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ASYMPTOTIC VALUES AND ANGULAR LIMITS OF QUASIREGULAR MAPPINGS OF A BALL

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1. Introduction

If f is a bounded analytic function of the unit disk and f has an asymptotic limit α at a boundary point b, then a wellknown theorem by E. Lindelöf says that f has the angular limit α at b, and hence α is the only asymptotic value f can have at b. In this paper we shall study the situation in dimensions $n\geqq 3$ for the natural generalization of analytic functions to Euclidean n-space, namely quasiregular mappings (for the definition, see [2]). It turns out that, although a radial limit implies the existence of an angular limit, a limit along a 1-dimensional set, like a path, is in general no longer sufficient to guarantee the existence of the angular limit, and hence Lindelöf's theorem is not true. In fact, there can be infinitely many asymptotic values at a boundary point as Theorem 1.1 shows. For the dimension $n=4$ we prove (Theorem 1.2) that here paths can as well be replaced by curved 2-dimensional half planes which have a common edge. On the other hand, Theorem 1.3 shows that $n-2$ is the maximum dimension for such results and gives a substitute for Lindelöf's theorem in a form where the asymptotic path is replaced by an $(n-1)$-dimensional set.

Let B^n be the unit ball in the Euclidean n-space R^n and $f\colon B^n\to B^n$ a quasiregular mapping. As we said above, if the radial limit

$$\alpha = \lim_{t\to 1-} f(ty)$$

exists for some y in the boundary ∂B^n, then the angular limit

$$\lim_{x\to y,\, x\in K} f(x)$$

exists and equals α for all cones $K=K(y,\varphi)=\{x\in R^n | (y|(y-x))>|y-x|\cos\varphi\}$, $0<\varphi<\pi/2$ [1, 5.8]. Here $(u|v)$ is the inner product of u and v in R^n. In this situation we say that f has the angular limit α at y.

1.1. Theorem. *For each $n\geqq 3$ there exists a quasiregular mapping $f\colon B^n\to B^n$ and a point $b\in\partial B^n$ such that*

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(1) *f has infinitely many asymptotic values at b,*
(2) *f has no angular limit at b.*

In dimension 4 we even have a stronger result.

1.2. Theorem. *There exists a sequence* $U_1, U_2, \ldots$ *of 2-dimensional submanifolds in* B^4 *such that the projection* $P\colon R^4 \to R^2$, $P(x)=(x_1, x_3)$, *maps each* U_j *homeomorphically onto the half disk* $\{y \in R^2 | |y|<1/2,\ y_1>0\}$, $\overline{U}_j \cap \partial B^4$ *is the arc* $\{x \in \partial B^4 | x_1=x_4=0,\ |x_3| \leqq 1/2,\ x_2<0\}$ *for all j, and the following holds:*

(1) *Given any sequence* $a^1, a^2, \ldots$ *in* $\{x \in R^4 | |x|<1/2\}$ *there exists a quasiregular mapping* $f\colon B^4 \to B^4$ *for which*

$$\lim_{x \to -e_2,\, x \in U_j} f(x) = a^j.$$

(2) *f has no angular limit at* $-e_2$. *(*e_i *is the* i^{th} *standard coordinate unit vector in* R^n.*)*

It seems likely that a construction can be carried out also for $n \geqq 5$ similar to the one used in the proof of Theorem 1.2 to produce an infinite number of asymptotic limits along $(n-2)$-dimensional curved half planes. However, the given proof depends on a certain 3-complex whose construction cannot canonically be generalized to higher dimensions.

Our positive result on the existence of angular limits is as follows.

1.3. Theorem. *Let* $n \geqq 2$, $A=\{y \in \partial B^n | y_n>0\}$, *and let* $f\colon B^n \to B^n$ *be a quasiregular mapping such that for some* $\alpha \in \overline{B}^n$

$$\lim_{y \to e_1,\, y \in A} \limsup_{x \to y} |f(x)-\alpha| = 0.$$

Then f has the angular limit α *at* e_1

The proof of Theorem 1.3 is a simple consequence of a two constant theorem for quasiregular mappings ([7]), which is based on estimates on solutions of quasilinear elliptic partial differential equations due to Maz'ja. The proofs of Theorems 1.1 and 1.2 are rather complicated. The basic idea is to use a method developed in [6] to deform a map through adjacent cylinders into a prescribed behavior. Since $n \geqq 3$, it is possible to place the "cylinders" so that their sizes tend to zero when they approach a part of the boundary.

1.4. Remarks. 1. The proofs of Theorems 1.1 and 1.2 can be modified so that the asymptotic limits form any given countable set in the closed unit ball. One can also obtain some related results on boundary behavior, for example from the proof of Theorem 1.2 the following: There exists a nonconstant quasiregular mapping $f\colon B^4 \to B^4$ such that

$$\lim_{x \to y} f(x) = c = \text{constant}$$

for y in the topological disk $\{z \in \partial B^4 | |z+e_2|<1,\ z_4=0\}$.

