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# PICARD SETS FOR MEROMORPHIC FUNCTIONS WITH A DEFICIENT VALUE

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## 1. Introduction

Let $F$ be a family of functions meromorphic in the complex plane $C$, and $S$ a subset of $C$. We call $S$ a Picard set for $F$ if every transcendental $f \in F$ assumes every complex value with at most two exceptions infinitely often in $C-S$. We use the usual notation of the Nevanlinna theory, the Nevanlinna deficiency is defined by

$$\delta(a,f) = \liminf_{r\to\infty} \frac{m(r,a,f)}{T(r,f)}$$

and the Valiron deficiency by

$$\Delta(a,f) = \limsup_{r\to\infty} \frac{m(r,a,f)}{T(r,f)}.$$

If $F$ is the family of all functions meromorphic in the plane, the corresponding class of Picard sets is denoted by $P(M)$. Let $P(P)$ be the class of Picard sets for those meromorphic functions which have at least one Picard exceptional value. By means of a linear transformation, we see that $P(P)$ is the class of Picard sets for entire functions. The class of Picard sets for those meromorphic functions which have at least one Nevanlinna (resp. Valiron) deficient value is denoted by $P(N)$ (resp. $P(V)$). We see immediately that

$$P(M) \subset P(V) \subset P(N) \subset P(P).$$

In this paper we shall consider the question, under which conditions a set $S$ belongs to the classes $P(N)$ or $P(V)$. First we shall consider countable sets and then the case when $S$ is a countable union of open discs.

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## 2. Countable sets of the class $P(N)$

We shall prove

Theorem 1. *Let* $E=\{a_n\}$ *be a countable set whose points converge to infinity. If there exists* $\varepsilon>0$ *such that*

(A) $$\left\{z\colon 0<|z-a_n|<\frac{\varepsilon|a_n|}{\log|a_n|}\right\}\cap E=\emptyset$$

*for all large* $n$, *then* $E\in P(N)$.

This theorem is best possible in the sense that, corresponding to each real-valued function $\varphi(r)$ with $\lim_{r\to\infty}\varphi(r)=\infty$, there exists $E=\{a_n\}$ satisfying

$$\left\{z\colon 0<|z-a_n|<\frac{|a_n|}{\varphi(|a_n|)\log|a_n|}\right\}\cap E=\emptyset$$

for all large $n$ such that $E\notin P(P)\supset P(N)$. The existence of such a set $E$ is proved in [11, pp. 7—8]. Since the condition (A) is the best possible one of this type for $P(P)$, too, there arises the question whether $P(P)=P(N)$. The answer to this question is negative.

Theorem 2. *There exists a countable set* $E=\{a_n\}$ *with* $\lim a_n=\infty$ *such that* $E\in P(P)-P(N)$.

The following theorem shows that the condition (A) is not optimal for linear sets.

Theorem 3. *Let* $E=\{a_n\}$ *be a sequence of points lying on the positive real axis and let* $a_n\to\infty$ *as* $n\to\infty$. *If there exists* $\varepsilon>0$ *such that*

(B) $$a_{n+1}>a_n\left(1+\frac{\varepsilon}{(\log a_n)^2}\right)$$

*for all large* $n$, *then* $E\in P(N)$.

The condition (B) here is optimal, even for $P(P)$, for it is proved in [12] that if $\varphi(r)$ is an increasing function such that $\varphi(r)\to\infty$ as $r\to\infty$, there exists a set $E=\{a_n\}$ with $\lim a_n=\infty$ lying on the positive real axis such that $E\notin P(P)$ and

$$a_{n+1}>a_n\left(1+\frac{1}{\varphi(a_n)(\log a_n)^2}\right)$$

for all large $n$.

## 3. Results for the class $P(V)$

Corresponding to Theorem 1, we shall prove the following result for $P(V)$.

Theorem 4. *If there exists* $\varepsilon>0$ *such that the set* $E=\{a_n\}$ *with* $\lim a_n=\infty$ *satisfies*

$$\text{(C)} \qquad \{z\colon 0<|z-a_n|<\varepsilon|a_n|\}\cap E=\emptyset$$

*for all large n, then* $E\in P(V)$.

This result is optimal, even for linear sets, for we shall prove

Theorem 5. *Given any increasing function* $\varphi(r)$ *such that* $\varphi(r)\to\infty$ *as* $r\to\infty$, *there exists a set* $E=\{a_n\}$ *lying on the positive real axis such that* $a_n\to\infty$ *as* $n\to\infty$, $E\notin P(V)$ *and*

$$\text{(D)} \qquad a_{n+1}>a_n\left(1+\frac{1}{\varphi(a_n)}\right)$$

*for all large n.*

If in Theorem 5 $\varphi(r)\to\infty$ sufficiently slowly as $r\to\infty$, then the corresponding set $E$ belongs to $P(N)$. Therefore $P(V)\neq P(N)$. Between these classes there is even a more essential difference. We denote by $U(a, r)$ the open disc $|z-a|<r$. Theorem 10 proves that there exists a denumerable collection of open discs $U(a_n, d_n)$ with $\lim|a_n|=\infty$ such that the union of these discs belongs to $P(N)$. The class $P(V)$ does not have this property. We prove

Theorem 6. *If* $U(a_n, d_n)$ *is any sequence of open discs such that* $\lim|a_n|=\infty$, *then the set*

$$\bigcup_{n=1}^{\infty} U(a_n, d_n)$$

*does not belong to* $P(V)$.

## 4. Comparison of $P(V)$ and $P(M)$

As in Theorems 1 and 4, it is proved in [10] that if the set $E=\{a_n\}$ satisfies

$$\text{(1)} \qquad |a_{n+1}|>\varepsilon|a_n|^2$$

for some $\varepsilon>0$ and for all large $n$, then $E\in P(M)$, and in [12] it is proved that if $\varphi(r)\to\infty$ as $r\to\infty$, there exists $E=\{a_n\}$ lying on the positive real axis such that $E\notin P(M)$ and

$$a_{n+1}>\frac{a_n^2}{\varphi(a_n)}$$

for all large $n$. We conclude that $P(V)\neq P(M)$. The conditions (C) and (1) are quite far from each other and therefore we try to characterize those functions which make the difference between $P(V)$ and $P(M)$ so large. We denote by $\Sigma$ the extended complex plane and prove

Theorem 7. *If $f$ is a transcendental meromorphic function such that the set $E=\{a_n\}=f^{-1}(\{0, 1, \infty\})$ satisfies the condition*

(E)
$$\lim_{n\to\infty} |a_{n+1}/a_n| = \infty,$$

*then*

$$\limsup_{r\to\infty} \big(\sup_{w\in\Sigma} n(r, w) - \inf_{w\in\Sigma} n(r, w)\big) \leqq 2,$$

*and for any two complex values $a$ and $b$,* $\limsup_{r\to\infty} |n(r, a)-n(r, b)| \leqq 1$.

Furthermore, we shall show that a meromorphic function may have at most three so thinly distributed values that $(E)$ is satisfied.

Theorem 8. *If $f$ is a transcendental meromorphic function in the plane and $w_4$ is different from $0, 1$ and $\infty$, then the set $E=\{a_n\}=f^{-1}(\{0, 1, w_4, \infty\})$ satisfies*

$$\liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < \infty.$$

In the other direction, we shall prove

Theorem 9. *For any $M>1$ there exists a transcendental meromorphic function $f$ such that the set*

(F)
$$E = \{a_n\} = f^{-1}(\{0, 1, M, \infty\})$$

*satisfies*

(G)
$$\liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = M.$$

## 5. Further results for the class $P(N)$

From the results of Anderson and Clunie [1] it follows that if $q>1$, the set $E=\{a_n\}$ satisfies

(a)
$$|a_{n+1}/a_n| \geqq q$$

for all $n$, and the radii $d_n$ are chosen such that

(b)
$$(\log |a_n|)^2 = o\left(\log \frac{1}{d_n}\right),$$

then the union of the discs $U(a_n, d_n)$ belongs to $P(N)$. It is proved in [15] that the condition (b) here can be replaced by

(b′)
$$\log \frac{1}{d_n} \geqq K(\log |a_n|)^2,$$

where $K>0$ depends only on $q$, and still $\cup U(a_n, d_n) \in P(N)$, and in the other direction, if $K$ in (b′) is taken too small,

(c) $$K = \frac{1}{2\log q},$$

there exist $U(a_n, d_n)$ satisfying (a) and (b′) such that $\cup U(a_n, d_n)$ does not belong to $P(P)$. We shall relieve (a) and prove

Theorem 10. *Let* $E=\{a_n\}$ *be a complex sequence such that* $\lim a_n=\infty$, $|a_n|>e$, *and*

(H) $$\left\{z\colon 0<|z-a_n|<\frac{|a_n|}{(\log|a_n|)^\alpha}\right\}\cap E=\emptyset$$

*for some* $\alpha, 0<\alpha<1$, *and for all* $n$. *If the radii* $d_n$ *are chosen by the equation*

(I) $$\log\frac{1}{d_n} = (\log|a_n|)^{2+\beta},$$

*where* $\beta>2\alpha$, *then the set*

$$S = \bigcup_{n=1}^{\infty} U(a_n, d_n)$$

*belongs to the class* $P(N)$.

Here $\beta$ cannot be smaller than $2\alpha$, for it is proved in [13] that if $\beta<2\alpha$, there exists $S=\cup U(a_n, d_n)$ satisfying (H) and (I), and not belonging to $P(P)$.

Theorem 3 follows as a special case from the following

Theorem 11. *Let* $E=\{a_n\}$ *lie on the positive real axis,* $e<a_1<a_2<\dots, a_n\to\infty$ *as* $n\to\infty, \varepsilon>0$, *and*

(J) $$a_{n+1} > a_n\left(1+\frac{\varepsilon}{(\log a_n)^\alpha}\right)$$

*for some* $\alpha, 0<\alpha\leqq 2$, *and for all* $n$. *If the radii* $d_n$ *are chosen by the equation*

(K) $$\log\frac{1}{d_n} = H(\log a_n)^{2+\alpha},$$

*where* $H=4800\,(1+\varepsilon^{-2})(100)^{2+\alpha}$, *then the union of the discs* $U(a_n, d_n)$ *belongs to* $P(N)$.

In the other direction, it is proved in [14] that if $\varepsilon=1/7$, $H=1/8$ and $0<\alpha\leqq 2$, then $E$ and $d_n$ satisfying (J) and (K) can be chosen such that the intersection of the positive real axis and the union of the discs $U(a_n, d_n)$ does not belong to $P(P)$.

