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HARNACK'S INEQUALITY IN THE BORDERLINE CASE

SEPPO GRANLUND

1. Introduction

Harnack's inequality for general quasi-linear elliptic equations has been proved by Serrin [13] and Trudinger [14]. In both papers the proof is based on the iteration method introduced by Moser in [7] and [8]. In this method the lemma of John and Nirenberg [4] is essential.

We consider variational integrals of the form

$$I(u) = \int_G F(x, \nabla u(x))\, dm(x), \tag{1.1}$$

where $G \subset R^n$ is a bounded domain, $u \in W_n^1(G)$, and the kernel $F\colon G \times R^n \to R$ satisfies the following conditions:

(1.2) The functions $x \mapsto F(x, \nabla u(x))$ are measurable for all $u \in W_n^1(G)$.

(1.3) For a.e $x \in G$ the function $z \to F(x, z)$ is convex and $\alpha |z|^n \leqq F(x, z) \leqq \beta |z|^n$ for all $z \in R^n$, where $\alpha, \beta > 0$ are constants.

Let $\varphi \in W_n^1(G)$ and write $\mathscr{F}_\varphi(G) = \{u \in W_n^1(G) | u - \varphi \in W_{n,0}^1(G)\}$. A function $u_0 \in \mathscr{F}_\varphi(G)$ is an extremal for the integral (1.1) if $I(u) \geqq I(u_0)$ for all $u \in \mathscr{F}_\varphi(G)$. It can be proved that u_0 is locally Hölder continuous in G; see [6, Theorem 4.3.1] and [3, Remark 5.7]. Let u_0 be an extremal for the integral (1.1) and $u_0(x) \geqq 0$ for $x \in G$. Harnack's inequality takes the following form:

1.4. Theorem. *Let* $\bar{B}^n(x_0, 2r) \subset G$. *Then* $\max u_0 \leqq c \min u_0$ *in* $B^n(x_0, r)$. *The constant c depends only on* α/β *and n.*

Our proof is based on an oscillation lemma for monotone functions in the space $W_n^1(G) \cap C(G)$. Lemmas of this type have been proved by Gehring [1, Lemma 1] and Mostow [9, Lemma 4.1]. This makes it possible to avoid the lemma of John and Nirenberg. In the case $n=2$ a similar method has been used earlier for linear elliptic equations; see Gilbarg—Trudinger [2, p. 200].

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2. Preliminary lemmas

We first give the definition of monotone functions.

2.1. Definition. Let $G\subset R^n$ be a bounded domain. A function $u\in C(G)$ is monotone in G if

$$\sup_{x\in D} u(x) = \sup_{x\in\partial D} u(x) \quad \text{and} \quad \inf_{x\in D} u(x) = \inf_{x\in\partial D} u(x)$$

for all domains D, $\bar{D}\subset G$.

The following lemma gives an estimate for the oscillation of monotone functions in the space $W_n^1(G)$.

2.2. Lemma. *Let* $u\in C(G)\cap W_n^1(G)$ *and* $B^n(x_0, 2r)\subset G$. *If* u *is monotone in* G, *we have*

$$\operatorname*{osc}_{B^n(x_0,r)} \{u\} \leqq \gamma\Big(\int_{B^n(x_0,3r/2)} |\nabla u|^n \, dm\Big)^{1/n}.$$

The constant γ *depends only on* n.

Proof. The inequality can be easily derived from the oscillation lemma proved in [9, Lemma 4.1]; see also [1, Lemma 1].

The next lemma gives a weak maximum principle for the extremals of the integral (1.1). In what follows $\varphi\in W_n^1(G)$ will be fixed.

2.3. Lemma. *Let* $u_0\in\mathscr{F}_\varphi(G)$ *be an extremal for the integral* (1.1). *Then* u_0 *is monotone in* G.

Proof. Suppose that there is a domain D, $\bar{D}\subset G$, and a point $x_0\in D$ such that

$$u_0(x_0) > \max_{x\in\partial D} u_0(x) = a.$$

Define $\delta(x)=\inf\{u_0(x), a\}$ for $x\in D$. Then $\delta\in\mathscr{F}_{u_0}(D)$ and $\nabla\delta(x)=0$ a.e. on the set $\{x\in D | u_0(x)\geqq a\}$. It follows from the condition (1.3) that

$$\int_D F(x, \nabla\delta)\, dm(x) < \int_D F(x, \nabla u_0)\, dm(x).$$

This is a contradiction since u_0 is an extremal in the class $\mathscr{F}_{u_0}(D)$. Thus

$$\sup_{x\in D} u_0(x) = \sup_{x\in\partial D} u_0(x).$$

We prove the corresponding equation for minimum values exactly by the same argument. It follows that u_0 is monotone in G.

Let $u_0\in\mathscr{F}_\varphi(G)$ be an extremal for the integral (1.1) and $u_0(x)>0$ for $x\in G$. Define $v(x)=\log u_0(x)$ for $x\in G$.

2.4. Lemma. *Let* $\bar{B}^n(x_0, 4r/3) \subset G$. *Then*

$$\int_{B^n(x_0,r)} |\nabla v|^n \, dm \leqq c_0 .$$

The constant c_0 *depends only on* α/β *and* n.