2. The Lindelöf's theorem quoted above holds for quasiregular mappings for $n=2$. Theorem 1.3 reduces to a special case of this if $n=2$.

3. Because of the connection of quasiregular mappings to solutions of quasilinear elliptic partial differential equations (see [7]), our constructions give examples for such solutions too.

4. The existence of angular limits of quasiregular mappings has been studied also by M. Vuorinen in [8—11]. For example he has shown that for closed quasiregular mappings of B^n into itself an asymptotic limit at $b\in\partial B^n$ implies the existence of the same angular limit at b, i.e. Lindelöf's theorem holds for these mappings.

2. Construction of the examples

2.1. In the proofs of Theorems 1.1 and 1.2 we shall use a construction similar to [6, pp. 542—546]. The presentation is selfcontained, hence no reference to [6] is needed.

We denote by $B^n(x, r)$ the open ball in R^n with center x and radius r, by q the spherical chordal metric in $\bar{R}^n=R^n\cup\{\infty\}$, and by $\langle a_1, \ldots, a_p\rangle$ the convex hull of $\{a_1, \ldots, a_p\}$. For $1\leqq k<n$, the sets R^k and $R^k\times\{0\}\subset R^n$ are identified by the embedding $(x_1, \ldots, x_k)\mapsto(x_1, \ldots, 0, \ldots, 0)$. We set

$$H_i = \{x\in R^n | x_i > 0\}, \quad H_{i_1,\ldots,i_k} = H_{i_1}\cap\ldots\cap H_{i_k},$$

$$I_\varepsilon = [-\varepsilon, \varepsilon] \text{ for } \varepsilon > 0, \quad I = I_1, \quad B^n(r) = B^n(0, r), \quad B^n = B^n(1), \quad S^{n-1} = \partial B^n.$$

All topological operations are with respect to $\bar{R}^n$ if not otherwise stated.

2.2. *Proof of Theorem* 1.1. Set

$$X = [0, 1]\times]0, 1] \subset R^2,$$

$$A^* = \{x\in R^n | (x_1, x_2)\in A, \ -x_2 < x_3, \ldots, x_n < x_2\} \text{ if } A\subset X.$$

We shall construct a quasimeromorphic mapping (see [3]) of int X^* omitting a ball and with infinitely many asymptotic values at 0. The construction is made separately in sets C^*, C running over the set γ of all (closed) 2-simplexes in the locally finite simplicial 2-complex L with underlying space X as shown in Figure 1.

Fix $C_0\in\gamma$ and an affine map h_{C_0} of C_0 onto the 2-simplex $W_1=\langle -e_1, e_1, e_2\rangle\subset R^2$. Then there exists a unique set of affine homeomorphisms $h_C\colon C\to W_1$ such that $h_C|C\cap D=h_D|C\cap D$ for $C, D\in\gamma$. This follows because each vertex of L in $X\setminus\dot{X}$ belongs to an even number of 2-simplexes of L where $\dot{X}$ is the boundary of X in R^2.

Let W be the closed square $\langle -e_1, -e_2, e_1, e_2\rangle\subset R^2$, for $0<\varepsilon<1$ let $w\colon W\times I_\varepsilon^{n-3}\to \bar{B}^{n-1}$ be the radial stretching (for $n=3$ we identify $W=W\times I_\varepsilon^{n-3}$), and let $\psi_1= w\times\mathrm{id}_I\colon (W\times I_\varepsilon^{n-3})\times I\to\bar{B}^{n-1}\times I$. We choose ε so that $w(\dot{W}\times I_\varepsilon^{n-3})\subset\bar{B}^{n-1}\cap (R^2\times I_{\varepsilon_1}^{n-3})$ where $\varepsilon_1=(2(n-3))^{-1/2}$. We let $\varphi_1\colon \bar{B}^{n-1}\times I\to\bar{H}_n$ be the mapping $\varphi_1(r, y, x_n)=(\varrho, y, \theta)$ defined by $\varrho=4e^{x_n+1}$, $\theta=\pi r/2$. Here cylindrical and spherical

