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NON-HOMOGENEOUS COMBINATIONS OF COEFFICIENTS OF UNIVALENT FUNCTIONS

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Introduction

G. Schober communicated us in 1978 the following problem: Determine $\max_{f\in S} \operatorname{Re}(a_3+ia_2)$ (see also [4], p. 84). In this paper we consider the general problem of finding $\max \operatorname{Re}(a_3+\lambda a_2)$ for an arbitrary complex parameter λ and for functions $f\in S(b)$. Löwner's parametric method shall be extensively used in the following considerations.

The case of $S_R(b)$

Let $b\in(0, 1)$ and let $\Delta=\{z||z|<1\}$. The class $S_R(b)$ consists of the univalent functions f: $\Delta\to\Delta$ for which $f(z)=b\{z+a_2z^2+a_3z^3+\dots\}$ with $a_k\in R$. The problem reduces to the study of $a_3+\lambda a_2$ for $\lambda\in R$. In [2] pp. 8, 9, 10 we have derived the following sharp estimates for functions $f\in S_R(b)$:

$$a_3 \geqq a_2^2-(1-b^2),$$

$$a_3 \leqq 1-b^2+a_2^2\left(1+\frac{1}{\log b}\right) \quad \text{if} \quad |a_2| \leqq -2b\log b,$$

$$a_3 \leqq a_2^2+1-b^2-2(\sigma^2-b^2)+4\sigma^2\log\sigma \quad \text{if} \quad |a_2| > -2b\log b.$$

The parameter σ is determined by $\sigma-\sigma\log\sigma=b+|a_2|/2$.

Taking into account that $|a_2|\leqq 2(1-b)$ we immediately obtain

$$\min(a_3+\lambda a_2)=\begin{cases}-(1-b^2)-\dfrac{1}{4}\lambda^2 & \text{if} \quad |\lambda| \leqq 4(1-b),\\ 3-8b+5b^2-2(1-b)|\lambda| & \text{if} \quad |\lambda| > 4(1-b).\end{cases}$$

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The upper bound is more complicated. We have to distinguish between the following two cases:

$$1)\quad 1+\frac{1}{\log b}\leqq 0,\quad 2)\quad 1+\frac{1}{\log b}>0.$$

In both these cases we have to deal separately with the possibilities $|a_2|\leqq -2b\log b$ and $|a_2|>-2b\log b$. After an elementary but rather long calculation we arrive at the following result.

Case 1: $e^{-1}\leqq b<1$.

$$\max(a_3+\lambda a_2)=\begin{cases}1-b^2-\frac{1}{4}\lambda^2\frac{\log b}{1+\log b} & \text{if}\quad |\lambda|\leqq 4b(1+\log b),\\ 1-b^2+|\lambda|\left(\sigma-\frac{1}{4}|\lambda|\right)+2(\sigma-b)^2 & \text{if}\quad 4b(1+\log b)<|\lambda|<4b,\\ 3-8b+5b^2+2(1-b)|\lambda| & \text{if}\quad |\lambda|\geqq 4b.\end{cases}$$

The number $\sigma\in[b,1]$ is determined by $\sigma\log\sigma+b=|\lambda|/4$.

Case 2: $0<b<e^{-1}$.

$$\max(a_3+\lambda a_2)=\begin{cases}1-b^2+|\lambda|\left(\sigma-\frac{1}{4}|\lambda|\right)+2(\sigma-b)^2 & \text{if}\quad |\lambda|<4b,\\ 3-8b+5b^2+2(1-b)|\lambda| & \text{if}\quad |\lambda|\geqq 4b.\end{cases}$$

The number $\sigma\in[b,1]$ is determined by $\sigma\log\sigma+b=|\lambda|/4$.

The general case $S(b)$

As usual, for $b\in(0,1)$, $S(b)$ consists of the univalent functions $f\colon \Delta\to\Delta$ for which $f(z)=b\{z+a_2z^2+a_3z^3+\dots\}$. Instead of S we shall sometimes write $S(0)$. We shall consider the dense subclass of slit-functions. For these the following Löwner expressions hold:

$$a_2=-2\int_b^1 \varkappa(u)\,du,\quad a_3=a_2^2-2\int_b^1 u\varkappa^2(u)\,du, \tag{1}$$

where $\varkappa(u)=e^{i\vartheta(u)}$ is a continuous function. For a piecewise continuous ϑ the formulae (1) still define coefficients of functions $f\in S(b)$, $b\in[0,1)$.