## 6. Some results needed in the proofs

We denote $U(\infty, \delta)=\{z\colon |z|>1/\delta\}$. We shall need the following

Lemma 1. *There exist positive constants $M_1$ and $M_2$ depending only on $w_3$ such that if $f$ is meromorphic in an annulus $r<|z|<R$ and omits there three different values* 0, 1 *and $w_3$, then, if $R>M_1 r$, the image of $|z|=\sqrt{rR}$ under $f$ is contained in*

$$U\big(a, M_2(\log (R/r))^{-1/4}\big)$$

*for some finite or infinite complex $a$.*

*Proof.* Let $f$ be meromorphic and omit the values 0, 1 and $w_3$ in $r<|z|<R$, where $\log (R/r)>8\pi$. We denote $z_0=\sqrt{Rr}=\exp(\zeta_0)$. We choose $g$ to be one of the functions $1/f$ and $1/(f-1)$ such that $|g(z_0)|\leqq 2$. The function $g(e^\zeta)$ is regular in $U\big(\zeta_0, (1/2)\log (R/r)\big)$ and omits there two finite values. Therefore it follows from Schottky's theorem that there exists $M_3>0$ depending only on $w_3$ such that $|g(e^\zeta)|\leqq M_3$ in $|\zeta-\zeta_0|\leqq(1/4)\log (R/r)$. The function

$$h(\zeta)=\frac{g(e^\zeta)-g(e^{\zeta_0})}{\zeta-\zeta_0}$$

is regular in $U\big(\zeta_0, (1/4)\log (R/r)\big)$, and on the boundary of this disc $h$ satisfies

$$\text{(i)} \qquad |h(\zeta)|\leqq \frac{8M_3}{\log (R/r)}.$$

It follows from the maximum principle that (i) holds on the segment $\zeta=\zeta_0+i\varphi$, $-\pi\leqq\varphi\leqq\pi$, and we get

$$\text{(ii)} \qquad |g(z)-g(z_0)|\leqq \frac{8\pi M_3}{\log (R/r)}$$

on $|z|=\sqrt{rR}$. Lemma 1 follows from (ii) by an easy computation.

Lemma 2. *Let $f$ be meromorphic in the plane and*

$$E=f^{-1}(\{0, 1, w_3\}),$$

*where $w_3$ is different from* 0 *and* 1. *For any $M>0$, there exists a constant $K=K(M, w_3)$ such that if $|f(b)|\geqq 2M$ and $|f(\zeta)|\leqq M$, then the disc*

$$U(\zeta, K|b-\zeta|)$$

*contains at least two points of $E$.*

*Proof.* Let $M_1$ and $M_2$ be as in Lemma 1. We choose $K>M_1^2$ so large that if $a$ is any complex point, the set $U(a, 2d)$, where $d=M_2\big((1/2)\log K\big)^{-1/4}$, contains at most one of the values $f(b)$ and $f(\zeta)$, and at most one of the points 0, 1 and $w_3$. Let us suppose that $U(\zeta, K|b-\zeta|)$ contains at most one point of $E$. If $U\big(\zeta, |b-\zeta|\sqrt{K}\big)\cap E$

$=\emptyset$, we set $r=|b-\zeta|$, and otherwise we set $r=|b-\zeta|\sqrt{K}$. Then $f$ omits the values 0, 1 and $w_3$ in the annulus $r<|z-\zeta|<r\sqrt{K}$, and it follows from Lemma 1 that the image of the circle $\gamma$: $|z-\zeta|=K^{1/4}r$ is contained in $U(a, d)$ for some complex $a$. Let $D$ be the open disc bounded by $\gamma$. Since the image of the boundary of $D$ is contained in $U(a, d)$ and at least one of the values $f(b)$ and $f(\zeta)$ lies outside $U(a, d)$, $f$ takes in $D$ all values lying outside $U(a, d)$. This implies that $f$ takes in $D$ at least two of the values 0, 1 and $w_3$, and we see that $D\subset U(\zeta, K|\zeta-b|)$ contains at least two points of $E$. So we have proved that the assumption that $U(\zeta, K|b-\zeta|)$ contains at most one point of $E$, leads to a contradiction. Lemma 2 is proved.

Let $f$ be meromorphic in the plane and let $w_1$, $w_2$ and $w_3$ be three different complex values. Let $a_n$ be the sequence of the distinct roots of the equations $f(z)=w_1$, $f(z)=w_2$ and $f(z)=w_3$. We denote by $n(r)$ the number of the $a_n$ lying in $|z|\leqq r$, and

$$N(r) = \int_0^r \frac{n(t)-n(0)}{t}\,dt+n(0)\log r.$$

From Theorem 2.5 of Hayman [5, p. 47] we get the following

Lemma 3. *Let $f$ and $N(r)$ be as above. Then*

$$T(r,f) \leqq \big(1+o(1)\big)N(r)$$

*as $r\to\infty$ outside a set $B$ of finite linear measure.*

Lemma 4. *Let $f$ be transcendental and meromorphic in the plane such that $\delta(\infty, f)>0$ and*

$$T(r,f) = O\big((\log r)^M\big) \tag{2}$$

*for some finite $M$. If there exists $\alpha>0$ such that*

$$E = \{a_n\} = f^{-1}(\{0, 1\})$$

*satisfies*

$$\left\{z\colon 0<|z-a_n|<\frac{|a_n|}{(\log|a_n|)^\alpha}\right\}\cap E = \emptyset \tag{3}$$

*for all large $n$, then there exists a real increasing sequence $\sigma_n$ such that $\sigma_n\to\infty$ as $n\to\infty$,*

$$\min\{|f(z)|\colon |z| = \sigma_n\} = 1,$$

*$|f(z)|>1$ in $\sqrt{\sigma_n}<|z|<\sigma_n$ and*

$$\log|f(z)| \geqq \left(\frac{1}{3}+o(1)\right)\delta(\infty,f)T(|z|,f) \tag{4}$$

*for all $z$ lying in $\sqrt{\sigma_n}\leqq|z|\leqq\sigma_n/2$.*

*Proof.* It follows from (2) that we may choose $\beta, 3/2 \leqq \beta \leqq M$, such that

(i) $$T(r,f) = O\big((\log r)^{\beta}\big)$$

and

(ii) $$T(r,f) \neq O\big((\log r)^{\beta - 1/2}\big).$$

Since

$$n(r, a) \log r \leqq \int_r^{r^2} \frac{n(t, a)}{t}\, dt = N(r^2, a) - N(r, a),$$

we deduce from (i) that

(iii) $$n(r, a) = O\big((\log r)^{\beta - 1}\big)$$

for any complex $a$, and from (ii) it follows that there exists a real sequence $R_n$ with $\lim R_n = \infty$ such that

(iv) $$T(R_n, f) > (\log R_n)^{\beta - 1/2}$$

for all $n$.

Let $b$ be a complex value such that $|b| < 1$ and

(v) $$N(r, b) = \big(1 + o(1)\big) T(r, f).$$

Let $b_n$ be the sequence of the $b$-points of $f$ and

$$B = \bigcup_{|b_k| > e} U\big(b_k, |b_k| (\log |b_k|)^{-2(\alpha + \beta)}\big).$$

We denote $d(z) = \min \{|z - b_k|\colon k = 1, 2, \ldots\}$. Using the Poisson—Jensen formula, we get for all $z = re^{i\varphi}$,

$$\log |f(z) - b| \geqq \frac{1}{2\pi} \int_0^{2\pi} \log |f(2re^{i\theta}) - b| \frac{(2r)^2 - r^2}{(2r)^2 - 4r^2 \cos(\theta - \varphi) + r^2}\, d\theta$$

$$- \sum_{|b_k| \leqq 2r} \log \left| \frac{(2r)^2 - \bar{b}_k z}{2r(z - b_k)} \right|.$$

This implies, together with the fact that $m(t, b) = o\big(T(t, f)\big)$, that

(vi) $$\log |f(z)| \geqq \left(\frac{1}{3} + o(1)\right) \delta(\infty, f) T(2r, f) - n(2r, b) \log \frac{4r}{d(z)}.$$

Let $z = re^{i\varphi}$ lie in $R_n < |z| < R_n^3$ outside $B$. Then

$$d(z) \geqq r (2 \log r)^{-2(\alpha + \beta)},$$

and we see from (iii) and (iv) that

$$n(2r, b) \log \big(4r/d(z)\big) = O\big((\log R_n)^{\beta - 1} \log\log R_n\big) = o\big(T(R_n, f)\big).$$

Therefore it follows from (vi) that $f$ satisfies (4) in $R_n < |z| < R_n^3$ outside the set $B$. Let us suppose that there exists some $b_k$ lying in $2R_n < |z| < R_n^3/2$. The sum of the radii of those discs of the set $B$ which meet the annulus $|b_k|/2<|z|<2|b_k|$ is at most

$$2n(R_n^3, b)\,|b_k|\,(\log|b_k|)^{-2(\alpha+\beta)} = o\big(|b_k|\,(\log|b_k|)^{-2\alpha}\big),$$

and we see that there exists $d_k$,

(vii) $$0 < d_k < |b_k|\,(\log|b_k|)^{-2\alpha},$$

such that the circle $|z-b_k|=d_k$ does not meet $B$. It follows from (3) and (vii) that $f$ takes at most one of the values 0 and 1 in $U(b_k, d_k)$, and since (4) is true on the boundary of this disc, we conclude from the minimum principle that

$$\log|b| = \log|f(b_k)| \geqq \left(\frac{1}{3}+o(1)\right)\delta(\infty, f)\,T(|b_k|, f).$$

This is not possible if $|b_k|$ is large, and we deduce that

$$B\cap\{z\colon\ 4R_n < |z| < R_n^3/4\} = \emptyset$$

for all large $n$.

For large values of $n$, we may choose $\sigma_n>R_n^3/4$ such that $|f(z)|>1$ in $4R_n<|z|<\sigma_n$ and

$$\min\{|f(z)|\colon\ |z| = \sigma_n\} = 1.$$

Let $\zeta=re^{i\varphi}$ lie in $\sqrt{\sigma_n}\leqq|z|\leqq\sigma_n/2$. Since $f$ has no $b$-points in $4R_n<|z|\leqq\sigma_n$, we conclude that $n(2r, b)=n(4R_n, b)$, $d(\zeta)\geqq r/2$ and

$$n(2r, b)\log\big(4r/d(\zeta)\big) = O\big((\log R_n)^{\beta-1}\log 8\big) = o\big(T(R_n, f)\big).$$

Now we see from (vi) that $f$ satisfies (4) in $\sqrt{\sigma_n}\leqq|z|\leqq\sigma_n/2$, and Lemma 4 is proved.

Following Hayman [6], we shall call an $\varepsilon$-set any countable set of circles not containing the origin, and subtending angles at the origin whose sum is finite. Hayman [6] has proved the following

Theorem A. *If an integral function $f$ satisfies* $\log M(r, f)=O\big((\log r)^2\big)$, *then*

$$\log|f(z)| = \big(1+o(1)\big)\log M(r, f)$$

*as* $z=re^{i\varphi}\to\infty$ *outside an $\varepsilon$-set.*

Valiron [16] has proved the following

Theorem B. *If a meromorphic function $f$ satisfies* $T(r, f)=O\big((\log r)^2\big)$, *then*

$$T(r, f) = \big(1+o(1)\big)\max\{N(r, a),\ N(r, b)\}$$

*for any two complex values $a$ and $b$.*

## 7. Proof of Theorem 1

Contrary to the assertion of Theorem 1, let us suppose that there exists a transcendental meromorphic function $f$ with a Nevanlinna deficient value $w$ such that the set

$$E = \{a_n\} = f^{-1}(\{w_1, w_2, w_3\})$$

satisfies (A) for some $\varepsilon > 0$ and for some choice of the three different values $w_1$, $w_2$ and $w_3$. We may suppose, without loss of generality, that $w = \infty$, $w_1 = 0$ and $w_2 = 1$, $w_3$ being an infinite or finite complex value, different from 0 and 1.