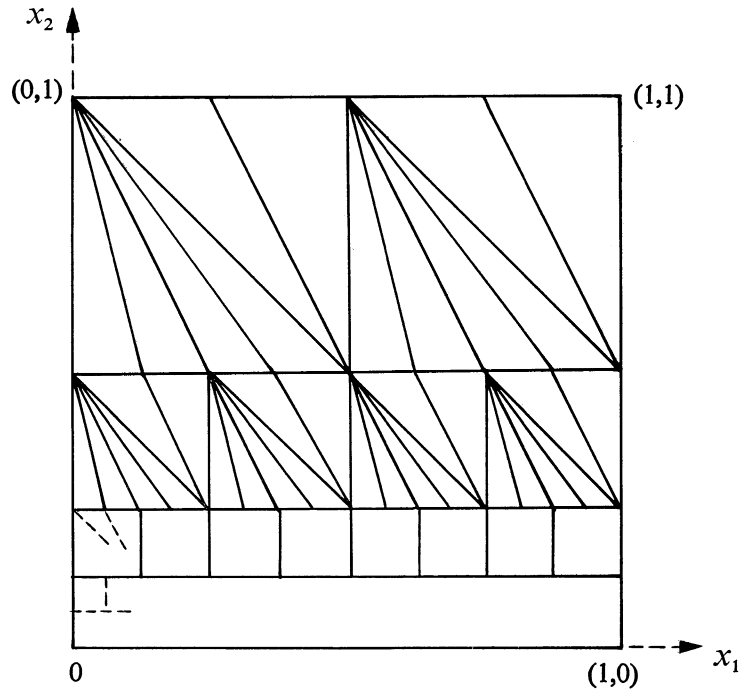

Figure 1

coordinates are used so that $y \in S^{n-2}$ and θ is the angle between the radius vector and the x_n-axis. Set $F = \varphi_1 \circ \psi_1$. Then $F|\mathrm{int}(W \times I_\varepsilon^{n-3} \times I)$ is quasiconformal.

Next let

$$A_\alpha = \{x \in R^n | \alpha_i < x_i < \alpha_i + 1, \; i = 2, \ldots, n\} \quad \text{for} \quad \alpha \in \{0\} \times Z^{n-1},$$

$$T_1 = \{x \in A_0 | x_1 < -2\},$$

and let $F_1 \colon A_0 \to H_n$ be a quasiconformal mapping such that $F_1(x)_1 < 0$ if $x_1 < 0$, $F_1(x)_1 > 0$ if $x_1 > 0$, $F_1 T_1 \subset B^n(-e_1, r_0)$ where we fix $r_0 = 1/10$, $F_1(A_0 \cap B^n(4e^2)) \subset \bar{R}^n \setminus B^n(-e_1, r_0/2)$, and

$$\lim_{x_1 \to -\infty} F_1(x) = -e_1,$$

$$\lim_{x_1 \to \infty} F_1(x) = e_1.$$

We extend F_1 to a quasimeromorphic mapping $F_0 \colon H_{2,n} \to \bar{R}^n \setminus \{-e_1, e_1\}$ by reflecting through the faces of the cylinders A_α and through ∂H_n. For $C \in \gamma$ we define $g_C \colon \bar{C}^* \to W_1 \times I_\varepsilon^{n-3} \times I$ by

$$g_C(x) = \big(h_C(x_1, x_2)_1, \; h_C(x_1, x_2)_2, \; \varepsilon x_3/x_2, \ldots, \varepsilon x_{n-1}/x_2, \; x_n/x_2\big).$$

Fix now $\beta \geq \beta_0 > 1$ and $M \in]1, 2[$ where the bound β_0 will be chosen sufficiently large later. Let σ be the hyperbolic metric in the upper half plane. Fix $1 < \mu_1 <$

ν_1 and set

$$Y_0 = \{x \in X \,|\, x_2 \geqq x_1^{\mu_1} \text{ or } x_2 \leqq x_1^{\nu_1}\},$$
$$Y_k = \{x \in X \,|\, \beta(k-1) < \sigma(x, Y_0) \leqq \beta k\}, \quad k = 1, 2, \ldots .$$

Let $\delta \in]0, 1/2[$ and $c^1 \in \bar{R}^n \setminus B^n(-e_1, \delta)$. In the first step we shall construct a mapping $f_1\colon X^* \to \bar{R}^n \setminus B^n(-e_1, r_1)$ which is K-quasimeromorphic in $\operatorname{int} X^*$, where K and $f_1 | Y_0^*$ are independent of c^1, where $r_1 > 0$ depends only on δ, and for which

$$\lim_{k \to \infty} \sup_{x \in Y_k^*} |f_1(x) - c^1| = 0.$$

Choose a sequence $z^0 = e_1, z^1, \ldots, z^s = z^{s+1} = \ldots = c^1$ in $\bar{R}^n$ such that

$$M^{-1} \leqq \frac{q(z^k, -e_1)}{q(z^{k+1}, -e_1)} \leqq M, \quad k = 0, \ldots, s,$$

and $q(z^k, -e_1) \geqq q(c^1, -e_1)$, $k = 1, 2, \ldots$. Also a sequence $x^0 = 0, x^1, x^2, \ldots$ is defined as follows. For $1 \leqq k \leqq s$ let l_k be the minimizing geodesic arc from $-e_1$ to z^k and let x^k be the middle point of l_k. (If $z^k = e_1$, we let l_k be the geodesic through 0.) If $k > s$, we let $x^k \in l_s$ be so that $q(x^k, c^1)/q(x^s, c^1) = 2^{s-k}$. Let S_k be a Möbius transformation of $\bar{R}^n$ such that $S_k(-e_1) = -e_1$, $S_k(e_1) = z^k$, $S_k(0) = x^k$, $k = 0, 1, \ldots$. There exists a K_1-quasiconformal mapping $\omega_k\colon \bar{R}^n \to \bar{R}^n$, K_1 independent of k, such that

$$\omega_k | B^n(-e_1, r_0) = S_k | B^n(-e_1, r_0),$$
$$\omega_k | H_1 = S_{k+1} | H_1 .$$