For a given number

$$\lambda=\mu+i\nu$$

we have

$$\text{(2a)}\quad \operatorname{Re}(a_3+\lambda a_2)=4\left(\int_b^1\cos\vartheta(u)\,du\right)^2-4\left(\int_b^1\sin\vartheta(u)\,du\right)^2-2\int_b^1 u\cos 2\vartheta(u)\,du$$

$$-2\mu\int_b^1\cos\vartheta(u)\,du+2\nu\int_b^1\sin\vartheta(u)\,du.$$

Consider first the case $\nu=0$, i.e. $\lambda\in R$. In this case we have

$$\operatorname{Re}(a_3+\mu a_2)\leqq 4\left(\int_b^1 \cos\vartheta(u)\,du\right)^2-2\int_b^1 u\cos 2\vartheta(u)\,du-2\mu\int_b^1 \cos\vartheta(u)\,du.$$

For the Löwner functions $f\in S_R(b)$ we have, according to [5] p. 10,

$$a_2=-2\int_b^1 \cos\vartheta(u)\,du,\quad a_3=a_2^2-2\int_b^1 u\cos 2\vartheta(u)\,du.$$

Thus in this case the maximum is attained in the subclass $S_R(b)$, for which the solution was determined above.

From now on we assume that

$$\nu\neq 0.$$

By considering $\overline{f(\bar z)}$ instead of $f(z)$ we see that $\nu\int_b^1 \sin\vartheta(u)du\geqq 0$ in the maximum case. Similarly, by considering $-f(-z)$ instead of $f(z)$ we find $\mu\int_b^1\cos\vartheta(u)du\leqq 0$ in the maximum case. For brevity, let us normalize

$$\nu>0,\quad \mu\leqq 0;\ \int_b^1 \sin\vartheta(u)\,du\geqq 0,\quad \int_b^1 \cos\vartheta(u)\,du\geqq 0.$$

Rewriting (2a) we obtain

$$\operatorname{Re}(a_3+\lambda a_2)=1-b^2+4\left(\int_b^1\cos\vartheta(u)\,du\right)^2-4\left(\int_b^1\sin\vartheta(u)\,du\right)^2 \tag{2}$$

$$-4\int_b^1 u\cos^2\vartheta(u)\,du-2\mu\int_b^1\cos\vartheta(u)\,du+2\nu\int_b^1\sin\vartheta(u)\,du.$$

Let us replace the maximizing ϑ by $\tilde\vartheta$ which is obtained from ϑ by changing ϑ into $\pi-\vartheta$ on an arbitrary subinterval l of $[b,1]$. The functional (2) is then altered in such a way that

$$\operatorname{Re}(a_3+\lambda a_2)-\operatorname{Re}(\tilde a_3+\lambda\tilde a_2)$$

$$=16\left(\int_b^1\cos\vartheta(u)\,du-\frac{\mu}{4}\right)\int_l\cos\vartheta(u)\,du-16\left(\int_l\cos\vartheta(u)\,du\right)^2.$$

We deduce from this that if $\cos\vartheta(u)\neq 0$ at some point, then $\cos\vartheta(u)>0$, i.e.

$$\cos\vartheta(u)\geqq 0$$

in the maximum case.

Similarly, we can deduce from (2) that in the maximum case $\cos\vartheta$ is decreasing (and hence $|\sin\vartheta|$ is increasing), since the only part depending on the arrangement of the values of $\cos\vartheta$ is $-\int_b^1 u\cos^2\vartheta(u)du$.

The part in (2) which depends explicitly on $\sin\vartheta$ is

$$-4\left(\int_b^1\sin\vartheta(u)\,du\right)^2+2\nu\int_b^1\sin\vartheta(u)\,du.$$

If $v \geqq 4(1-b)$, it follows from (2) that in the maximum case

$$\sin \vartheta(u) \geqq 0.$$

For, if $\sin \vartheta$ assumed negative values, we could change its sign without affecting $\cos \vartheta$ and thus increase $\int_b^1 \sin \vartheta(u) du$, and by doing so, we would increase the above mentioned part determined by $\sin \vartheta$. — Therefore, if in the maximum case $\sin \vartheta$ assumes negative values, we must have $v<4(1-b)$ and $\int_b^1 \sin \vartheta(u) du = v/4$.

A necessary condition for the function ϑ to be extremal is that the first order variation of (2) is zero. This leads to the condition

(3)

$$\left(\int_b^1 \cos \vartheta(u)\, du - \frac{1}{4}\mu\right) \sin \vartheta(u) + \left(\int_b^1 \sin \vartheta(u)\, du - \frac{1}{4}v\right) \cos \vartheta(u) = u \sin \vartheta(u) \cos \vartheta(u).$$

If ϑ has to give rise to the maximum, then the second order variation has to be non-positive. This leads to the following condition: For all piecewise continuous functions φ we have