Proof. We may assume that $u_0(x) \geqq (n-1)^{1/n}$ for $x \in B^n(x_0, 4r/3)$. If this is not the case we consider the function λu_0, where $\lambda > 0$ is large enough. The function λu_0 is an extremal for a variational integral, which is of the form (1.1) and satisfies the structure condition (1.3) with the same constants α and β.

Let $\xi \in C_0^\infty(B^n(x_0, 4r/3))$ be a non-negative function such that $\xi(x)=1$ for $x \in B^n(x_0, r)$ and $|\nabla \xi(x)| \leqq c_1/r$. We choose

$$h(x) = u_0(x) + \frac{\xi^n(x)}{u_0(x)^{n-1}} .$$

Then $h \in \mathscr{F}_{u_0}(B^n(x_0, 4r/3))$, and it has the generalized derivatives

$$h_{x_i} = u_{0x_i} + \frac{n\xi^{n-1}}{u_0^{n-1}} \xi_{x_i} - (n-1)\frac{\xi^n}{u_0^n} u_{0x_i} = \left(1-(n-1)\frac{\xi^n}{u_0^n}\right) u_{0x_i} + n\frac{\xi^{n-1}}{u_0^{n-1}} \xi_{x_i} .$$

Write $S = \{x \in B^n(x_0, 4r/3) \mid \xi(x) = 0\}$. Suppose that $x \in B^n(x_0, 4r/3) \setminus S$. The convexity and growth condition (1.3) yields

$$\begin{aligned} F(x, \nabla h) &\leqq \left(1-(n-1)\frac{\xi^n}{u_0^n}\right) F(x, \nabla u_0) + (n-1)\frac{\xi^n}{u_0^n} F\left(x, \frac{n}{n-1}\frac{u_0}{\xi}\nabla\xi\right) \\ &\leqq \left(1-(n-1)\frac{\xi^n}{u_0^n}\right) F(x, \nabla u_0) + \beta\frac{n^n}{(n-1)^{n-1}} |\nabla\xi|^n . \end{aligned}$$

Since u_0 is minimizing for the integral (1.1), we get

$$\begin{aligned} \int_{B^n(x_0,4r/3)} F(x, \nabla u_0)\, dm(x) &\leqq \int_{B^n(x_0,4r/3)} F(x, \nabla h)\, dm(x) \\ &= \int_{B^n(x_0,4r/3)\setminus S} F(x, \nabla h)\, dm(x) + \int_S F(x, \nabla h)\, dm(x) \\ &\leqq \int_{B^n(x_0,4r/3)\setminus S} \left(1-(n-1)\frac{\xi^n}{u_0^n}\right) F(x, \nabla u_0)\, dm(x) \\ &\quad + \beta\frac{n^n}{(n-1)^{n-1}} \int_{B^n(x_0,4r/3)\setminus S} |\nabla\xi|^n \, dm + \int_S F(x, \nabla u_0)\, dm(x). \end{aligned}$$

It follows from this that

$$(n-1) \int_{B^n(x_0,4r/3)\setminus S} \frac{\xi^n}{u_0^n} F(x, \nabla u_0)\, dm(x) \leqq \beta\frac{n^n}{(n-1)^{n-1}} \int_{B^n(x_0,4r/3)} |\nabla\xi|^n \, dm.$$

Notice that $B^n(x_0, r) \cap S = \emptyset$. The condition (1.3) implies

$$\int_{B^n(x_0, r)} \frac{|\nabla u_0|^n}{u_0^n}\, dm \leqq \frac{\beta}{\alpha}\left(\frac{n}{n-1}\right)^n \int_{B^n(x_0, 4r/3)} |\nabla \xi|^n\, dm \leqq c_0.$$

The constant c_0 depends only on α/β and n. Our lemma is proved.

3. Proof for Theorem 1.4

Let u_0 be an extremal for the integral (1.1) and $u_0(x) \geqq 0$ for $x \in G$. Let $\varepsilon > 0$ and define $v(x) = \log\left(u_0(x) + \varepsilon\right)$ for $x \in G$. The function v is monotone in G since u_0 is monotone by Lemma 2.3. We obtain a bound for the oscillation of v from Lemma 2.2 and Lemma 2.4:

$$\operatorname*{osc}_{B^n(x_0, r)} \{v\} \leqq \gamma \Bigg(\int_{B^n(x_0, 3r/2)} |\nabla v|^n\, dm \Bigg)^{1/n} \leqq \gamma c_0^{1/n}.$$

Then we have

$$\log(\max u_0 + \varepsilon) - \log(\min u_0 + \varepsilon) \leqq \gamma c_0^{1/n}$$

on the set $B^n(x_0, r)$. Finally we get

$$\max u_0 + \varepsilon \leqq e^{\gamma c_0^{1/n}} (\min u_0 + \varepsilon).$$

Harnack's inequality follows if we let $\varepsilon \to 0$.

3.1. Remark. Let $f = (f_1, \ldots, f_n)\colon G \to R^n$ be a quasiregular mapping; see [5], [10]. Rešetnjak [11], [12] has shown that each of the coordinate functions of f minimizes a variational integral of the form (1.1). Then it follows from Theorem 1.4 that Harnack's inequality is valid for the coordinate functions. This fact has been proved earlier by Rešetnjak, who used Serrin's paper [13].

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Helsinki University of Technology
Institute of Mathematics
SF-02150 Espoo 15
Finland

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