We need a map $\sigma_1\colon \bar{H}_{2,n} \to \bar{H}_{2,n}$ defined by

$$\sigma_1(x) = (x_1 - x_2, x_2, x_3, \ldots, x_n) \quad \text{if} \quad 0 \leqq x_2 < 2,$$
$$\sigma_1(x) = (x_1 - 2, x_2, x_3, \ldots, x_n) \quad \text{if} \quad x_2 \geqq 2.$$

If u is a vertex of the complex L, we define

$$P_u = \{h_D^{-1}(e_1) | u \in D, \ D \in \gamma\}.$$

We are now in a position to be able to define for each $C \in \gamma$ the restriction to C^* of the map f_1 to be constructed, call it τ_C. Fix $C \in \gamma$. Denote $\xi = h_C^{-1}(-e_1)$, $\eta = h_C^{-1}(e_1)$, $a = h_C^{-1}(e_2)$. Let $\bar{F}$ be F_0 if h_C is sense preserving and $v \circ F_0$ if h_C is sense reversing where v is the reflection in ∂H_n. Set

$$Q_C = \cup \{P_u | u \in D, \ \xi \in D, \ D \in \gamma\}.$$

For sufficiently large β_0, independent of C, there exists a largest integer k such that $Q_C \subset Y_k \cup Y_{k+1}$. In different cases we define τ_C as follows:

(1) $\quad \tau_C = S_k \circ \bar{F} \circ F \circ g_C \quad$ if $\quad P_a \subset Y_k$,

(2) $\quad \tau_C = S_k \circ \bar{F} \circ \sigma_1 \circ F \circ g_C \quad$ if $\quad \eta \in Y_k, \ P_a \cap Y_{k+1} \neq \emptyset$,

(3) $\quad \tau_C = \omega_k \circ \bar{F} \circ \sigma_1 \circ F \circ g_C \quad$ if $\quad \eta \in Y_{k+1}, \ P_a \cap Y_k \neq \emptyset$,

(4) $\quad \tau_C = \omega_k \circ \bar{F} \circ F \circ g_C \quad$ if $\quad P_a \subset Y_{k+1}$.

We want to check that the maps τ_C coincide on common boundaries. It is enough to consider $D\in\gamma$ such that $C\cap D$ is a 1-simplex. Let p be the largest integer such that $Q_D\subset Y_p\cup Y_{p+1}$.

Suppose first that $C\cap D=\langle\xi,\eta\rangle$. Then $p=k$. It follows that τ_C and τ_D may differ only in the appearance of the map σ_1 in the formulas (1)—(4). But $F\circ g_C(C^*\cap D^*)\subset\partial H_2$ and σ_1 is the identity in ∂H_2.

Suppose next that $C\cap D=\langle\xi,a\rangle$. Also here $p=k$. We claim that $\bar F\circ\sigma_1\circ F\circ g_C(C^*\cap D^*)\subset\bar B^n(-e_1,r_0)$. It is enough to show $\sigma_1(z)_1\leqq -2$ whenever $z\in Fg_C(C^*\cap D^*)$. Such a point is of the form $z=(\varrho y_1,\dots,\varrho y_{n-1},0)$, $y_3,\dots,y_{n-1}\in I_{\varepsilon_1}$, $|y|=1$, $y_1\leqq 0$, $y_2\geqq 0$, $4\leqq\varrho\leqq 4e^2$. We may assume $0\leqq z_2\leqq 2$, whence $\sigma_1(z)_1=z_1-z_2$. We have $y_2\leqq 1/2$ and $y_1^2\geqq 1-\varepsilon_1^2(n-3)-y_2^2\geqq 1/2-y_2^2$. This implies $y_1\leqq -1/2$ and $z_1\leqq -2$, hence $\sigma_1(z)_1\leqq -2$. In the cases (1) and (4) we have the same definition for τ_D, so the assertion is clear. Suppose that we have (2) for C and (3) for D. In this case we use the fact that $\omega_k|B^n(-e_1,r_0)=S_k|B^n(-e_1,r_0)$, and similarly if we have (2) for D and (3) for C.

Finally, let $C\cap D=\langle\eta,a\rangle$. If $p=k$, the cases for D coincide with those of C in (1)—(4). By symmetry with respect to C and D it is enough to consider the case $p=k-1$. For a sufficiently large β_0, independent of C, we must have $Q_C\subset Y_k$, and we have case (1) for C and case (4) for D which fit together on $C^*\cap D^*$.