(3a)
$$\begin{aligned} &\left(\int_b^1 \varphi(u) \sin \vartheta(u)\, du\right)^2 - \left(\int_b^1 \varphi(u) \cos \vartheta(u)\, du\right)^2 \\ &+ \left(\int_b^1 \sin \vartheta(u)\, du - \frac{1}{4}v\right) \int_b^1 \varphi^2(u) \sin \vartheta(u)\, du \\ &- \left(\int_b^1 \cos \vartheta(u)\, du - \frac{1}{4}\mu\right) \int_b^1 \varphi^2(u) \cos \vartheta(u)\, du \\ &+ \int_b^1 u\varphi^2(u) \cos^2 \vartheta(u)\, du - \int_b^1 u\varphi^2(u) \sin^2 \vartheta(u)\, du \leqq 0. \end{aligned}$$

The perfect square representation

Let C be an arbitrary parameter. The identity

$$a_3 - a_2^2 - Ca_2 + \frac{1}{2} C^2 \log b = -2 \int_b^1 A^2(u)\, du;$$

$$A(u) = \sqrt{u}\left(\varkappa(u) - \frac{C}{2u}\right)$$

follows from the formulae (1). Hence

$$\begin{aligned} &\operatorname{Re}\left(a_3 - a_2^2 - Ca_2 + \frac{1}{2} C^2 \log b\right) \\ &= -2 \int_b^1 \operatorname{Re} A^2(u)\, du = 2 \int_b^1 |A(u)|^2\, du - 4 \int_b^1 (\operatorname{Re} A(u))^2\, du \\ &= 1 - b^2 - \frac{1}{2} |C|^2 \log b + \operatorname{Re}(\bar{C} a_2) - 4 \int_b^1 (\operatorname{Re} A(u))^2\, du. \end{aligned}$$

Let us make use of the choice

$$C = -\left(a_2 + \frac{1}{2}\lambda\right)$$

which gives

$$\text{(4)} \qquad \operatorname{Re}(a_3 + \lambda a_2) = 1 - b^2 + \frac{1}{4}\nu^2 - \frac{1}{4}\mu^2 \log b - \left(\operatorname{Im} a_2 + \frac{1}{2}\nu\right)^2$$

$$-(1 + \log b)(\operatorname{Re} a_2)^2 - \mu \operatorname{Re} a_2 \log b - 4\int_b^1 (\operatorname{Re} A(u))^2\, du.$$

If $b \neq e^{-1}$, this can be written as

$$\text{(4a)} \qquad \operatorname{Re}(a_3 + \lambda a_2) = 1 - b^2 + \frac{1}{4}\nu^2 - \frac{1}{4}\mu^2 \frac{\log b}{1 + \log b} - \left(\operatorname{Im} a_2 + \frac{1}{2}\nu\right)^2$$

$$-(1 + \log b)\left(\operatorname{Re} a_2 + \frac{\mu \log b}{2(1 + \log b)}\right)^2 - 4\int_b^1 (\operatorname{Re} A(u))^2\, du.$$

We can also rewrite (4) in the form

$$\text{(4b)} \qquad \operatorname{Re}(a_3 + \lambda a_2) = 1 - b^2 + \frac{1}{4}\nu^2 - (\operatorname{Re} a_2)^2 - \left(\operatorname{Re} a_2 + \frac{1}{2}\mu\right)^2 \log b$$

$$-\left(\operatorname{Im} a_2 + \frac{1}{2}\nu\right)^2 - 4\int_b^1 (\operatorname{Re} A(u))^2\, du.$$

The representation (4) is closely related to those used by Haario and Jokinen in [1].

Extremals of type 2:2

Suppose that $e^{-1} < b < 1$ (hence $1 + \log b > 0$) and obtain from (4a)

$$\text{(5)} \qquad \operatorname{Re}(a_3 + \lambda a_2) \leqq 1 - b^2 + \frac{1}{4}\nu^2 - \frac{1}{4}\mu^2 \frac{\log b}{1 + \log b}.$$

Equality is possible if and only if

i) $\operatorname{Im} a_2 + \frac{1}{2}\nu = 0,$

ii) $\operatorname{Re} a_2 + \frac{\mu \log b}{2(1 + \log b)} = 0,$

iii) $\operatorname{Re} A(u) = 0$ i.e. $\cos \vartheta(u) = -\frac{\operatorname{Re} a_2 + \mu/2}{2u}.$

We shall show that (5) is sharp for some numbers $\lambda = \mu + i\nu$.

