_{t_\nu}\frac{d\zeta}{(f(\zeta)-a)(\zeta-z)}\right|\leqq\sum_{\nu=1}^{P_n}l(t_\nu)/\omega(t_\nu, a)$$

if $z\in G$, $a\in B$, and this implies together with (3) that

$$\sum_{\nu=1}^{P_n}l(t_\nu)/\omega(t_\nu, a)\geqq\beta \tag{4}$$

for any $a\in B$.

We may assume that $l(t_1)\geqq l(t_2)\geqq\ldots\geqq l(t_{P_n})$, and that the arcs t_1 and t_{P_n} belong to the largest circle C_ν. Then $l(t_1)=2^{S-1}l(t_{P_n})$ for some positive integer S and the condition

$$l(t_1)/2^{k-1}\geqq l(t_\nu)\geqq l(t_1)/2^k=s_k \tag{5}$$

is satisfied for at least 4 different values of ν when $1\leqq k\leqq S$. The arcs t_ν satisfying (5) are denoted by $\gamma_{k,s}$, $s=1, 2, \ldots, K_k$.

5. Let k be fixed, $1\leqq k\leqq S$. We denote $d=2^{1/\lambda}$, $\varepsilon=1-\alpha/\lambda$, and p is a positive integer satisfying $2^{p+1}>K_k\geqq 2^p$. If possible, we choose $b_{0,1}\in B$ such that $\omega(\gamma_{k,s}, b_{0,1})\leqq d_0=d^p s_k^{1-\varepsilon}$ happens at least for 2^p different values of s, and we set

$$C_{k,0,1}=U(b_{0,1}, 2Kd_0)$$

and

$$\Gamma_{0,1}=\{\gamma_{k,s}: \omega(\gamma_{k,s}, b_{0,1})\leqq d_0\}.$$

If $b\in B-C_{k,0,1}$ and $\gamma_{k,s}\in\Gamma_{0,1}$, it follows from the lemma that $\omega(\gamma_{k,s}, b)\geqq Kd_0\geqq 4d_0$. Because $K_k<2^{p+1}$, it is not possible to choose $b_{0,2}\in B-C_{k,0,1}$ such that the condition $\omega(\gamma_{k,s}, b_{0,2})\leqq d_0$ is satisfied at least for 2^p different values of s. If $C_{k,0,1}$ exists, we set $T_0=1$, otherwise we set $T_0=0$.

Inductively, let us suppose that $C_{k,m-1,t}$ and $\Gamma_{m-1,t}$ are determined for $t=1, 2, \ldots, T_{m-1}$. We choose

$$b_{m,1} \in B - \bigcup_{\nu=0}^{m-1} \bigcup_{t=1}^{T_\nu} C_{k,\nu,t}$$

such that $\omega(\gamma_{k,s}, b_{m,1}) \leqq d_m = d^{p-m} s_k^{1-\varepsilon}$ is true for 2^{p-m} values of s, and we set

(6) $$C_{k,m,j} = U(b_{m,j}, 2Kd_m)$$

and

(7) $$\Gamma_{m,j} = \{\gamma_{k,s}\colon\ \omega(\gamma_{k,s}, b_{m,j}) \leqq d_m\},$$

where $j=1$. If $C_{k,m,j-1}$ and $\Gamma_{m,j-1}$ are determined ($j \geqq 2$), we choose, if possible,

$$b_{m,j} \in \left(B - \bigcup_{\nu=0}^{m-1} \bigcup_{t=1}^{T_\nu} C_{k,\nu,t}\right) - \bigcup_{t=1}^{j-1} C_{k,m,t}$$

such that $\omega(\gamma_{k,s}, b_{m,j}) \leqq d_m$ at least for 2^{p-m} values of s and $C_{k,m,j}$ and $\Gamma_{m,j}$ are defined by (6) and (7). It follows from the lemma that if $\gamma_{k,s} \in \Gamma_{m,j}$, then

$$\gamma_{k,s} \notin \left(\bigcup_{\nu=0}^{m-1} \bigcup_{t=1}^{T_\nu} \Gamma_{\nu,t}\right) \cup \bigcup_{t=1}^{j-1} \Gamma_{m,t}.$$

Therefore our method produces only a finite number of discs $C_{k,m,j}$, $j=1, 2, \ldots, T_m$.