We have shown that there exists a map f_1 of X^* such that $f_1|C^*=\tau_C$ for $C\in\gamma$. It follows from the definition of the maps τ_C that $f_1\colon X^*\to\bar R^n\setminus B^n(-e_1,r_1)$ for some $r_1>0$ and has the other required properties for the first step; $f_1|\operatorname{int}X^*$ has c^1 as an asymptotic value at 0. For this we observe that

$$\lim_{x\to 0,\,x\in E_1^*} f_1(x)=c^1,$$

where $E_1=\{x\in X\,|\,x_2=x_1^{\lambda_1}\}$, $\mu_1<\lambda_1<\nu_1$, and $E_1^*\cap\operatorname{int}X^*$ contains paths tending to 0.

Let then (c^j) be any sequence in $\bar R^n\setminus B^n(-e_1,\delta)$, and let $1<\lambda_1<\lambda_2<\dots$. If $1<\mu_1<\lambda_1<\nu_1<\mu_2<\lambda_2<\nu_2<\dots$, we may deform f_1 in the sets Y^{j*}, $Y^j=\{x\in X\,|\,x_1^{\nu_j}<x_2<x_1^{\mu_j}\}$, $j=2,3,\dots$, by applying the construction from the first step to obtain a quasimeromorphic mapping $f_0\colon\operatorname{int}X^*\to\bar R^n\setminus B^n(-e_1,r_1)$ such that

$$\lim_{x\to 0,\,x\in E_j^*} f_0(x)=c^j,$$

where $E_j=\{x\in X\,|\,x_2=x_1^{\lambda_j}\}$. The required mapping satisfying (1) and (2) in Theorem 1.1 is then $f=h\circ f_0\circ g$ where g is a suitable quasiconformal mapping of B^n onto $\operatorname{int}X^*$ and h is a Möbius transformation.

2.3. Remark. The paper [6] was for simplicity written for $n=3$. The above proof shows how to do the appropriate modifications to obtain the result in [6] for general $n\geqq 3$.

2.4. *Proof of Theorem* 1.2. We replace the square $[0,1]\times]0,1]$ in the proof of 1.1 by the cube $X'=[0,1]\times]0,1]\times[0,1]$ and form the locally finite simplicial

3-complex L' as follows. In Figure 2 we let the square $\langle p_0, p_3, p_2, p_1\rangle$ be in the plane $x_2=1/2$ with $p_0=(0, 1/2, 0)$, $p_1=(1/2, 1/2, 0)$, $p_2=(1/2, 1/2, 1/2)$, $p_3=(0, 1/2, 1/2)$. Let $\langle q_0, q_3, q_2, q_1\rangle$ be the square obtained from $\langle p_0, p_3, p_2, p_1\rangle$ by the translation $x\mapsto x+e_2/2$, hence for example $q_0=(0, 1, 0)$. The 3-simplexes of L' in the part $\langle q_3, p_0, p_3, p_2\rangle$ of X' are the 3-simplexes of the form $\langle q_3, s, t, u\rangle$ where $\langle s, t, u\rangle$ is a 2-simplex in the 2-complex in Figure 2 with underlying space $\langle p_0, p_3, p_2\rangle$. The 3-simplexes of L' in the part $\langle p_0, p_2, q_3, q_2, q_0\rangle$ of X' are given in Figure 3. We

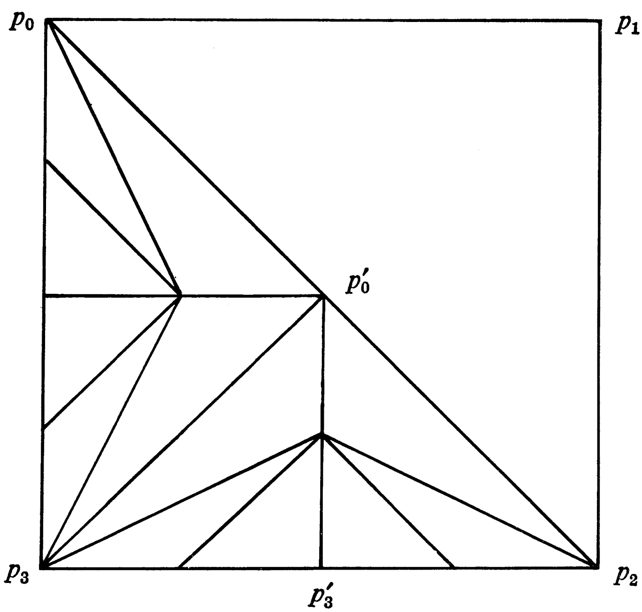

Figure 2

complete the construction of L' in the cube $\langle p_0, p_1, p_2, p_3, q_0, q_1, q_2, q_3\rangle$ by reflecting through $\langle p_0, p_2, q_2, q_0\rangle$. We continue the construction to three similar cubes with side length $1/2$ by first reflecting through the squares $\langle p_3, p_2, q_2, q_3\rangle$ and $\langle p_1, p_2, q_2, q_1\rangle$, and then through the square $\langle p_2, q_2, p_4, q_4\rangle$ where $p_4=(1/2, 1/2, 1)$, $q_4=(1/2, 1, 1)$. We have completed the construction of L' in the layer $\{x\in X' \mid 1/2\leqq x_2\leqq 1\}$. The construction is continued in the layer $\{x\in X' \mid 1/2^2\leqq x_2\leqq 1/2\}$ by a similar construction in cubes with side length $1/2^2$. Observe that for example the 2-simplexes in $\langle p_0', p_3', p_2\rangle$ (Figure 2) are obtained from the 2-simplexes in $\langle q_0, q_3, q_2\rangle$ (Figure 3) by a similarity mapping. This gives a canonical way to continue the construction of L' to all layers $\{x\in X' \mid 1/2^k\leqq x_2\leqq 1/2^{k-1}\}$.