Let us choose $|\mu|\leqq 4b(1+\log b)$ and let

$$\sigma=\frac{|\mu|}{4(1+\log b)};$$

therefore $0\leqq\sigma\leqq b$. Define ϑ in such a way that

$$\cos\vartheta(u)=\frac{\sigma}{u},$$

$$\sin\vartheta(u)=\begin{cases}\sqrt{1-\dfrac{\sigma^2}{u^2}} & \text{for} \quad b\leqq u\leqq c,\\ -\sqrt{1-\dfrac{\sigma^2}{u^2}} & \text{for} \quad c<u\leqq 1.\end{cases}$$

The point c will be chosen later. For this ϑ we have

$$\operatorname{Re} a_2=-2\int_b^1\cos\vartheta(u)\,du=\frac{|\mu|\log b}{2(1+\log b)}=-\frac{\mu\log b}{2(1+\log b)};$$

thus ii) is satisfied. Now it follows that

$$\operatorname{Re} a_2+\frac{1}{2}\mu=\frac{\mu}{2(1+\log b)}=-2\sigma,$$

which means that iii) holds. In order to show i) we choose c such that $\operatorname{Im} a_2=-2\int_b^1\sin\vartheta(u)\,du=-\nu/2$. This is possible so far as

$$\frac{1}{2}|\nu|\leqq 2\int_b^1|\sin\vartheta(u)|\,du,$$

i.e. $|\nu|\leqq 4\left(\sqrt{1-\sigma^2}-\sqrt{b^2-\sigma^2}+\sigma\,\overline{\operatorname{arc}}\cos\sigma/b-\sigma\,\overline{\operatorname{arc}}\cos\sigma\right)$.

The equality case for $\mu>0$ can be handled similarly. Collecting the results we arrive at

Theorem 1. *Let* $e^{-1}<b<1$, $\lambda=\mu+i\nu$, $\sigma=|\mu|/4(1+\log b)$. *If*

$$\begin{cases}|\mu|\leqq 4b(1+\log b),\\ |\nu|\leqq 4\left(\sqrt{1-\sigma^2}-\sqrt{b^2-\sigma^2}+\sigma\,\overline{\operatorname{arc}}\cos\dfrac{\sigma}{b}-\sigma\,\overline{\operatorname{arc}}\cos\sigma\right),\end{cases}$$

then

$$\max_{f\in S(b)}\operatorname{Re}(a_3+\lambda a_2)=1-b^2+\frac{1}{4}\nu^2-\frac{1}{4}\mu^2\frac{\log b}{1+\log b}.$$

The maximum is reached for a function mapping Δ onto Δ minus two slits.

Note. If $b=e^{-1}$, then similar arguments show that to each v with $|v|<4(1-e^{-1})$ there belongs a one-parametric family of extremal functions parametrized by $\operatorname{Re} a_2 \in [-2e^{-1}, 2e^{-1}]$.

Extremals of type 1:2

Now we take (4b) as a starting point. Let $|\mu| \leqq 4b$, $t=-(\operatorname{Re} a_2+\mu/2)/2$. From $|a_2| \leqq 2(1-b)$ it follows that $0 \leqq t \leqq 1$. In this notation we have

$$\operatorname{Re} A(u) = \sqrt{u}\left(\cos \vartheta(u) - \frac{t}{u}\right).$$

For all functions ϑ the following holds. If $t \leqq b$, we have the trivial estimate $|\cos \vartheta(u) - t/u| \geqq 0$ for $b \leqq u \leqq 1$. If $t > b$, we can say more:

$$\left|\cos \vartheta(u) - \frac{t}{u}\right| \geqq \begin{cases} \left|1-\frac{t}{u}\right| & \text{for} \quad b \leqq u \leqq t, \\ 0 & \text{for} \quad t \leqq u \leqq 1. \end{cases}$$

Therefore, we have

$$-(\operatorname{Re} A(u))^2 \geqq \begin{cases} -u\left(1-\frac{t}{u}\right)^2 & \text{for} \quad b \leqq u \leqq t, \\ 0 & \text{for} \quad t \leqq u \leqq 1, \end{cases}$$

and thus

$$-4\int_b^1 (\operatorname{Re} A(u))^2\, du \leqq 6t^2 - 4t^2 \log t + 2b^2 - 8tb + 4t^2 \log b,$$

with the equality if and only if

$$\cos \vartheta(u) = \begin{cases} 1 & \text{for} \quad b \leqq u \leqq t, \\ \frac{t}{u} & \text{for} \quad t \leqq u \leqq 1. \end{cases}$$

From (4b) we obtain now

$$\operatorname{Re}(a_3 + \lambda a_2) \leqq g(t),$$

where

$$g(t) = \begin{cases} 1-b^2+\frac{1}{4}v^2-\frac{1}{4}\mu^2-4t^2-2t\mu-4t^2\log b-\left(\operatorname{Im} a_2+\frac{1}{2}v\right)^2 & \text{for} \quad 0 \leqq t \leqq b, \\ 1-b^2+\frac{1}{4}v^2-\frac{1}{4}\mu^2-4t^2-2t\mu+6t^2-4t^2\log t+2b^2-8tb-\left(\operatorname{Im} a_2+\frac{1}{2}v\right)^2 & \text{for} \quad b \leqq t \leqq 1. \end{cases}$$