Let $n(r)$ be the counting function of $E$. It follows from (A) that

$$n(e^s) - n(e^{s-1}) = O(s^2),$$

and we conclude that

$$n(e^s) = O(1) + \sum_{k=1}^{s} (n(e^k) - n(e^{k-1})) = O\left(\sum_{k=1}^{s} k^2\right) = O(s^3).$$

This implies that $n(r) = O((\log r)^3)$, and therefore the integrated counting function of $E$ satisfies $N(r) = O((\log r)^4)$. It follows from Lemma 3 that $T(r, f) \leqq (1 + o(1))N(2r)$ for all large values of $r$, and we deduce that $f$ satisfies $T(r, f) = O((\log r)^4)$. This implies that we may apply Lemma 4.

Let the sequence $\sigma_n$ be as in Lemma 4. We choose $b_n$ lying on the circle $|z| = \sigma_n$ such that $|f(b_n)| = 1$. Since $f$ is transcendental, we conclude that

$$\liminf_{r \to \infty} \frac{T(r, f)}{\log r} = \infty.$$

Therefore we deduce from (4) that there exists a sequence $K_n$ with $\lim K_n = \infty$ such that

(i) $$\log |f(z)| \geqq K_n^2 \log \sigma_n$$

on $|z| = \sigma_n/e$. The function $\omega(z) = \log(\sigma_n/|z|)$ is harmonic in the annulus $\sigma_n/e \leqq |z| \leqq \sigma_n$, and on the boundary of this annulus we have

(ii) $$\log |f(z)| \geqq K_n^2 \omega(z) \log \sigma_n .$$

Since $\log |f(z)|$ is superharmonic in $\sigma_n/e \leqq |z| \leqq \sigma_n$, it follows from the minimum principle that (ii) holds in this annulus. We set

$$z_n = b_n \left(1 - \frac{1}{K_n \log \sigma_n}\right).$$

Then it follows from (ii) that $\log |f(z_n)| \geqq K_n$, and we see from Lemma 2 that the disc

$$C_n = U\left(b_n, \frac{K|b_n|}{K_n \log |b_n|}\right)$$

contains at least two points of $E$. However, since $\lim K_n = \infty$, it follows from (A) that if $n$ is large, then $C_n$ contains at most one point of $E$. We are led to a contradiction and Theorem 1 is proved.

## 8. Proof of Theorem 2

Let $r_1 = e^{10}$ and $r_{n-1} = \log\log\log r_n$ for $n \geqq 2$. We set

$$f(z) = z \prod_{n=1}^{\infty} \left(\left(1 - \frac{z}{r_n}\right)^2 \left(1 - \frac{z}{r_n - \sqrt{r_n}}\right)^{-1}\right).$$

Then $n(r, \infty, f) = (1/2 + o(1))\, n(r, 0, f)$, and we see that $\delta(\infty, f) \geqq 1/2$. This implies that the set $E = \{a_n\} = f^{-1}(\{0, 1, \infty\})$ does not belong to the class $P(N)$. We assume that the sequence $a_n$ is arranged in the order of increasing moduli. We see by an easy computation that if $k$ is large, then $a_{4k-1} = r_k - \sqrt{r_k}$ and $a_{4k+p} \in U(r_k, r_k^{-2})$ for $p = 0, 1, 2$. Let $f(\zeta) = 1$ and $\zeta \in U(r_k, r_k^{-2})$. Then

$$(-1)^k + o(1) = r_k^{k-2} \sqrt{r_k}\, (\zeta - r_k)^2 \prod_{t=1}^{k-1} \left(r_t^{-2}(r_t - \sqrt{r_t})\right),$$

and we conclude that

$$\text{(i)} \qquad \log |\zeta - z|^{-1} = \frac{1}{2}\left(k - 2 + \frac{1}{2}\right) \log r_k + o(\log\log r_k)$$

for any choice $\zeta \neq z$, $\{\zeta, z\} \subset \{a_{4k}, a_{4k+1}, a_{4k+2}\}$.

Let us suppose now that $E \notin P(P)$. Then there exists a trancendental entire function $g$ such that

$$E(g) = g^{-1}(\{0, 1\}) \subset E \cup U(0, r_0)$$

for some $r_0 > 0$. Since $M(r, g) \to \infty$ as $r \to \infty$, we may conclude from Schottky's theorem that $|g(z)| \geqq 4$ on the circles $\gamma_k$: $|z| = r_k/2$ and $\Gamma_k$: $|z| = 2r_k$ for all large $k$. We denote by $D_k$ the annulus which is bounded by $\gamma_k$ and $\Gamma_k$.

Let us suppose that $a_n \in D_k$ is a multiple root of the equation $g(z) = 0$ with multiplicity $m \geqq 4$. Since $|g(z)| \geqq 4$ on the boundary of $D_k$ and $D_k$ contains only four points of $E$, there exists a region $G \subset C$ such that the image of the boundary of $G$ is contained in the segment $w = u + iv$: $0 \leqq u \leqq 1$, $v = 0$. This implies together with the maximum principle that $\operatorname{Im} g(z) \equiv 0$ on $G$, and therefore $g(z) \equiv$ constant on $G$. This is a contradiction, and we conclude that the equation $g(z) = 0$ may have only a finite number of roots with multiplicity $m \geqq 4$. It follows from Lemma 3 that $T(r, g)=O((\log r)^2)$, and we may write

(ii) $$g(z) = P(z) \prod_{n=1}^{\infty} (1-z/a_n)^{s_n},$$

where $P$ is a polynomial and $s_n \in \{0, 1, 2, 3\}$ for any $n$.

If $k$ is large and $z$ is a boundary point of the disc $U(r_k-\sqrt{r_k}, (1/2)\sqrt{r_k})$, then

$$\log|g(z)| \geqq 100 \log|z| + (n(2r_k, 0, g) - n(r_k/2, 0, g)) \log(8r_k^{-1/2}),$$

and since $n(2r_k, 0, g)-n(r_k/2, 0, g) \leqq 12$, we conclude that $|g(z)| \geqq 4$. Since $g$ omits at least one of the values 0 and 1 in $U(r_k-\sqrt{r_k}, (1/2)\sqrt{r_k})$ and $|g(z)| \geqq 4$ on the boundary of this disc, it follows from the minimum principle that $|g(z)| \geqq 2$ in this disc. This implies that $r_k-\sqrt{r_k} \notin E(g)$, and therefore $D_k$ contains at most three points of $E(g)$. As before, we see now that if $n(2r_k, 0, g)-n(r_k/2, 0, g) \geqq 3$, then there exists a region $G$ contained in the open disc bounded by $\Gamma_k$ such that the image of the boundary of $G$ is contained in the real axis. However, this is impossible, and we conclude that $n(2r_k, 0, g)-n(r_k/2, 0, g) \leqq 2$ for all large $k$. We denote by $p_k$ the number of the roots of the equation $g(z)=0$ in $D_k$ when the multiple roots are counted according to multiplicity. Then $p_k \leqq 2$ for all large $k$, and it follows from Rouche's that the equation $g(z)=1$ has $p_k$ roots in $D_k$, too.

Let $k$ be large and $p_k>0$. If $p_k=2$, then one of the functions $g$ and $1-g$ has a double zero at one of the points $a_{4k}$, $a_{4k+1}$ and $a_{4k+2}$, and takes the value 1 at the two remaining points. We may suppose that in this case $g$ has this property. In both cases, $p_k=2$ or $p_k=1$, we denote by $\zeta$ the zero of $g$ lying in $D_k$ and let $z \in D_k$ be such a point that $g(z)=1$. Then $\{\zeta, z\} \subset \{a_{4k}, a_{4k+1}, a_{4k+2}\}$. It follows from (ii) and the choice of the sequence $r_n$ that there exists a positive integer $m(k)$ such that

$$0 = \log|g(z)| = m(k) \log r_k + O(\log r_{k-1}) + p_k \log\left|\frac{\zeta-z}{r_k}\right|.$$

This implies that

$$\log|\zeta-z|^{-1} = \frac{1}{p_k}(m(k)-p_k) \log r_k + o(\log\log r_k),$$

and comparing this with (i) we get $(1/4)\log r_k = o(\log\log r_k)$. This is impossible for large values of $k$, and therefore $g$ has only a finite number of zeros. This implies, together with the facts that $g$ is entire and has order zero, that $g$ is a polynomial. We are led to a contradiction, and therefore $E \in P(P)$. This completes the proof of Theorem 2.

## 9. Proof of Theorem 4

Contrary to the assertion of Theorem 4, let us suppose that there exists a meromorphic transcendental function $f$ with $\Delta(\infty, f)>0$ such that

$$E = \{a_n\} = f^{-1}(\{0, 1, w_3\})$$

satisfies (C) for some $\varepsilon>0$, $w_3$ being different from 0 and 1. It follows from (C) that the integrated counting function of $E$ satisfies $N(r)=O((\log r)^2)$, and we conclude from Lemma 3 that $f$ satisfies $T(r, f)=O((\log r)^2)$. Therefore we may write $f(z)=f_1(z)/f_2(z)$, where $f_1$ and $f_1$ are entire functions with no zeros in common and both of them satisfying $T(r, f_k)=O((\log r)^2)$. It follows from Theorem B that

$$N(r,0,f) = N(r,0,f_1) = (1+o(1))T(r,f_1)$$

and

$$N(r,\infty,f) = N(r,0,f_2) = (1+o(1))T(r,f_2),$$

and from Theorem A it follows that

$$T(r,f_k) = (1+o(1))\log M(r,f_k).$$

Now we deduce from Theorem A that

$$\text{(i)}\qquad \begin{aligned}\log|f(z)| &= \log|f_1(z)|-\log|f_2(z)|\\ &= \log M(r,f_1)-\log M(r,f_2)+o(T(r,f))\\ &= N(r,0,f)-N(r,\infty,f)+o(T(r,f))\end{aligned}$$

outside an $\varepsilon$-set.

We choose a sequence $R_n$ with $\lim R_n=\infty$ such that

$$\text{(ii)}\qquad N(R_n,\infty,f) < \left(1-\frac{1}{2}\Delta(\infty,f)\right)T(R_n,f)$$

for all $n$. It follows from Theorem B that

$$\text{(iii)}\qquad N(R_n,0,f) = (1+o(1))T(R_n,f).$$

For large values of $n$, we may choose $r_n$ such that $R_n/2<r_n\leq R_n$ and that the circle $|z|=r_n$ lies outside the $\varepsilon$-set which is the exceptional set for the formula (i). Since $T(r, f)=O((\log r)^2)$, we conclude that $n(r, 0, f)=O(\log r)$ and

$$\begin{aligned}N(r_n,0,f) &= N(R_n,0,f)-\int_{r_n}^{R_n}\frac{n(t,0,f)}{t}\,dt\\ &= N(R_n,0,f)+O(\log R_n).\end{aligned}$$

This implies together with (iii) that

$$N(r_n,0,f) = (1+o(1))T(R_n,0,f),$$

and since $N(r_n, \infty, f) \leqq N(R_n, \infty, f)$, we see from (i) and (ii) that

$$\text{(iv)} \qquad \log|f(z)| \geqq \left(\frac{1}{2}+o(1)\right) \Delta(\infty, f) T(R_n, f)$$

on $|z|=r_n$. Since $f$ is non-rational, it follows from (iv) that there exists a sequence $K_n$ with $\lim K_n = \infty$ such that

$$\text{(v)} \qquad \log|f(z)| \geqq K_n^2 \log r_n$$

on $|z|=r_n$.