Let γ' be the set of 3-simplexes of the complex L'. Now we set

$$A^* = \{x\in R^4 \mid (x_1, x_2, x_3)\in A,\ -x_2 < x_4 < x_2\},\quad A\subset X'.$$

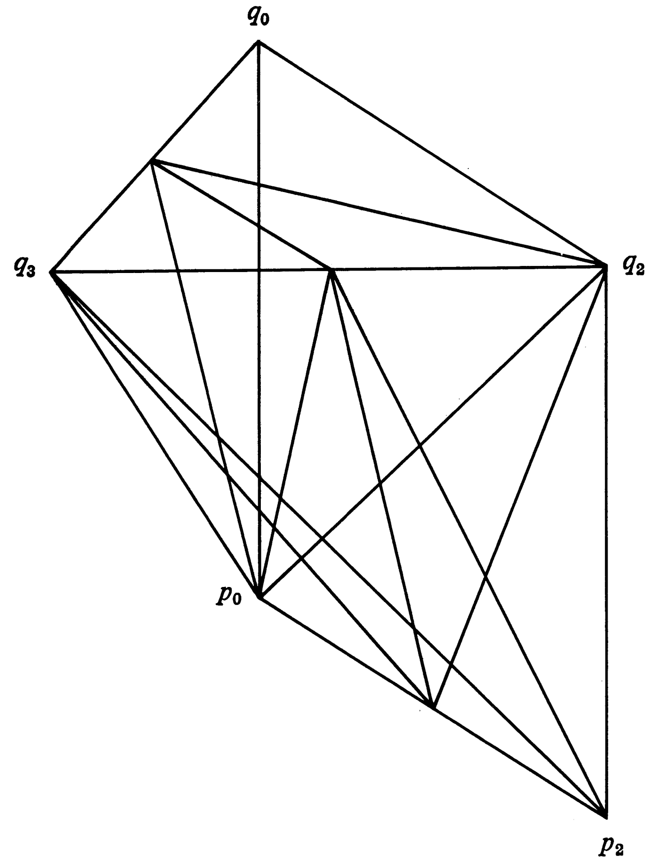

Figure 3

We fix $C_0 \in \gamma'$ and an affine map h_{C_0} of C_0 onto the simplex $W_1' = \langle -e_1, e_1, e_2, e_3 \rangle \subset R^3$. It follows from the construction of L' that there exists a unique set of affine homeomorphisms $h_C \colon C \to W_1'$ such that $h_C | C \cap D = h_D | C \cap D$ for $C, D \in \gamma'$.

Let W' be the closed octahedron $\langle -e_1, e_1, -e_2, e_2, -e_3, e_3 \rangle \subset R^3$, let $w' \colon W' \to \bar{B}^3$ be the radial stretching, and let $\psi_1' = w' \times \mathrm{id}_I \colon W' \times I \to \bar{B}^3 \times I$ where $I = [-1, 1]$. The maps φ_1, F_0 and $\bar{F}$ are defined as in the proof of 1.1. We set $F' = \varphi_1 \circ \psi_1'$ and define for $C \in \gamma'$ $g_C \colon \bar{C}^* \to W_1' \times I$ by

$$g_C(x) = \big(h_C(x_1, x_2, x_3)_1,\ h_C(x_1, x_2, x_3)_2,\ h_C(x_1, x_2, x_3)_3,\ x_4/x_2\big).$$

As in the proof of 1.1 let $\delta \in]0, 1/2[$ and let (c^j) be any sequence in $\bar{R}^4 \setminus B^4(-e_1, \delta)$. Let us only consider the first step and give the definition of a map $f_1 \colon \mathrm{int}\, X'^* \to \bar{R}^4 \setminus B^4(-e_1, r_1)$, $r_1 > 0$. The rest is similar to the end of the proof of 1.1. The sets Y_k are defined as before, so are the Möbius transformations S_k and the quasiconformal mappings ω_k.