This function g is differentiable on $[0, 1]$ and

$$g'(t) = \begin{cases} -8t-2\mu-8t\log b & \text{for} \quad 0 \leqq t \leqq b, \\ -2\mu-8t\log t-8b & \text{for} \quad b \leqq t \leqq 1. \end{cases} \tag{6}$$

Consider first the case $e^{-1} < b < 1$ and take

$$4b(1+\log b) \leqq |\mu| \leqq 4b.$$

Now $g'(t) \geqq 0$ on $[0, b)$ and hence g has its maximum on $[b, 1]$, where g' has one zero σ. This σ is determined by the condition $-4\sigma \log \sigma = 4b + \mu$. We obtain

$$\max_{0 \leqq t \leqq 1} g(t) = g(\sigma) = 1 + b^2 + \frac{1}{4} \nu^2 - \frac{1}{4} \mu^2 + 2\sigma^2 - \sigma\mu - 4\sigma b$$

and thus

$$\operatorname{Re}(a_3 + \lambda a_2) \leqq 1 + b^2 + \frac{1}{4} \nu^2 - \frac{1}{4} \mu^2 + 2\sigma^2 - \sigma\mu - 4\sigma b,$$

where the equality occurs if and only if

i) $$\cos \vartheta(u) = \begin{cases} 1 & \text{for} \quad b \leqq u \leqq \sigma, \\ \dfrac{\sigma}{u} & \text{for} \quad \sigma \leqq u \leqq 1, \end{cases}$$

ii) $$\sigma = -\frac{1}{2}\left(\operatorname{Re} a_2 + \frac{1}{2} \mu\right)$$

iii) $$-4\sigma \log \sigma = 4b + \mu,$$

iv) $$\operatorname{Im} a_2 + \frac{1}{2} \nu = 0.$$

In order to show that these conditions can be satisfied simultaneously we consider μ with

$$-4b \leqq \mu \leqq -4b(1 + \log b).$$

There is one $\sigma \geqq e^{-1}$ with $-4\sigma \log \sigma = 4b + \mu$. Define

$$\cos \vartheta(u) = \begin{cases} 1 & \text{for} \quad b \leqq u \leqq \sigma, \\ \dfrac{\sigma}{u} & \text{for} \quad \sigma \leqq u \leqq 1. \end{cases}$$

So far the conditions i), ii) and iii) are satisfied. In order to make iv) hold we have to require

$$\frac{1}{4} |\nu| \leqq \int_b^1 |\sin \vartheta(u)| \, du.$$

The equality case $\mu > 0$ is treated similarly. The results collected give

Theorem 2. *Let* $e^{-1} < b < 1$, $\lambda = \mu + i\nu$ *and* $\sigma \in [e^{-1}, 1]$ *be determined by*

$$-4\sigma \log \sigma = 4b - |\mu|.$$

If

$$\begin{cases} 4b(1 + \log b) \leqq |\mu| \leqq 4b, \\ |\nu| \leqq 4\left(\sqrt{1 - \sigma^2} - \sigma \,\overline{\operatorname{arc}} \cos \sigma\right), \end{cases}$$

we have

$$\max_{f \in S(b)} \operatorname{Re}(a_3 + \lambda a_2) = 1 + b^2 + \frac{1}{4} \nu^2 - \frac{1}{4} \mu^2 + 2\sigma^2 + \sigma |\mu| - 4\sigma b.$$

The maximum is reached for a function mapping Δ onto Δ minus a forked slit.

Next consider the case $0<b\leqq e^{-1}$. From (6) we see that $g'(t)\geqq 0$ on $[0, b]$. Thus, again, g has its maximum on $[b, 1]$. Arguments similar to those in the previous case lead to

Theorem 3. *Let* $0<b\leqq e^{-1}$, $\lambda=\mu+iv$ *and* $\sigma\in[e^{-1}, 1]$ *is determined by*

$$-4\sigma\log\sigma = 4b-|\mu|.$$

If

$$\begin{cases}|\mu|\leqq 4b,\\ |v|\leqq 4(\sqrt{1-\sigma^2}-\sigma\,\overline{\mathrm{arc}}\cos\sigma),\end{cases}$$

we have

$$\max_{f\in S(b)}\operatorname{Re}(a_3+\lambda a_2) = 1+b^2+\frac{1}{4}v^2-\frac{1}{4}\mu^2+2\sigma^2+\sigma|\mu|-4\sigma b.$$

The maximum is reached for a function mapping Δ onto Δ minus a forked slit.