We choose $\varrho_n < r_n$ such that $|f(z)|>1$ in $\varrho_n<|z|\leqq r_n$ and that there exists a point $\zeta_n$ lying on $|z|=\varrho_n$ such that $|f(\zeta_n)|=1$. On the boundary of the annulus $H_n$: $\varrho_n<|z|<r_n$ we have

$$\text{(vi)} \qquad \log|f(z)| \geqq K_n^2 \log r_n \frac{\log(|z|/\varrho_n)}{\log(r_n/\varrho_n)},$$

and from the superharmonicity of $\log|f(z)|$ we conclude that (vi) holds in $H_n$. Let $t$ be defined by the equation

$$\frac{\log(t/\varrho_n)}{\log(r_n/\varrho_n)} = \frac{1}{K_n \log r_n},$$

and let $z_n$ be the point on $|z|=t$ which satisfies $\arg z_n = \arg \zeta_n$. Then it follows from (vi) that $\log|f(z_n)| \geqq K_n$, and for large values of $n$ we get

$$|z_n - \zeta_n| \leqq \frac{2|\zeta_n|}{K_n}.$$

Applying Lemma 2, we deduce that

$$U\left(\zeta_n, \frac{2K|\zeta_n|}{K_n}\right)$$

contains at least two points of $E$ for all large $n$. This is a contradiction with (C), and Theorem 4 is proved.

## 10. Proof of Theorem 5

Let $\varphi(r)$ be an increasing function such that $\varphi(r)\to\infty$ as $r\to\infty$. We denote by $(a, b)$ the open segment $a<x<b$ on the positive real axis. We set

$$f_n(z) = \prod_{p=1}^{n} \left(1 - \frac{z}{1+(p/n)}\right)^2$$

and let $x_p$, $p=1, 2, \ldots, 2n-1$, be the roots of the equation $f_n'(z)=0$ arranged such that $x_{p+1}>x_p$. We see easily that

$$x_1 = 1+\frac{1}{n} < x_2 < x_3 = 1+\frac{2}{n} < x_4 < \ldots < x_{2n-2} < x_{2n-1} = 2.$$

We denote

$$d_n = \min\{f_n(x_{2p}):\ p = 1, 2, \ldots, n-1\}.$$

Then $d_n > 0$, and we note that if $0 < b \leqq d_n/4$ and the points $y_s, s=1, 2, \ldots, 2n$, are chosen such that $b/2 < f(y_s) < 2b$, $y_1 \in (0, x_1)$, $y_s \in (x_{s-1}, x_s)$ for $s=2, 3, \ldots, 2n-1$, and $y_{2n} \in (2, 3)$, then

(i) $$\min\{|y_s - x_p|:\ p = 1, 2, \ldots, 2n-1,\ s = 1, 2, \ldots, 2n\} \geqq \alpha(n, b)$$

for some $\alpha(n, b) > 0$ depending only on $n$ and $b$.

We set $t_k = k!$ and the sequences $r_k$ and $\varrho_k$ of positive real numbers are chosen such that

(ii) $$t_k \leqq \log\log\log r_k$$

and

(iii) $$r_k < \log \varrho_k < \log\log\log r_{k+1}$$

for every $k$. We set

$$f(z) = \prod_{k=1}^{\infty} \left(f_{t_k}(z/r_k)(1 - z/\varrho_k)^{-2t_k}\right).$$

We may write

$$f(z) = \frac{\prod_{s=1}^{\infty}(1 - z/z_s)}{\prod_{s=1}^{\infty}(1 - z/b_s)},$$

where the sequences $z_s$ and $b_s$ are increasing.

We denote $n_k = n(r_k/2, 0, f)$. Then $n(r_k/2, \infty, f) = n_k$, and we may assume that the sequences $r_s$ and $\varrho_s$ are chosen such that if $\sqrt{r_k} < |z| < r_k^2$ and $k \geqq 2$, then

(iv) $$f(z) = (1 + o(1)) f_{t_k}(z/r_k) A_k,$$

where

(v) $$A_k = \prod_{s=1}^{n_k} \frac{b_s}{z_s} > \frac{4}{d_{t_k}}$$

and $o(1)$ satisfies $|o(1)| < 1/100$ in $\sqrt{r_k} < |z| < r_k^2$.

Let $x_p$, $x_1 < x_2 < \ldots < x_{2t_k - 1}$, be the zeros of $f'_{t_k}$. From (iv) and (v) we get $f(r_k/2) > 2$, $f(x_{2p-1} r_k) = 0$ for $p = 1, 2, \ldots, t_k$, $f(x_{2p} r_k) > 2$ for $p = 1, 2, \ldots, t_k - 1$, and $f(3r_k) > 2$. Therefore $f$ has real 1-points $\xi_k$, $k = 1, \ldots, 2t_k$, such that $\xi_1 \in (r_k/2, x_1 r_k)$, $\xi_p \in (x_{p-1} r_k, x_p r_k)$ for $p = 1, 2, \ldots, 2t_k - 1$, and $\xi_{2t_k} \in (2r_k, 3r_k)$. It follows from (iv) that the points $y_p = \xi_p / r_k$ satisfy

$$\frac{1}{2A_k} < f_{t_k}(y_p) < \frac{2}{A_k},$$

and we conclude from (i) and (v) that

(vi) $$|x_p - y_s| \geqq \alpha(t_k, 1/A_k)$$

for all $p$ and $s$. Since $|f(z)| \geqq 5$ on the circles $|z| = \sqrt{r_k}$ and $|z| = \sqrt{r_{k+1}}$, it follows from Rouche's theorem that $f$ has exactly $2t_k$ 1-points in $\sqrt{r_k} \leqq |z| \leqq \sqrt{r_{k+1}}$, and we deduce that the only 1-points of $f$ lying in $\sqrt{r_k} \leqq |z| \leqq \sqrt{r_{k+1}}$ are the points $\xi_p$, $p=1, 2, \ldots, 2t_k$.

Let $E=\{a_n\}$ be the set of the zeros, 1-points and poles of $f$. Then $E$ lies on the positive real axis. We assume that $E$ is arranged such that $0=a_1<a_2<a_3<\ldots$. It follows from (vi) that those points $a_n$ which lie in $\sqrt{r_k} \leqq |z| \leqq \sqrt{r_{k+1}}$ satisfy

$$a_{n+1} > a_n \left(1+\frac{1}{4}\alpha(t_k, 1/A_k)\right).$$

Since the value of $A_k$ does not depend on the choice of $r_k$, we may assume that $r_k$ is chosen so large that

$$\alpha(t_k, 1/A_k) \geqq \left(\varphi(\sqrt{r_k})\right)^{-1/2}$$

Then $E$ satisfies (D) for all large values of $n$.

If $n$ is large, then

$$N(r_k^3, 0, f) \geqq t_k \log r_k$$

and

$$N(r_k^3, \infty, f) \leqq \left(6+o(1)\right) t_{k-1} \log r_k.$$

Since $t_{k-1}=t_k/k=o(t_k)$, we deduce now that $\Delta(\infty, f)=1$. This implies that the set $E$ does not belong to the class $P(V)$, and Theorem 5 is proved.

## 11. Proof of Theorem 6

Let $U(a_n, d_n)$ be as in Theorem 6. Taking a subset of the union of the discs $U(a_n, d_n)$, if necessary, we may assume that $|a_1|>100$, $|a_n^2|<|a_{n+1}|$ and $0<d_n<1$ for all $n$. We set

$$f(z) = \prod_{n=1}^{\infty} \left(1-\left(\frac{d_n}{2(z-a_n)}\right)^{t_n}\right),$$

where the sequence $t_n$ grows at least so rapidly that $|f(z)|<2$ outside the union of the discs $U(a_n, d_n)$. Furthermore, we assume that

(i) $$2nt_{n-1}\log|a_n| < \frac{1}{8} t_n \log\frac{|a_n|}{|a_n|-d_n/8}$$

for $n \geqq 2$. We have

$$\begin{aligned} N(|a_n|, 0, f) &\geqq N(|a_n|, 0, f) - N(|a_n|-d_n/8, 0, f) \\ &\geqq n(|a_n|-d_n/8, 0, f) \log\frac{|a_n|}{|a_n|-d_n/8} \\ &\geqq \frac{1}{8} t_n \log\frac{|a_n|}{|a_n|-d_n/8}, \end{aligned}$$

and

$$N(|a_n|, \infty, f) \leqq n(|a_n|-1, \infty, f) \log |a_n| \leqq 2t_{n-1} \log |a_n|,$$

and these estimates imply together with (i) that

$$N(|a_n|, \infty, f) \leqq \frac{1}{n} N(|a_n|, 0, f)$$

for $n \geqq 2$. Therefore $\Delta(\infty, f)=1$, and since $f$ is bounded in the complement of the union of the discs $U(a_n, d_n)$, it omits at least three values outside the discs $U(a_n, d_n)$. Therefore the set

$$\bigcup_{n=1}^{\infty} U(a_n, d_n)$$

cannot belong to the class $P(V)$, and Theorem 6 is proved.

## 12. Proof of Theorem 7

Let $f$ be transcendental and meromorphic in the plane and and let the set $E=\{a_n\}=f^{-1}(\{0, 1, \infty\})$ satisfy the condition $\lim |a_{n+1}/a_n|=\infty$.

We denote

$$\gamma_n = \{z\colon |z| = \sqrt{|a_n a_{n+1}|}\},$$
$$s_n = \{z\colon |z| = |a_n|/2\},$$
$$S_n = \{z\colon |z| = 2|a_n|\},$$

and let $D_n$ be the annulus which is bounded by $\gamma_{n-1}$ and $\gamma_n$.

It follows from Lemma 1, applied in the annuli $2|a_n|<|z|<|a_{n+1}|/2$, that there exists a sequence $U(b_n, d_n)$ such that $\lim d_n=0$ and that $f(\gamma_n)\subset U(b_n, d_n)$ for all large $n$. It does not mean any restriction to assume that the sequence $d_n$ is decreasing.

Let $n_0$ be so large that $d_{n_0}<1/100$. Let $n>n_0$ and let us suppose that

$$U(b_{n-1}, d_{n-1}) \cap U(b_n, d_n) = \emptyset.$$

Joining $\gamma_{n-1}$ to $\gamma_n$ by a path $\gamma\subset D_n$ we see that $f$ takes in $D_n$ at least one value lying outside the union of the discs $U(b_n, d_n)$ and $U(b_{n-1}, d_{n-1})$. Since the image of the boundary of $D_n$ is contained in this union, we deduce that $f$ takes in $D_n$ all values lying in the complement of this union. This is possible only in the case that for some combination $\{w_1, w_2, w_3\}=\{0, 1, \infty\}$ we have $w_1\in U(b_{n-1}, d_{n-1})$, $f(a_n)=w_2$, and $w_3\in U(b_n, d_n)$. We note that $U(b_{n-1}, d_{n-1})\subset U(w_1, 2d_{n-1})$. Let us suppose that

$$U(b_{n-1}, d_{n-1}) \cap U(b_n, d_n) \neq \emptyset.$$

In this case the union of the discs $U(b_{n-1}, d_{n-1})$ and $U(b_n, d_n)$ contains at most one of the values 0, 1 and $\infty$, and since $f$ omits at least two of these values in $D_n$, we conclude from the maximum principle that $f(D_n)$ is contained in the union of these discs. We note that

$$f(D_n) \subset U(b_{n-1}, d_{n-1}) \cup U(b_n, d_n) \subset U(f(a_n), 4d_{n-1}).$$

Combining the estimations above, we conclude that, if $n > n_0$, then $f(\gamma_n) \subset U(c_n, 4d_n)$, where $c_n \in \{0, 1, \infty\}$, and just one of the following two cases for $D_n$ occurs:

(i) $$\{c_{n-1}, f(a_n), c_n\} = \{0, 1, \infty\}, \quad \text{or}$$

(ii) $$f(D_n) \subset U(f(a_n), 4d_{n-1}) \quad \text{and} \quad c_{n-1} = f(a_n) = c_n.$$

Let us suppose that the case (ii) happens for all large $n$, say for $n \geqq n_1 > n_0$. Then we have

$$c_{n_1-1} = f(a_{n_1}) = c_{n_1} = f(a_{n_1+1}) = c_{n_1+1} = f(a_{n_1+2}) = \dots,$$

and we see that the image of the set $|z| \geqq |a_{n_1}|$ is contained in $U(f(a_{n_1}), 4d_{n_1-1})$. This is impossible, and we conclude that there exist arbitrarily large values of $n$ such that the case (i) happens.