In each C^*, $C\in\gamma'$, we shall now define a map τ_C. We need sliding maps σ_i: $\bar{H}_{2,3,4}\to\bar{H}_{2,3,4}$, $i=1, 2, 3$, defined as follows:

$$\sigma_1(x) = (x_1-x_2, x_2, x_3, x_4) \quad \text{if} \quad 0 \leqq x_2 < 2,$$

$$\sigma_1(x) = (x_1-2, x_2, x_3, x_4) \quad \text{if} \quad x_2 \geqq 2,$$

$$\sigma_2(x) = (x_1-x_3, x_2, x_3, x_4) \quad \text{if} \quad 0 \leqq x_3 < 2,$$

$$\sigma_2(x) = (x_1-2, x_2, x_3, x_4) \quad \text{if} \quad x_3 \geqq 2,$$

$$\sigma_3(x) = (x_1-x_2-x_3, x_2, x_3, x_4) \quad \text{if} \quad 0 \leqq x_2+x_3 < 2,$$

$$\sigma_3(x) = (x_1-2, x_2, x_3, x_4) \quad \text{if} \quad x_2+x_3 \geqq 2.$$

For a vertex u in the 3-complex L' we set as before $P_u=\{h_D^{-1}(e_1)|u\in D,\ D\in\gamma'\}$. Fix $C\in\gamma'$ and denote $\xi=h_C^{-1}(-e_1)$, $\eta=h_C^{-1}(e_1)$, $a=h_C^{-1}(e_2)$, $b=h_C^{-1}(e_3)$, and

$$Q_C = \cup\{P_u|u\in D,\ \xi\in D,\ D\in\gamma'\}.$$

Let π: $R^3\to R^2$ be the orthogonal projection. For a sufficiently large β_0, independent of C, there exists a largest k such that $\pi Q_C\subset Y_k\cup Y_{k+1}$.

The possible cases are as follows:

(1) $\pi P_a, \pi P_b \subset Y_k$,

(2) $\pi P_a\cap Y_{k+1} \neq \emptyset, \quad \pi P_b \subset Y_k$,

(3) $\pi P_a \subset Y_k, \quad \pi P_b\cap Y_{k+1} \neq \emptyset$,

(4) $\pi(\eta)\in Y_k, \quad \pi P_a\cap Y_{k+1} \neq \emptyset, \quad \pi P_b\cap Y_{k+1} \neq \emptyset$

(5) $\pi P_a\cap Y_k \neq \emptyset, \quad \pi P_b \subset Y_{k+1}$,

(6) $\pi P_a \subset Y_{k+1}, \quad \pi P_b\cap Y_k \neq \emptyset$,

(7) $\pi(\eta)\in Y_{k+1}, \quad \pi P_a\cap Y_k \neq \emptyset, \quad \pi P_b\cap Y_k \neq \emptyset$,

(8) $\pi P_a, \pi P_b \subset Y_{k+1}$.

We define τ_C in these cases to be the following maps:

(1) $S_k\circ\bar{F}\circ F'\circ g_C$,

(2) $S_k\circ\bar{F}\circ\sigma_1\circ F'\circ g_C$,

(3) $S_k\circ\bar{F}\circ\sigma_2\circ F'\circ g_C$,

(4) $S_k\circ\bar{F}\circ\sigma_3\circ F'\circ g_C$,

(5) $\omega_k\circ\bar{F}\circ\sigma_1\circ F'\circ g_C$,

(6) $\omega_k\circ\bar{F}\circ\sigma_2\circ F'\circ g_C$,

(7) $\omega_k\circ\bar{F}\circ\sigma_3\circ F'\circ g_C$,

(8) $\omega_k\circ\bar{F}\circ F'\circ g_C$.

Let us now check that $\tau_C|C^*\cap D^*=\tau_D|C^*\cap D^*$ for all $D\in\gamma'$. We may suppose that $C\cap D$ is a 2-simplex. Let p be the largest integer such that $\pi Q_D\subset Y_p\cup Y_{p+1}$.

Suppose first $C\cap D=\langle\xi,\eta,a\rangle$. Then $p=k$. The maps τ_C and τ_D may differ only in the appearance of the maps σ_i. We observe that $F'\circ g_C(C^*\cap D^*)\subset\partial H_3$. Since $h_D^{-1}(e_2)=h_C^{-1}(e_2)=a$, all possible maps σ_i coincide on ∂H_3. In fact, if the formula for τ_C differs from that of τ_D, then the possible pairings are the following:

(1) for C and (3) for D,

(2) for C and (4) for D,

(5) for C and (7) for D,

(6) for C and (8) for D,

plus the interchange of C and D. The case $C\cap D=\langle\xi,\eta,b\rangle$ is similar.

Suppose next that $C\cap D=\langle\xi,a,b\rangle$. Also here $p=k$. If the defining formulas for τ_C and τ_D are different, the only pairing which can occur is (4) for C and (7) for D plus the interchange of C and D. But $\bar{F}\circ\sigma_3\circ F'\circ g_C(C^*\cap D^*)\subset B^4(-e_1,r_0)$, and we use the fact $\omega_k|B^4(-e_1,r_0)=S_k|B^4(-e_1,r_0)$ to conclude that τ_C and τ_D coincide on the common part $C^*\cap D^*$.

Suppose finally that $C\cap D=\langle\eta,a,b\rangle$. If $p=k$, the cases (1)—(8) coincide for C and D. By symmetry we may assume $p=k-1$. For a sufficiently large β_0, independent of C, we must have $\pi Q_C\subset Y_k$, and we have case (1) for C and (8) for D. These fit together on $C^*\cap D^*$.