Extremals of type 1:1

A particular case of extremals of type 1:1 is obtained if $\mu=0$. From (3) we see that for such an extremal we have

$$\sin\vartheta(u)\int_b^1\cos\vartheta(u)\,du+\cos\vartheta(u)\int_b^1\sin\vartheta(u)\,du-u\sin\vartheta(u)\cos\vartheta(u) = \frac{1}{4}v\int_b^1\cos\vartheta(u)\,du.$$

Integration over $[b, 1]$ gives

$$2\int_b^1\sin\vartheta(u)\,du\int_b^1\cos\vartheta(u)\,du-\int_b^1 u\sin\vartheta(u)\cos\vartheta(u)\,du = \frac{1}{4}v\int_b^1\cos\vartheta(u)\,du.$$

We consider only those cases where $v\geqq 4(1-b)$. We know that in the maximum case $\sin\vartheta(u)\geqq 0$ and thus

$$v\int_b^1\cos\vartheta(u)\,du\leqq 8\int_b^1\sin\vartheta(u)\,du\int_b^1\cos\vartheta(u)\,du\leqq 8(1-b)\int_b^1\cos\vartheta(u)\,du.$$

Therefore, if $\int_b^1\cos\vartheta(u)du\neq 0$, we must have $v\leqq 8(1-b)$. It is clear that we have even $v<8(1-b)$.

Theorem 4. *For* $0<b<1$ *assume that* $|v|\geqq 8(1-b)$. *Then*

$$\max_{f\in S(b)}\operatorname{Re}(a_3+iva_2) = -3+8b-5b^2+2(1-b)|v|.$$

The maximum is reached for a function f for which

$$b\left(f-\frac{1}{f}\right) = z-\frac{1}{z}\pm 2(1-b)i.$$

This function maps Δ onto Δ minus a rectilinear slit.

If $e^{-1} \leqq b < 1$, we can say more. From (4) we see that

$$\operatorname{Re}(a_3 + iva_2) \leqq 1 - b^2 + \frac{1}{4} v^2 - \left(\operatorname{Im} a_2 + \frac{1}{2} v\right)^2.$$

If $v \geqq 4(1-b)$, it follows from $|a_2| \leqq 2(1-b)$ that

$$\left|\operatorname{Im} a_2 + \frac{1}{2} v\right| \geqq \frac{1}{2} v - 2(1-b);$$

$$\operatorname{Re}(a_3 + iva_2) \leqq -3 + 8b - 5b^2 + 2(1-b)v.$$

The equality sign holds if $\sin \vartheta(u) \equiv 1$. Negative values of v can be treated similarly.

Theorem 5. *For* $e^{-1} \leqq b < 1$ *assume that* $|v| \geqq 4(1-b)$. *Then*

$$\max_{f \in S(b)} \operatorname{Re}(a_3 + iva_2) = -3 + 8b - 5b^2 + 2(1-b)|v|.$$

The maximum is reached for a function f for which

$$b\left(f - \frac{1}{f}\right) = z - \frac{1}{z} \pm 2(1-b)i.$$

This function maps Δ onto Δ minus a rectilinear slit.

The general cases 1:1 remain to be discussed. Let

$$p = \int_b^1 \cos \vartheta(u)\, du - \frac{1}{4} \mu \geqq 0, \quad q = \int_b^1 \sin \vartheta(u)\, du - \frac{1}{4} v.$$

The variational formula (3) thus assumes the form

$$p \sin \vartheta(u) + q \cos \vartheta(u) = u \sin \vartheta(u) \cos \vartheta(u). \tag{7}$$

We have to consider four alternatives with respect to $\sin \vartheta(u)$ and $\cos \vartheta(u)$.

1° There exists a value u for which $\sin \vartheta(u) = 0$.

From (7) if follows that $q=0$ and hence $v \leqq 4(1-b)$. We can say even more. Because $|\sin \vartheta|$ is increasing, there exists a number $c \in [b, 1]$ such that $\sin \vartheta(u) = 0$ on $[b, c)$, $\sin \vartheta(u) \neq 0$ on $(c, 1]$. Therefore we see from (7) that

$$\cos \vartheta(u) = \begin{cases} 1 & \text{on} \quad [b, c), \\ \dfrac{p}{u} & \text{on} \quad (c, 1]; \ c \geqq p. \end{cases}$$

From $q=0$ it follows further that

$$\frac{1}{4} v \leqq \int_b^1 |\sin \vartheta(u)|\, du = \int_c^1 \sqrt{1 - p^2/u^2}\, du \leqq \int_p^1 \sqrt{1 - p^2/u^2}\, du = \sqrt{1-p^2} - p \,\overline{\operatorname{arc}} \cos p;$$

thus

$$v \leqq 4\left(\sqrt{1-p^2} - p \,\overline{\operatorname{arc}} \cos p\right).$$