Let the case (i) occur for $D_n$ with $n > n_0$. We assume first that $c_{n-1} = 0$, $f(a_n) = 1$ and $c_n = \infty$. Let us suppose that $a_n$ is a multiple root of the equation $f(z) = 1$. Let $J$ be the segment on the positive real axis which joins the points 0 and 1. Then there exists a region $G \subset D_n$ such that the boundary of $G$ is contained in $\gamma_{n-1} \cup f^{-1}(J)$ and that $a_n$ is a boundary point of $G$. Then the image of the boundary of $G$ is contained in $J \cup U(0, 4d_{n-1})$, and since $f$ takes in $G$ near the point $a_n$ at least one value lying outside $J \cup U(0, 4d_{n-1})$, we conclude that $f$ takes in $G$ all values lying outside $J \cup U(0, 4d_{n-1})$. This implies that $f$ takes the value $\infty$ in $D_n$, and we are led to a contradiction. Therefore $a_n$ is a simple 1-point of $f$. Since $f$ has no zeros or poles in $|z - a_n| \leqq |a_n|/2$, we conclude from the maximum and minimum principles that $|f(z)|$ takes the value 1 at some point of $|z - a_n| = |a_n|/2$. Applying Schottky's theorem, we see that there exists an absolute constant $q > 0$ such that $|f(z)| > q$ on $s_n$ and $|f(z)| < 1/q$ on $S_n$. Then it follows from Rouche's theorem that $n(|a_n|/2, b) = n(|a_n|/2, 0)$ for $b \in U(0, q)$ and $n(2|a_n|, b) = n(2|a_n|, \infty)$ for $b \in U(\infty, q)$. Modifying these results for the general case, we get the following conclusion: If the case (i) happens for $D_n$ and $n > n_0$, then $a_n$ is a simple root of the equation $f(z) = f(a_n)$ and

(iii) $$n(|a_n|/2, b) = n(|a_n|/2, c_{n-1})$$

for $b \in U(c_{n-1}, q)$, and

(iv) $$n(2|a_n|, b) = n(2|a_n|, c_n)$$

for $b \in U(c_n, q)$. Here $q > 0$ is an absolute constant.

We denote by $t_n$ the radius of the circle $\gamma_n$. Let the case (i) happen for $n$, $n > n_0$, and let $p > n$ be the smallest integer such that the case (i) happens for $p$, too. In order to simplify the notations, we assume that $c_{n-1} = 0$, $f(a_n) = 1$ and $c_n = \infty$.

It follows from (ii) that $c_{p-1}=\infty$ and that the image of the set $t_n \leqq |z| \leqq t_{p-1}$ is contained in $U(\infty, 4d_n)$. Furthermore, we deduce from (i) that $\{f(a_p), c_p\}=\{0, 1\}$.

We denote $k_n=n(t_n, 1)$. Applying Rouche's theorem, we get

$$n(t_{n-1}, 0) = n(t_n, 0) = n(t_n, 1) = k_n,$$

and we conclude that

(v) $$n(r, 0) = n(r, 1) = k_n$$

for $|a_n| \leqq r < |a_p|$.

Let $|a_n| \leqq r < |a_p|$. Let us suppose first that $w \notin U(0, q) \cup U(1, q)$. We apply Rouche's theorem repeatedly, and conclude that

(vi) $$n(r, w) \geqq n(t_{n-1}, w) = n(t_{n-1}, 1) = k_n - 1,$$

and

(vii) $$n(r, w) \leqq n(t_p, w) = n\big(t_p, f(a_p)\big) = k_n + 1$$

because of (v). Combining (vi) and (vii) we conclude that

(viii) $$|n(r, w) - k_n| \leqq 1$$

for $w \notin U(0, q) \cup U(1, q)$. Let us suppose now that $w \in U(0, q)$. Then it follows from (iii) that

(ix) $$n(r, w) \geqq n(|a_n|/2, w) = n(|a_n|/2, 0) = n(t_{n-1}, 0) = k_n.$$

If $f(a_p)=0$, we get

(x) $$n(r, w) \leqq n(t_p, w) = n(t_p, 0) = k_n + 1,$$

and if $f(a_p)=1$, it follows from (iv) that

(xi) $$n(r, w) \leqq n(2|a_p|, w) = n(2|a_p|, 0) = k_n.$$

Combining the estimates (ix), (x) and (xi), we deduce that (viii) holds for $w \in U(0, q)$. By a similar consideration, we conclude that (viii) holds for $w \in U(1, q)$, too.

Since (viii) is valid for all $w \in \Sigma$ and for all $r$, $|a_n| \leqq r < |a_p|$, we conclude that for any large $r$, there exists a positive integer $k(r)$ such that

(xii) $$|n(r, w) - k(r)| \leqq 1$$

for all $w \in \Sigma$. This implies that

$$\limsup_{r \to \infty} \big(\sup_{w \in \Sigma} n(r, w) - \inf_{w \in \Sigma} n(r, w)\big) \leqq 2.$$

It follows from (v) that two of the numbers $n(r, 0)$, $n(r, 1)$ and $n(r, \infty)$ are equal to $k(r)$ and that the third of these numbers satisfies (xii) for all large $r$. This implies that

(xiii) $$|n(r, w_1) - n(r, w_2)| \leqq 1$$

if $r \geqq r_0$ and $\{w_1, w_2\} \subset \{0, 1, \infty\}$.

Let $a$ be any complex value different from 0, 1 and $\infty$, and let $n$ be so large that $a$ cannot belong to any of the discs $U(0, 4d_n)$, $U(1, 4d_n)$ and $U(\infty, 4d_n)$. If the case (ii) happens, then

(xiv) $$n(r, a) = n(r, w_1) = n(r, w_2)$$

for $t_{n-1} \leqq r \leqq t_n$, $\{w_1, w_2\} = \{0, 1, \infty\} - \{f(a_n)\}$. If the case (i) happens and $t_{n-1} \leqq r \leqq t_n$, then we see from (v) that

$$n(r, a) \leqq n(t_n, a) = n(t_n, f(a_n)) = n(r, c_{n-1})$$

and

$$n(r, a) \geqq n(t_{n-1}, a) = n(t_{n-1}, f(a_n)) = n(r, c_n).$$

Since $n(r, c_{n-1}) = 1 + n(r, c_n)$, we conclude that either $n(r, a) = n(r, c_n)$ or $n(r, a) = n(r, c_{n-1})$. This implies together with (xiv) that for all large values of $r$, there exists $w(r, a) \in \{0, 1, \infty\}$ such that $n(r, a) = n(r, w(r, a))$. If $a \in \{0, 1, \infty\}$, we set $w(r, a) = a$ for all $r$.

Let $a$ and $b$ be two complex values. Then we get for all large $r$,

$$|n(r, a) - n(r, b)| = |n(r, w(r, a)) - n(r, w(r, b))|,$$

and we deduce from (xiii) that $|n(r, a) - n(r, b)| \leqq 1$. This implies that

$$\limsup_{r \to \infty} |n(r, a) - n(r, b)| \leqq 1,$$

which completes the proof of Theorem 7.

## 13. Proof of Theorem 8

Let $f$ and $E$ be as in Theorem 8. Contrary to the assertion of Theorem 8, let us suppose that

$$\lim_{n \to \infty} |a_{n+1}/a_n| = \infty.$$

Let $\gamma_n$, $D_n$ and $U(b_n, d_n)$ be as in the proof of Theorem 7. If $n$ is large, then the set

$$A_n = U(b_{n-1}, d_{n-1}) \cup U(b_n, d_n)$$

contains at most two of the points 0, 1, $w_4$ and $\infty$. Then $f$ omits in $D_n$ at least one value lying in the complement of $A_n$, and since the image of the boundary of $D_n$ is contained in $A_n$, we conclude from the maximum principle that $f(D_n) \subset A_n$. Since $D_n$ is a connected set, $f(D_n)$ is connected, and we deduce that

$$U(b_{n-1}, d_{n-1}) \cap U(b_n, d_n) \neq \emptyset.$$

This implies together with the fact that $f(a_n) \in A_n$ that

$$f(D_n) \subset A_n \subset U(f(a_n), 4d_n)$$

for all large $n$, and in the same manner as in the proof of Theorem 7, we see now that if $n$ is large, then the image of the set $|z|>|a_n|$ is contained in $U(f(a_n), 4d_n)$. This is impossible, and therefore we may conclude that

$$\liminf_{n\to\infty} |a_{n+1}/a_n| < \infty.$$

This proves Theorem 8.

## 14. Proof of Theorem 9

Let $M>1$, $r_1=e$ and $r_{n-1}=\log\log r_n$ for $n\geqq 2$. We set

$$f(z) = z \prod_{n=1}^{\infty} (1-z/r_n)^{2(-1)^n}.$$

It follows from the considerations made in [11, p. 16] that all except a finite number of the 1-points and $M$-points of $f$ lie on the positive real axis on the union of the segments $I_n$: $r_n^{3/2}\leqq x\leqq r_n^3$, and that if $n$ is large, then $I_n$ contains exactly one 1-point and one $M$-point of $f$. Let these points be $b_n$ and $z_n$, arranged such that $b_n<z_n$.

If $x\in I_n$, then the logarithmic derivative of $f$ satisfies

$$\frac{f'(x)}{f(x)} = \big(1+o(1)\big)\frac{(-1)^n}{x},$$

and we get

$$\log M = |\log f(z_n)-\log f(b_n)| = \left|\int_{b_n}^{z_n} \frac{f'(x)}{f(x)}\, dx\right|$$

$$= \big(1+o(1)\big)\int_{b_n}^{z_n}\frac{dx}{x} = \big(1+o(1)\big)\log(z_n/b_n).$$

This implies that

(i) $$z_n/b_n = M^{1+o(1)} \to M$$

as $n\to\infty$. Since $r_n^{3/2}<b_n<z_n<\sqrt{r_{n+1}}$ for all large $n$, we conclude from (i) that the set $E$ defined by (F) satisfies the condition (G). This proves Theorem 9.