We have shown that there exists a map f_1 of X'^* such that $f_1|C^*=\tau_C$ for all $C\in\gamma'$, $f_1|\text{int}\,X'^*$ is K-quasimeromorphic K not depending on c^1, and f_1 omits a ball $B^4(-e_1,r_1)$. For $A\subset X=[0,1]\times]0,1]$ we denote now

$$\tilde{A}=\{x\in X'|(x_1,x_2)\in A\}.$$

We observe that $f_1|\tilde{Y}_0^*$ is independent of c^1 and

$$\lim_{x\to x_0,\,x\in\tilde{E}_1^*}f_1(x)=c^1,$$

where $x_0=e_3/2$. E_j is as before a set $\{x\in X|x_2=x_1^{\lambda_j}\}$. This completes the first step of the proof. The construction is completed as in the proof of 1.1 to obtain a quasimeromorphic mapping f_0': $\text{int}\,X'^*\to\bar{R}^4\setminus B^4(-e_1,r_1)$ such that

$$\lim_{x\to x_0,\,x\in\tilde{E}_j^*}f_0'(x)=c^j.$$

The required mapping is $f=h\circ f_0'\circ g$ where g: $B^4\to\text{int}\,X'^*$ and h: $\bar{R}^4\to\bar{R}^4$ are suitable quasiconformal mappings. The 2-manifold U_i is the preimage by g of a part of the 2-manifold $\{x\in\tilde{E}_j^*|x_4=0\}$. The theorem is proved.

3. Existence of angular limits

To prove Theorem 1.3 we need a two constant theorem for quasiregular mappings. If $f\colon G\to R^n$ is quasiregular, then the function $u(x)=\log|f(x)|$ is a solution of a quasilinear elliptic partial differential equation [5]. By using estimates for such solutions due to Maz'ja [4] we proved in [7] the following result which is a local two constant theorem.

3.1. Theorem [7, 4.22]. *Let $f\colon G\to R^n$ be a quasiregular mapping of a domain G in R^n, let $0<m<M$, let U be a ball $B^n(z,\varrho)$, and suppose*

(i) $|f(x)|\leqq M$ *if* $x\in G\cap U$,

(ii) $\limsup_{x\to y}|f(x)|\leqq m$ *if* $y\in\partial G\cap U$.

Then

$$|f(x)|\leqq\exp(\beta\log m+(1-\beta)\log M)\quad \textit{if}\quad x\in G\cap B^n(z,\gamma\varrho)$$

where $0<\gamma\leqq\gamma_0$,

$$\beta=C\operatorname{cap}(U,\bar{B}^n(z,\gamma\varrho)\setminus G)^{1/(n-1)}\log(1/\gamma),$$

and $\gamma_0<1/2$ *and C are positive constants depending only on n and the maximal dilatation $K(f)$ of f. The capacity in the expression of β is the n-capacity for condensers* [2].

3.2. *Proof of Theorem* 1.3. We shall use Theorem 3.1 in a simple iterative method. We first form a sequence $D_1, D_2, \dots$ of subdomains of B^n inductively by setting

$$D_1=B^n\cap\bigcup_{z\in A}B^n(z,\gamma d(z,\partial B^n\setminus A)),$$

$$D_{i+1}=B^n\cap\bigcup_{z\in D_i}B^n(z,\gamma d(z,\partial B^n\setminus A)),$$

where $\gamma=\gamma_0$ is the constant in 3.1 and d is the Euclidean metric. Let K be a cone $K(e_1,\varphi)$, $0<\varphi<\pi/2$. There exists a positive integer k and $s>0$ such that $K\cap B^n(e_1,s)\subset D_k$. Given $\varepsilon>0$ let $\delta>0$ be such that $\delta<s$ and

$$\limsup_{x\to y}|f(x)-\alpha|<\varepsilon\quad\text{if}\quad y\in A\cap B^n(e_1,\delta).$$

Then Theorem 3.1 applied to balls $B^n(z,d(z,\partial B^n\setminus A))$, $z\in A\cap B^n(e_1,\delta/2)$, and to the mapping $f-\alpha$ gives

$$|f(x)-\alpha|<e^{\beta\log\varepsilon}\quad\text{if}\quad x\in D_1\cap B^n(e_1,\delta/3)$$

where $\beta=C\varkappa^{1/(n-1)}\log(1/\gamma)$, C being the constant in 3.1 and $\varkappa$ the capacity of the Grötzsch ring $B^n\setminus\{te_n|0\leqq t\leqq\gamma\}$. By repeated use of this we obtain

$$|f(x)-\alpha|<e^{\beta^k\log\varepsilon}\quad\text{if}\quad x\in D_k\cap B^n(e_1,\delta/3^k).$$

Hence

$$\lim_{x\to e_1,\,x\in K}f(x)=\alpha.$$

3.3. Remark. By the method in the proof of Theorem 1.3 it is also possible to prove a global two constant theorem for general domains.

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