By using (2) we decide that for a prescribed p, $\operatorname{Re}(a_3+\lambda a_2)$ is maximal if

$$\int_b^1 u\cos^2\vartheta(u)\,du = \frac{1}{2}(c^2-b^2)-p^2\log c$$

is minimal, i.e. if $c=p$. Thus the maximizing choice of c and $\cos\vartheta(u)$ is

$$\cos\vartheta(u) = \begin{cases} 1 & \text{on} \quad [b, p], \\ \dfrac{p}{u} & \text{on} \quad [p, 1], \end{cases}$$

which gives

$$p = \int_b^1 \cos\vartheta(u)\,du - \frac{1}{4}\mu = p-b-p\log p-\frac{1}{4}\mu,$$

i.e.

$$-4p\log p = 4b+\mu.$$

If $e^{-1}<b<1$, the previous condition implies, because $p\in[b, 1]$, that

$$4b(1+\log b) \leqq |\mu| \leqq 4b.$$

Similarly, if $0<b\leqq e^{-1}$, we obtain

$$|\mu| \leqq 4b.$$

Therefore, in the case 1° $\operatorname{Re}(a_3+\lambda a_2)$ is maximized, according to Theorems 1 and 3, by extremal functions of the type 1:1.

Next, consider the remaining cases where

2° $\sin\vartheta(u)\neq 0$.

The following alternatives are to be checked.

1) $p=0$.

$$0 \geqq \frac{1}{4}\mu = \int_b^1 \cos\vartheta(u)\,du \geqq 0.$$

Thus

$$\mu = 0, \quad \cos\vartheta(u)\equiv 0 \quad \text{and} \quad |\sin\vartheta(u)|\equiv 1.$$

There are two possibilities available.

If $\nu\geqq 4(1-b)$, we know that in the maximum case $\sin\vartheta(u)\geqq 0$, i.e. $\sin\vartheta(u)\equiv 1$, and therefore we are led to the cases of Theorem 5, where

$$\operatorname{Re}(a_3+i\nu a_2) = -3+8b-5b^2+2\nu(1-b).$$

If $\nu<4(1-b)$, we see from (2) that

$$\operatorname{Re}(a_3+\lambda a_2) = 1-b^2+\frac{1}{4}\nu^2-\left(\operatorname{Im} a_2+\frac{\nu}{2}\right)^2 \leqq 1-b^2+\frac{1}{4}\nu^2,$$

where the equality, making $\operatorname{Im} a_2 = -v/2$, is reached for

$$\sin \vartheta(u) = \begin{cases} 1 & \text{on} \quad \left[b, \dfrac{1+b}{2} + \dfrac{v}{8}\right], \\ -1 & \text{on} \quad \left(\dfrac{1+b}{2} + \dfrac{v}{8}, 1\right]. \end{cases}$$

This maximum thus belongs to the cases of Theorem 1.

2) $p>0$, $q=0$.

From (7) we see that $\cos \vartheta(u) = p/u$, $p \leqq b$ and $|\sin \vartheta(u)| = \sqrt{1-p^2/u^2}$. Further,

$$\frac{1}{4} v \leqq \int_b^1 |\sin \vartheta(u)|\, du = \int_b^1 \sqrt{1-p^2/u^2}\, du$$

$$= \sqrt{1-p^2} - \sqrt{b^2-p^2} + p\, \overline{\operatorname{arc}} \cos \frac{p}{b} - p\, \overline{\operatorname{arc}} \cos p.$$

Because $p = \int_b^1 \cos \vartheta(u)\, du - \mu/4$, we have

$$p + \frac{1}{4}\mu = -p \log b; \quad p = \frac{-\mu}{4(1+\log p)}.$$

From $p>0$ it follows now that $b \in (e^{-1}, 1)$, and therefore we are in the cases of Theorem 1.

3) $p>0$, $q>0$.

Now we have $v/4 < \int_b^1 \sin \vartheta(u)\, du \leqq 1-b$. Here one can repeat the conclusions on pp. 132—134, i.e. changing the signs of ϑ propely without affecting $\cos \vartheta$ we can always diminish $\int_b^1 \sin \vartheta(u)\, du$ into the value $v/4$. This new ϑ increases Re $(a_3 + \lambda a_2)$ to its maximum. Because for the new ϑ $q=0$, we see that 3) is not the maximum case.

3° $\sin \vartheta(u)$ obtains negative values.

According to the remark on p. 134 we know that in the maximum case necessarily $\int_b^1 \sin \vartheta(u)\, du = v/4$, i.e. $q=0$. From (7) we see that we can now go back to the function ϑ which was defined in 1° and thus we end up with the same conclusions as in 1°.

4° There exists a value u for which $\cos \vartheta(u) = 0$.

In this case $p=0$, but because $\cos \vartheta(u) \geqq 0$ and $\mu \leqq 0$, we must have $\cos \vartheta(u) \equiv 0$ and $\mu=0$. We are led back to the beginning of 2°, where this case was handled under the assumption $p=0$.