Let us suppose that there exists $b \in B - D_k$, where

$$D_k = \bigcup_{m=0}^{p} \bigcup_{t=1}^{T_m} C_{k,m,t}.$$

Then $\omega(\gamma_{k,s}, b) > d_p = s_k^{1-\varepsilon}$ for any s and the condition

$$d^{m-1} s_k^{1-\varepsilon} < \omega(\gamma_{k,s}, b) \leqq d^m s_k^{1-\varepsilon}$$

is satisfied at most for $2^m - 1$ different values of s. This implies that (because $l(\gamma_{k,s}) \leqq 2s_k$)

$$\sum_{s=1}^{K_k} l(\gamma_{k,s})/\omega(\gamma_{k,s}, b) \leqq 2s_k \sum_{s=1}^{K_k} 1/\omega(\gamma_{k,s}, b)$$
$$\leqq 2s_k \left(\sum_{m=1}^{p} \frac{2^m - 1}{d^{m-1} s_k^{1-\varepsilon}} + \frac{K_k}{d^p s_k^{1-\varepsilon}} \right).$$

Here $K_k < 2^{p+1}$, and we get

(8) $$\sum_{s=1}^{K_k} l(\gamma_{k,s})/\omega(\gamma_{k,s}, b) < \frac{4ds_k^{\varepsilon}}{d-2}.$$

The radius of the disc $C_{k,m,t}$ is $2Kd_m$, and because $d^\lambda = 2$, $\alpha = \lambda(1-\varepsilon)$, we get

$$(2Kd_m)^\lambda = (2K)^\lambda (d^{p-m} s_k^{1-\varepsilon})^\lambda = (2K)^\lambda 2^{p-m} s_k^{\alpha}.$$

The set $\Gamma_{m,t}$ contains at least 2^{p-m} different arcs $\gamma_{k,s}$ and $l(\gamma_{k,s}) \geqq s_k$. Therefore we get

$$(2Kd_m)^\lambda \leqq (2K)^\lambda \sum_{\gamma_{k,s} \in \Gamma_{m,t}} (l(\gamma_{k,s}))^\alpha.$$

If $m\neq\mu$ or $t\neq\tau$, then $\Gamma_{m,t}\cap\Gamma_{\mu,\tau}=\emptyset$, and we see that D_k consists of discs $C_{k,v}$ with radii $\varrho_{k,v}$, $v=1, 2, \ldots, L_k$, satisfying

(9)
$$\sum_{v=1}^{L_k} \varrho_{k,v}^{\lambda} \leqq (2K)^{\lambda} \sum_{s=1}^{K_k} (l(\gamma_{k,s}))^{\alpha}.$$

6. Let us suppose now that $b\in B-\bigcup_{k=1}^{S} D_k$. It follows from (8) that

$$\sum_{s=1}^{P_n} \frac{l(t_s)}{\omega(t_s, b)} = \sum_{k=1}^{S} \sum_{s=1}^{K_k} \frac{l(\gamma_{k,s})}{\omega(\gamma_{k,s}, b)} \leqq \frac{4d}{d-2} \sum_{k=1}^{S} s_k^{\varepsilon}$$
$$= \frac{4d}{d-2} \sum_{k=1}^{S} (l(t_1)/2^k)^{\varepsilon} < \frac{4d(l(t_1)/2)^{\varepsilon}}{(d-2)(1-(1/2)^{\varepsilon})} < \beta$$

if $l(t_1)$ is sufficiently small, and we see from (4) that

$$B\subset \bigcup_{k=1}^{S} D_k = \bigcup_{k=1}^{S} \bigcup_{v=1}^{L_k} C_{k,v}$$

for all large values of n. We see from (9) and (2) that

$$\sum_{k=1}^{S} \sum_{v=1}^{L_k} (\varrho_{k,v})^{\lambda} \leqq (2K)^{\lambda} \sum_{k=1}^{S} \sum_{s=1}^{K_k} l(\gamma_{k,s})^{\alpha} = (2K)^{\lambda} \sum_{v=1}^{P_n} (l(t_v))^{\alpha} < (2K)^{\lambda}/n \to 0$$

as $n\to\infty$. Therefore $\dim(B)\leqq\lambda$, and the theorem is proved.

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