## 15. Two lemmas

Lemma 5. *Let $f$ be meromorphic in the half disc*

$$D = \{z\colon\ |z|\leqq r,\ \operatorname{Im} z\geqq 0\}$$

*and satisfy $|f(z)|\geqq 1$ there. There exists an absolute constant $K_1>0$ such that*

$$\log|f(z)| \geqq \frac{K_1}{r}\int_{-r/2}^{r/2}\log|f(x)|\, dx$$

*for any $z\in U(ir/2, r/8)$.*

*Proof.* Let $z(w)$ map the unit disc $|w|<1$ conformally on to $D$ such that $w(0)=ir/2$. From the superharmonicity of $\log|f(z(w))|$ it follows that

$$\text{(i)}\qquad \log|f(z(\varrho e^{i\alpha}))| \geqq \frac{1}{2\pi}\int_0^{2\pi} \log|f(z(e^{i\varphi}))| \frac{1-\varrho^2}{1-2\varrho\cos(\varphi-\alpha)+\varrho^2}\,d\varphi$$

for $\varrho<1$. Since there exist absolute constants $m_1>0$ and $m_2<1$ such that

$$\left|\frac{dz(w)}{dw}\right| \leqq m_1 r$$

if $z$ lies on the segment $[-r/2, r/2]$, and $|w|\leqq m_2$ if $z(w)\in U(ir/2, r/8)$, it follows from (i) that

$$\log|f(\zeta)| \geqq \frac{1-m_2}{2\pi(1+m_2)m_1 r}\int_{-r/2}^{r/2} \log|f(x)|\,dx$$

for $\zeta\in U(ir/2, r/8)$, and Lemma 5 is proved.

Lemma 6. *Let $u$ be harmonic in the annulus $H$: $r<|z|<R$, non-negative and continuous on its closure, and $u(z)=0$ on $|z|=r$. Let*

$$\mu(R, u) = \frac{1}{2\pi}\int_0^{2\pi} u(Re^{i\varphi})\,d\varphi.$$

*Then*

$$\text{(5)}\qquad u(z) \leqq \mu(R, u)\frac{R+|z|}{R-|z|}$$

*and*

$$\text{(6)}\qquad u(z) \geqq \mu(R, u)\left(\frac{R-|z|}{R+|z|} - \frac{(R+r)\log(R/|z|)}{(R-r)\log(R/r)}\right)$$

*for all $z\in H$, and if $R\geqq re^{36}$, then*

$$\text{(7)}\qquad u(z) \geqq \frac{1}{6}\mu(R, u)$$

*for those $z$ which lie in $R/4\leqq|z|\leqq R/2$.*

*Proof.* The function

$$v(\varrho e^{i\alpha}) = \frac{1}{2\pi}\int_0^{2\pi} u(Re^{i\varphi})\frac{R^2-\varrho^2}{R^2-2R\varrho\cos(\varphi-\alpha)+\varrho^2}\,d\varphi$$

is harmonic in $|z|<R$, continuous on its closure, and $v(z)=u(z)$ on $|z|=R$. On $|z|=r$ we have $u(z)=0\leqq v(z)$, and since

$$\text{(i)}\qquad v(z) \leqq \mu(R, u)\frac{R+|z|}{R-|z|}$$

in $|z|<R$, (5) follows from the maximum principle. We see from (i) that

(ii) $$u(z) \geqq v(z)-\mu(R, u)\frac{(R+r)\log(R/|z|)}{(R-r)\log(R/r)}$$

on the boundary of $H$, and we conclude from the maximum principle that (ii) holds in $H_n$. Since

$$v(z) \geqq \mu(R, u)\frac{R-|z|}{R+|z|}$$

in $|z|<R$, we get (6) from (ii). The condition (7) is a direct consequence of (6). Lemma 6 is proved.

## 16. Proof of Theorem 10

Contrary to the assertion of Theorem 10, let us suppose that there exist $\alpha, \beta, E, d_n$ and $S$ as in Theorem 10 and a transcendental meromorphic function $f$ with $\delta(\infty, f)>0$ such that

$$f^{-1}(\{0, 1, w_3\}) \subset U(0, r_0)\cup S$$

for some $r_0$, $w_3$ being different from 0 and 1.

It follows from (H) that the number of the $a_n$ which lie in the annulus $r<|z|<2r$ is at most $4(\log r)^{2\alpha}$. Therefore, if $r$ is large, we may choose $\varrho$, $r<\varrho<2r$, such that

(i) $$\left\{z\colon \varrho-\frac{\varrho}{16(\log\varrho)^{2\alpha}}<|z|<\varrho+\frac{\varrho}{16(\log\varrho)^{2\alpha}}\right\}\cap S=\emptyset.$$

Since $f$ is transcendental and $\delta(\infty, f)>0$, we have

$$\frac{1}{2\pi}\int_0^{2\pi}\log^+|f(re^{i\varphi})|\,d\varphi>10\log r$$

for all large $r$. This implies that we may choose $\varphi_1$ and $\varphi_2$, $\varphi_2=\varphi_1+(\log\varrho)^{-3}$, such that

(ii) $$\int_{\varphi_1}^{\varphi_2}\log^+|f(\varrho e^{i\varphi})|\,d\varphi>10(\log\varrho)^{-2}.$$

We set $g(\zeta)=f(e^\zeta)$. It follows from (ii) that

(iii) $$\int_{\varphi_1}^{\varphi_2}\log^+|g(\log\varrho+i\varphi)|\,d\varphi\geqq 10(\log\varrho)^{-2}.$$

We denote $E_\zeta=\{\zeta\colon e^\zeta\in E\}$. We see from (H) that if $b\in E_\zeta$ and $\operatorname{Re} b$ is large, then

(iv) $$\left\{\zeta\colon 0<|\zeta-b|<\frac{1}{2(\operatorname{Re} b)^\alpha}\right\}\cap E_\zeta=\emptyset,$$

and from (I) we deduce that $g$ omits the values 0, 1 and $w_3$ in

$$G = \{\zeta\colon \operatorname{Re}\zeta \geqq \gamma_0\} - \bigcup_{b \in E_\zeta} U(b, d(b))$$

if $\gamma_0 > 0$ is chosen sufficiently large and $d(b)$ is defined by the equation

$$\log \frac{1}{d(b)} = \frac{1}{2} (\operatorname{Re} b)^{2+\beta}.$$

We assume that $\log \varrho > 100 + \gamma_0$, and we conclude from (i) that

(v) $$\left\{\zeta\colon |\operatorname{Re}\zeta - \log \varrho| \subset \frac{1}{32 (\log \varrho)^{2\alpha}}\right\} \subset G.$$

Let $J$ be the segment $\operatorname{Re}\zeta = \log \varrho$, $\varphi_1 \leqq \operatorname{Im}\zeta \leqq \varphi_2$. It follows from (iii) that there exists $\zeta_0 \in J$ such that $|g(\zeta_0)| \geqq 1$. Since

$$U\left(\zeta_0, \frac{1}{32 (\log \varrho)^{2\alpha}}\right) \subset G,$$

we deduce from Schottky's theorem that there exists $M_5 > 0$ depending only on $w_3$ such that $|g(\zeta)| \geqq M_5$ in

$$U\left(\zeta_0, \frac{1}{64 (\log \varrho)^{2\alpha}}\right).$$

Applying Lemma 5 in the half disc

$$D = \{\zeta\colon \operatorname{Re}\zeta \leqq \log \varrho,\ |\zeta - (i/2)(\varphi_1 + \varphi_2) - \log \varrho| \leqq (\log \varrho)^{-3}\},$$

we conclude now from (iii) that there exists $\zeta_1 \in D$ such that

(vi) $$\log |g(\zeta)| \geqq K_1 \log \varrho$$

in $|\zeta - \zeta_1| \leqq (\log \varrho)^{-5}$.

Let $b \in E_\zeta \cap \{\zeta\colon (3/4) \log \varrho < \operatorname{Re}\zeta < (4/3) \log \varrho\}$. It follows from (iv) that

(vii) $$U\left(b, \frac{1}{3 (\operatorname{Re} b)^{\alpha}}\right) - U(b, d(b)) \subset G.$$

We choose $d$, $0 < d < 1/1000$, such that the set $U(a, 8d)$ cannot contain two of the points 0, 1 and $w_3$ for any $a \in \Sigma$. We see from Lemma 1 and (vii) that there exists $M_6 > 0$ depending only on $w_3$ such that the image of the circle

$$\Gamma_b = \left\{\zeta\colon |\zeta - b| = \frac{1}{M_6 (\operatorname{Re} b)^{\alpha}}\right\}$$

is contained in some set $U(w_1(b), d)$ and the image of

$$\gamma_b = \{\zeta\colon |\zeta - b| = M_6 d(b)\}$$

is contained in some $U(w_2(b), d)$. If

$$U(w_1(b), d) \cap U(w_2(b), d) = \emptyset,$$

then $g$ would take at least one of the values 0, 1 and $w_3$ in

$$D_b = \left\{\zeta\colon\ M_6\, d(b) \leqq |\zeta - b| \leqq \frac{1}{M_6 (\operatorname{Re} b)^\alpha}\right\}.$$

This is impossible, and we deduce that there exists $w(b) \in \Sigma$ such that

(viii) $$f(D_b) \subset U(w(b), 2d).$$

Let $C_b$ be the disc $|\zeta - b| < (\log \varrho)^{-3}$. Let us suppose that there exists $R$, $(\log \varrho)^{-5} < R < 100$, such that $|g(\zeta)| \geqq 2$ in

$$\{\zeta\colon\ (\log \varrho)^{-6} \leqq |\zeta - \zeta_1| \leqq R\} - \bigcup_{b \in E_\zeta} C_b$$

and that there exists $\zeta_2$ such that $|g(\zeta_2)| = 2$ and that

$$\zeta_2 \in \{\zeta\colon\ |\zeta - \zeta_1| = R\} - \bigcup_{b \in E_\zeta} C_b.$$

Let

$$E_1 = \{b \in E_\zeta\colon\ C_b \cap U(\zeta_1, R) \neq \emptyset\}.$$

It follows from (viii) that $|g(\zeta)| \geqq 1$ on $D_b$ for $b \in E_1$, and from (iv) we conclude that the number of the points of $E_1$ is at most

(ix) $$q = 320\,000 (\log \varrho)^{2\alpha}.$$

The function

$$\omega(\zeta) = \frac{\log (R/|\zeta - \zeta_1|)}{\log (R (\log \varrho)^6)} - \sum_{b \in E_1} \frac{\log (3R/|\zeta - b|)}{\log (3R/(M_6\, d(b)))}$$

is harmonic in

$$A = \{\zeta\colon\ (\log \varrho)^{-6} \leqq |\zeta - \zeta_1| \leqq R\} - \bigcup_{b \in E_1} U(b, M_6\, d(b)),$$

$\omega(\zeta) \leqq 1$ on $|\zeta - \zeta_1| = (\log \varrho)^{-6}$, and $\omega(\zeta) \leqq 0$ at the other boundary points of $A$. Since $|g(\zeta)| \geqq 1$ in $A$, it follows from (vi) that

(x) $$\log |g(\zeta)| \geqq K_1\, \omega(\zeta) \log \varrho$$

on the boundary of $A$, and from the superharmonicity of $\log |g(\zeta)|$ in $A$ we conclude that (x) holds in $A$.