From 1°—4° we decide now that in the cases not handled yet there are

$$p>0, \quad q<0, \quad \cos \vartheta > 0, \quad \sin \vartheta > 0.$$

Rewrite (7) in the form

(7a) $$F(\vartheta, u) = \frac{p}{\cos \vartheta} + \frac{q}{\sin \vartheta} - u = 0.$$

Because $F_\vartheta(\vartheta, u) > 0$ we see that (7a) determines ϑ as a differentiable function of u and

$$du = \left(\frac{p \sin \vartheta}{\cos^2 \vartheta} - \frac{q \sin \vartheta}{\sin^2 \vartheta}\right) d\vartheta.$$

It follows from the Löwner theory [3] that such a function ϑ determines a solution of type 1:1.

If we denote

$$\alpha = -\vartheta(1), \quad \omega = -\vartheta(b),$$

we obtain from (3) the equations

$$\text{(8a)} \qquad \begin{cases} p \sin \alpha - q \cos \alpha = \sin \alpha \cos \alpha, \\ p \sin \omega - q \cos \omega = b \sin \omega \cos \omega. \end{cases}$$

Two more equations can be obtained from

$$\text{(8b)} \quad \begin{cases} p = \int_b^1 \cos \vartheta(u)\, du - \frac{1}{4}\mu = \int_{-\omega}^{-\alpha} \cos \vartheta \left(\frac{p \sin \vartheta}{\cos^2 \vartheta} - \frac{q \cos \vartheta}{\sin^2 \vartheta}\right) d\vartheta - \frac{1}{4}\mu \\ \qquad = -p \log \frac{\cos \alpha}{\cos \omega} - q(\cot \alpha - \cot \omega + \alpha - \omega) - \frac{\mu}{4}, \\ q = \int_b^1 \sin \vartheta(u)\, du - \frac{1}{4}\nu = -p(\tan \alpha - \tan \omega - \alpha + \omega) - q \log \frac{\sin \alpha}{\sin \omega} - \frac{\nu}{4}. \end{cases}$$

From (2) we see that

$$\max_{f \in S(b)} \mathrm{Re}\,(a_3 + \lambda a_2) = 1 - b^2 - \frac{1}{4}\mu^2 + \frac{1}{4}\nu^2 - p\mu + q\nu + 4pq(\tan \alpha - \tan \omega)$$
$$-2q^2\left(\frac{1}{\sin^2 \alpha} - \frac{1}{\sin^2 \omega}\right).$$

For the original problem $\max_{f \in S} \mathrm{Re}\,(a_3 + i a_2)$ we have determined the following numerical solution:

$$\begin{cases} \alpha = -0.528{\cdot}513{\cdot}532, \\ \omega = -0.066{\cdot}344{\cdot}080; \end{cases}$$

$$\max_{f \in S} \mathrm{Re}\,(a_3 + i a_2) = 3.190{\cdot}298{\cdot}109.$$

Note 1. From (3a) it follows that $\vartheta \equiv \pi/2$ gives at least a local maximum if and only if

$$\nu \geqq 4(1-b) + 4\,\frac{\left(\int_b^1 \varphi(u)du\right)^2 - \int_b^1 u\varphi^2(u)\, du}{\int_b^1 \varphi^2(u)\, du}$$

for all piecewise continuous functions φ on $[b, 1]$. If $b \in [e^{-1}, 1)$, we have from Schwarz's inequality

$$\Big(\int_b^1 \varphi(u)\,du\Big)^2 \leqq \int_b^1 \frac{du}{u} \int_b^1 u\varphi^2(u)\,du = -\log b \int_b^1 u\varphi^2(u)\,du \leqq \int_b^1 u\varphi^2(u)\,du.$$

The condition for v is thus in accordance with Theorem 5.

Note 2. By solving the system (8) with the aid of power series in the neighbourhood of $\alpha=\omega=\pi/2$ we find that for $|v| \geqq 4(1-b)+4(1-eb)/(e-1)$ the functions with one rectilinear slit give $\max_{f\in S(b)} \operatorname{Re}(a_3+iva_2)$.

Note 3. The problem of determining $\min_{f\in S(b)} \operatorname{Re}(a_3+\lambda a_2)$ is easily reduced to the problem studied here. By considering $-if(iz)$ instead of $f(z)$ we see that

$$\min_{f\in S(b)} \operatorname{Re}(a_3+\lambda a_2) = -\max_{f\in S(b)} \operatorname{Re}(a_3+i\lambda a_2).$$

Note 4. By using the same arguments as before one can also determine the part of the coefficient body (a_2, a_3) of $S(b)$ where the boundary functions are of the type 1:1.

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