It follows from (v) that $U(\zeta_1, (\log \varrho)^{-2}) \cap E_\zeta = \emptyset$. Therefore we may choose $s$, $1 \leqq s \leqq 2$, such that the point

$$\zeta_3 = \zeta_1 + (\zeta_2 - \zeta_1)\left(1 - s (\log \varrho)^{-(1+\alpha)/2}\right)$$

lies outside the union of the discs $C_b$, and we deduce from the definition of $d(b)$ and (ix) that

$$\omega(\zeta_3) \geqq \frac{1}{7 (\log \varrho)^{(1+\alpha)/2} \log \log \varrho} - O\left(\frac{q \log \log \varrho}{(\log \varrho)^{2+\beta}}\right)$$

$$\geqq \frac{(\log \varrho)^{(1-\alpha)/4}}{\log \varrho}$$

if $\varrho$ is large. This implies together with (x) that $|g(\zeta_3)|>10$, and since $g(\zeta_2)\leqq 2$, we deduce from Lemma 2 that there exists $b\in E_\zeta$ such that

$$|b-\zeta_2| \leqq 2K|\zeta_2-\zeta_3| \leqq 4KR(\log \varrho)^{-(1+\alpha)/2}.$$

This implies that both of the points $\zeta_2$ and $\zeta_3$ lie in

$$U\left(b, \frac{1}{M_6(\operatorname{Re} b)^\alpha}\right),$$

and since $\zeta_2$ and $\zeta_3$ lie outside the union of the discs $C_b$, we conclude that $\zeta_3, \zeta_2\in D_b$. This contradicts (viii) because $|g(\zeta_2)|\leqq 2$ and $|g(\zeta_3)|>10$. Therefore we deduce now that $|g(\zeta)|\geqq 2$ in

$$U(\zeta_1, 100)-\bigcup_{b\in E_\zeta} C_b.$$

Combining this with (viii) and letting $\varrho$ grow, we see that there exists $\log \varrho_0>0$ such that $|g(\zeta)|\geqq 1$ in

$$\{\zeta\colon \operatorname{Re}\zeta > \log\varrho_0\}-\bigcup_{b\in E_\zeta} U(b, M_6 d(b)),$$

which, written for $f$, means that $|f(z)|\geqq 1$ in

$$\{z\colon |z| > \varrho_0\}-\bigcup_{n=1}^{\infty} U(a_n, t_n),$$

where the radii $t_n$ are chosen by the equation

$$\text{(xi)} \qquad \log\frac{1}{t_n} = \frac{1}{4}(\log|a_n|)^{2+\beta}.$$

We choose a sequence $r_n, r_1>(4+\varrho_0)^{100}$, such that $r_{n-1}^2<r_n<2r_{n-1}^2$, there are no poles of $f$ on $|z|=r_n$, and

$$\text{(xii)} \qquad \left\{z\colon \big|r_n-|z|\big| < \frac{r_n}{16(\log r_n)^{2\alpha}}\right\}\cap S_0 = \emptyset,$$

where $S_0$ is the union of the discs $U(a_k, t_k)$, and $x_n$ is chosen such that $r_n^{1/100}/2< x_n<r_n^{1/100}$ and that (xii) is satisfied if $r_n$ is replaced by $x_n$.

Let $u$ be the function harmonic in $B_n$: $x_n<|z|<r_n$ which satisfies $u(z)=\log|f(z)|$ on $|z|=r_n$ and $u(z)=0$ on $|z|=x_n$. For $a_k$ lying in $B_n$ we set

$$\omega_k(z) = \frac{\log(2r_n/|z-a_k|)}{\log(2r_n/t_k)}.$$

It follows from Lemma 6 that

$$u(z) \leqq 2m(r_n, \infty)\frac{r_n+|a_k|}{r_n-|a_k|}$$

on $|z-a_k|=t_k$, and therefore we may conclude that

$$\text{(xiii)} \qquad \log|f(z)| \geqq u(z)-2m(r_n, \infty) \sum_{a_k \in B_n} \frac{r_n+|a_k|}{r_n-|a_k|}\,\omega_k(z)$$

on the boundary of $B_n-S_0$, and from the superharmonicity of $\log|f(z)|$ it follows that (xiii) holds in $B_n-S_0$, especially on $|z|=r_{n-1}$.

Let $|z|=r_{n-1}$. If $r_n/2 \leqq |a_k| < r_n$, then we see from (xii) and (xi) that

$$\frac{r_n+|a_k|}{r_n-|a_k|}\,\omega_k(z) = O\left(\frac{(\log r_n)^{2\alpha}}{(\log r_n)^{2+\beta}}\right) = o\big((\log r_n)^{-2}\big),$$

and since the number of these points $a_k$ is $O\big((\log r_n)^{2\alpha}\big)$, we deduce that

$$\sum\nolimits_1 \frac{r_n+|a_k|}{r_n-|a_k|}\,\omega_k(z) = o(1),$$

where the sum $\sum_1$ is taken over those $a_k$ which lie in $r_n/2 \leqq |z| < r_n$. If $x_n<|a_k|<r_n/2$, then (xi) implies that

$$\frac{r_n+|a_k|}{r_n-|a_k|}\,\omega_k(z) = O\left(\frac{\log r_n}{(\log r_n)^{2+\beta}}\right),$$

and since the number of these points $a_k$ is $O\big((\log r_n)^{1+2\alpha}\big)$ and $\beta>2\alpha$, we conclude that

$$\sum\nolimits_2 \frac{r_n+|a_k|}{r_n-|a_k|}\,\omega_k(z) = o(1),$$

where the sum $\sum_2$ is taken over those $a_k$ which lie in $x_n<|z|<r_n/2$. Combining these estimates with (xiii), we get

$$\log|f(z)| \geqq u(z)+o\big(m(r_n, \infty)\big),$$

and using the condition (6) of Lemma 6, we get

$$\text{(xiv)} \qquad \log|f(z)| \geqq \left(\frac{98}{198}+o(1)\right) m(r_n, \infty)$$

on $|z|=r_{n-1}$.

From (xiv) we conclude that

$$\text{(xv)} \qquad m(r_n, \infty) \leqq \frac{199}{98}\, m(r_{n-1}, \infty)$$

for all large values of $n$, say for $n \geqq p$. Then, if we write $\delta=\delta(\infty, f)$,

$$T(r_p^{2^k}, f) \leqq T(r_{p+k}, f) \leqq \frac{2}{\delta}\, m(r_{p+k}, \infty)$$

$$\leqq \frac{2}{\delta}\left(\frac{199}{98}\right)^k m(r_p, \infty),$$

and we deduce that if $R_k = r_p^{2^k}$, then $T(R_k, f) = O((\log R_k)^{9/8})$ as $k \to \infty$. This implies that

(xvi) $$T(r,f) = O((\log r)^{9/8}).$$

We denote by $z_k$ and $b_k$ the zeros and poles of $f$, and by $u_n$ (resp. $v_n$) the number of zeros (resp. poles) of $f$ lying in $|z-a_n| < 1/|a_n|$. We choose $\zeta$ lying on $|z-a_n| = t_n$ such that $|\zeta - b_k| \geqq t_n/v_n$ for any $k$. Applying Poisson—Jensen formula with $R = 2|a_n|$ we obtain, since $|f(\zeta)| \geqq 1$, that

(xvii) $$\begin{aligned} 0 &\leqq \log|f(\zeta)| \\ &\leqq 4m(R, \infty) + \sum_{|z_k|<R} \log\left|\frac{R(\zeta - z_k)}{R^2 - \bar{z}_k \zeta}\right| - \sum_{|b_k|<R} \log\left|\frac{R(\zeta - b_k)}{R^2 - \bar{b}_k \zeta}\right| \\ &\leqq (v_n - u_n)\log\frac{1}{t_n} + O(T(R,f)) + v_n \log v_n + O(n(R, \infty)\log R). \end{aligned}$$

It follows from (xvi) that $n(R, \infty) = O((\log R)^{1/8})$, and therefore we may conclude from (xvii) and (xi) that

$$(u_n - v_n)(\log R)^2 \leqq O((\log R)^{9/8}).$$

This implies that $u_n \leqq v_n$ for all large $n$, and therefore

(xviii) $$n(r, 0) \leqq O(1) + n(r, \infty)$$

for those large values of $r$ which lie outside the union of the intervals $|a_k| - 1/|a_k| < r < a_k + 1/|a_k|$. Therefore we may deduce from (xviii) that $\delta(0, f) \geqq \delta(\infty, f)$. This is impossible, since the growth condition (xvi) quarantees that $f$ has at most one deficient value. We are led to a contradiction, and Theorem 10 is proved.

## 17. A lemma needed in the proof of Theorem 11

Schottky's theorem is proved by Ahlfors in the following form.

Schottky's theorem. *If $g$ is regular in $|z| < 1$ and omits there the values* 0 *and* 1, *then*

$$\log^+|g(z)| \leqq \frac{1+|z|}{1-|z|}(7 + \log^+|g(0)|).$$

We shall need the following

Lemma 7. *Let $E$ and $d_n$ be as in Theorem* 11 *and let $f$ be transcendental and meromorphic in the plane such that $\delta(\infty, f) > 0$ and that*

$$f^{-1}(\{0, 1, w_3\}) \subset U(0, r_0) \cup \bigcup_{n=1}^{\infty} U(a_n, d_n)$$

*for some* $r_0>0$, $w_3$ *being different from* 0 *and* 1. *Then there exist sequences* $R_n$, $R_n\to\infty$ *as* $n\to\infty$, *and* $K_n$, $K_n\to\infty$ *as* $n\to\infty$, *such that*

(8) $$\log|f(z)| \geqq K_n^2 \log R_n$$

*for any* $z$ *lying on* $|z|=R_n$.

*Proof.* If $f$ satisfies $T(r, f)=O((\log r)^m)$ for some finite $m$, it follows from the proof of Lemma 4 that there exist large values of $r$ such that

$$\log|f(z)| \geqq \left(\frac{1}{3}+o(1)\right)\delta(\infty,f)T(r,f)$$

on $|z|=r$, and we may choose the desired sequences $R_n$ an $K_n$.

Let us suppose that $T(r, f)\neq O((\log r)^{100})$. Let $r$ be large and chosen such that $m(r, \infty)\geqq(\log r)^{100}$. We choose $t$, $r\leqq t\leqq 2r$, such that $U(t, 2t(\log t)^{-3})\cap S=\emptyset$, where

$$S=\bigcup_{n=1}^{\infty} U(a_n, d_n).$$

Then, if $r$ is large, $m(t, \infty)>(\log t)^{99}$.

We set $g(\zeta)=f(e^\zeta)$ and $S_\zeta=\{\zeta\colon e^\zeta\in S\}$. Since

$$\frac{1}{2\pi}\int_0^{2\pi} \log^+|g(\log t+i\varphi)|\,d\varphi > (\log t)^{99},$$

we may choose $\varphi_1$ and $\varphi_2$, $0\leqq\varphi_1\leqq\varphi_2=\varphi_1+(\log t)^{-5}\leqq 2\pi$, such that

(i) $$\int_{\varphi_1}^{\varphi_2} \log^+|g(\log t+i\varphi)|\,d\varphi \geqq (\log t)^{92}.$$

We may assume that $0\leqq\varphi_1\leqq\pi$, for the case when $\pi\leqq\varphi_2\leqq 2\pi$, is symmetric.

We choose $\varphi_0$, $\varphi_1<\varphi_0<\varphi_2$, such that $|g(\log t+i\varphi_0)|>1$. Applying Schottky's theorem in the disc

$$U(\log t+i\varphi_0, (\log t)^{-3}),$$

we conclude that there exists $H_1>0$ depending only on $w_3$ such that $\log|g(\zeta)|\geqq -H_1$ in

$$U(\log t+i\varphi_0, (\log t)^{-4}).$$

Therefore we may deduce from Lemma 5 and (i) that there exists

$$\zeta_0\in U(\log t+i\varphi_0, (\log t)^{-5})$$

such that $\operatorname{Im}\zeta_0\geqq 0$ and

(ii) $$\log|g(\zeta)| \geqq (\log t)^{92}$$

in the disc $U(\zeta_0, (2\log t)^{-5})$.

We denote $\varrho = \operatorname{Re} \zeta_0$. We choose $h$ to be one of the functions $g$ and $1-g$ such that $|h(\varrho+i\pi)| \geqq 1/2$. We apply Schottky's theorem in the disc

$$U(\varrho+i\pi, \pi-(\log t)^{-7})$$

to the function $1/h$, and get

(iii)
$$\log|h(\zeta)| \geqq -(\log t)^{-8}$$

for all $\zeta$ lying in

$$U(\varrho+i\pi, \pi-(\log t)^{-6}).$$

Let us suppose that $\varphi_1 < 4\pi/5$. We denote

$$R = \pi - \operatorname{Im} \zeta_0 - (\log t)^{-6}.$$

The length of the arc of the circle $|\zeta-(\varrho+i\pi)| = R$ which lies in $U(\zeta_0, (2\log t)^{-5})$ is at least $(\log t)^{-6}$, and since $\log|h(\zeta)|$ is superharmonic in $U(\varrho+i\pi, R)$, we conclude from (ii) and (iii) for $\zeta = \varrho+i\pi+re^{i\alpha}$ lying in $U(\varrho+i\pi, \pi/10)$ that

(iv)
$$\log|h(\zeta)| \geqq \frac{1}{2\pi}\int_0^{2\pi} \log|h(\varrho+i\pi+Re^{i\varphi})| \frac{(R^2-r^2)\,d\varphi}{R^2-2rR\cos(\varphi-\alpha)+r^2}$$
$$\geqq (\log t)^{85}.$$

If $4\pi/5 \leqq \varphi_1 \leqq \pi$, the same argument as above shows that (iv) holds on $|\zeta-(\varrho+i\pi)| = 2\pi/5$, and then it follows from the superharmonicity of $\log|h(\zeta)|$ that (iv) is valid in $U(\varrho+i\pi, \pi/10)$, in this case, too.

We set

$$\omega(\zeta) = \frac{\log \dfrac{\pi-(\log t)^{-6}}{|\zeta-(\varrho+i\pi)|}}{\log\log t}.$$

Then $\omega(\zeta)$ is harmonic in

$$G = \{\zeta:\ \pi-(\log t)^{-6} > |\zeta-(\varrho+i\pi)| > \pi/10\},$$

$\omega(\zeta) = 0$ on $|\zeta-(\varrho+i\pi)| = \pi-(\log t)^{-6}$ and $\omega(\zeta) < 1$ on $|\zeta-(\varrho+i\pi)| = \pi/10$. From (iv) and (iii) we deduce that

(v)
$$\log|h(\zeta)| \geqq (\log t)^{85}\,\omega(\zeta) - (\log t)^{8}$$

on the boundary of $G$, and from the superharmonicity of $\log|h(\zeta)|$ we conclude that (v) holds in $G$. Especially, if $\zeta$ lies in the disc

$$A = \{\zeta:\ |\zeta-(\varrho+i\pi)| < \pi-2(\log t)^{-6}\},$$

then

(vi)
$$\log|h(\zeta)| \geqq (\log t)^{78}.$$

It follows from Schottky's theorem, applied in

$$U(\varrho+i(\log t)^{-5}, 4(\log t)^{-4}),$$

that $\log|h(\zeta)|>-\log t$ in

$$U(\varrho+i(\log t)^{-5}, 2(\log t)^{-4}).$$

We set

$$w(\zeta)=\frac{\log\dfrac{(\log t)^{-4}}{|\zeta-(\varrho+i(\log t)^{-5})|}}{\log\log t}.$$

It follows from (vi) that

$$\log|h(\zeta)| \geqq (\log t)^{78} w(\zeta)-\log t$$

on the boundary of the annulus

$$(\log t)^{-5} \leqq |\zeta-(\varrho+i(\log t)^{-5})| < (\log t)^{-4},$$

and therefore we conclude that

(vii) $$\log|h(\zeta)| \geqq (\log t)^{77}$$

in $U(\varrho+i(\log t)^{-5}, (\log t)^{-4}/2)$.

Combining the estimates (vi) and (vii) we deduce that $\log|h(\zeta)|\geqq(\log t)^{77}$ on the segment

$$\{\zeta\colon \operatorname{Re}\zeta=\varrho,\ -(\log t)^{-4}/4 \leqq \operatorname{Im}\zeta \leqq 2\pi-(\log t)^{-5}\}.$$

This implies that $\log|f(z)|\geqq(\log t)^{77}$ for all $z$ lying on the circle $|z|=e^{\varrho}$, and we conclude that, if $T(r, f)\neq O((\log r)^{100})$, then the desired sequences $R_n$ and $K_n$ exist in this case, too. This completes the proof of Lemma 7.

## 18. Proof of Theorem 11

Let $a_n$ and $d_n$ be as in Theorem 11. We denote

(1) $$S=\bigcup_{n=1}^{\infty} U(a_n, d_n).$$

Contrary to the assertion of Theorem 11, let us suppose that there exists a transcendental meromorphic function $f$ such that $\delta(\infty, f)>0$ and

(2) $$f^{-1}(\{0, 1, w_3\}) \subset U(0, r_0)\cup S$$

for some $r_0>0$, $w_3$ being different from 0 and 1.

Using Lemma 7, we choose sequences $R_n$, $R_n\to\infty$ as $n\to\infty$, and $K_n$, $K_n\to\infty$ as $n\to\infty$, such that

(3) $$\log|f(z)| \geqq K_n^2\log R_n$$

for any $z$ lying on $|z|=R_n$.

Let $d \leqq 1/1000$ and $M_6$ be as in the proof of Theorem 10,

$$\Gamma_n = \left\{ z\colon \; |z-a_n| = \frac{\varepsilon a_n}{M_6 (\log a_n)^\alpha} \right\},$$

$$\gamma_n = \{ z\colon \; |z-a_n| = M_6 d_n \},$$

and $w_n$ is chosen such that

$$f(D_n) \subset U(w_n, 2d), \tag{4}$$

where $D_n$ is the annulus bounded by $\gamma_n$ and $\Gamma_n$.

We set

$$\omega_k(z) = \frac{1000 + \log (a_k/|z-a_k|)}{1000 + \log (a_k/(M_6 d_k))}.$$

Then $\omega_k(z)=0$ on $|z-a_k| = a_k e^{1000}$ and $\omega_k(z)=1$ on $\gamma_n$. If $|z-a_k| \geqq a_k (\log a_k)^{-9}$, then

$$\omega_k(z) \leqq \frac{19 \log \log a_k}{(\log a_k)^{2+\alpha}}.$$

Let $z$ lie outside the union of the discs $U(a_s, a_s (\log a_s)^{-9})$. We denote

$$E_z = \{ a_k\colon \; |z| e^{-500} \leqq a_k \leqq |z| e^{500} \}.$$

From (J) it follows that there exists $B_1$ depending only on $\varepsilon$ such that the number of the points of $E_z$ is at most $B_1 (\log |z|)^\alpha$, and we conclude that there exists $B_2$ depending only on $\varepsilon$ such that

$$\lambda(z) = \sum_{a_k \in E_z} \omega_k(z) \leqq \frac{B_2 \log \log |z|}{(\log |z|)^2} \tag{5}$$

if $|z|$ is sufficiently large and $z$ lies outside the union of the discs $U(a_s, a_s (\log a_s)^{-9})$.

Let $k$ be fixed. Let us suppose that there exists $R$, $R_k \leqq R < R_k e^{37}$, such that $|f(z)| \geqq 2$ in

$$\{ z\colon \; R_k \leqq |z| \leqq R \} - \bigcup_{n=1}^{\infty} U\left(a_n, \frac{\varepsilon a_n}{M_6 (\log a_n)^\alpha}\right)$$

and that there exists

$$z_1 \in \{ z\colon \; |z| = R \} - \bigcup_{n=1}^{\infty} U\left(a_n, \frac{\varepsilon a_n}{M_6 (\log a_n)^\alpha}\right) \tag{6}$$

such that $|f(z_1)|=2$. From (4) we see that $|f(z)| \geqq 1$ in

$$\{ z\colon \; R_k \leqq |z| \leqq R \} - \bigcup_{n=1}^{\infty} U(a_n, M_6 d_n).$$

We set $g(\zeta) = f(R+\zeta^2)$ and let $h(\zeta) = R + \zeta^2$ be the function which maps $\operatorname{Im} \zeta \geqq 0$ onto the $z$-plane. We denote

$$A = h^{-1}(\{ z\colon \; |z| < R \}),$$

$$A_1 = h^{-1}(\{ z\colon \; |z| \leqq R_k \})$$

and $\zeta_1 = h^{-1}(z_1)$. Then $\zeta_1$ lies on the boundary of $A$ and $\pi/4 < \arg \zeta_1 < 3\pi/4$.

If $t<(\sqrt{2}-1)\sqrt{R}$, then $U(i\sqrt{R}, t)\subset A$. Let us suppose that there exists $t$,

$$(7)\qquad (\sqrt{2}-1)\sqrt{R} \leqq t < \sqrt{R}\left(1-\frac{1}{B_4 \log R}\right),$$

where $B_4=\min\{(\log R)^{1/8}, \sqrt{K_k}\}$ such that $|g(\zeta)|\geqq 2$ in $U(i\sqrt{R}, t)-A$ and that there exists $\zeta_3$ lying on $|\zeta-i\sqrt{R}|=t$ outside $A$ satisfying $|g(\zeta_3)|=2$.

We denote $G=U(i\sqrt{R}, t)-A_1$, and let $\omega(\zeta)$ be the harmonic measure with respect to $G$ of that part of the boundary of $G$ which is common with $A_1$. There exists an absolute constant $B_3>0$ such that $U(i\sqrt{R}, 4B_3\sqrt{R})\subset A_1$. On the boundary of $G$ we have

$$(8)\qquad \omega(\zeta) \geqq \frac{\log\left(t/|\zeta-i\sqrt{R}|\right)}{\log\left(t/(B_3\sqrt{R})\right)},$$

and from the harmonicity we conclude that (8) holds in $G$.

Let $p$ be the greatest integer such that $\gamma_p\subset U(0,R)$. For $n\leqq p$, we denote

$$Q_n = \{\zeta\colon\ R+\zeta^2\in\gamma_n,\ \operatorname{Im}\zeta \geqq 0\},$$

and $V_n$ is the open domain bounded by $Q_n$. If $V_n\cap G\neq\emptyset$, we denote by $v_n$ the harmonic measure of $Q_n\cap G$ with respect to $G$, and

$$v(\zeta) = \sum_{V_n\cap G\neq\emptyset} v_n(\zeta).$$

Let $G_0=G-\bigcup_{n=1}^{p} V_n$. We note that $v_n(\zeta)\leqq\omega_n(R+\zeta^2)$ in $G_0$ and conclude from (5) that

$$(9)\qquad v(\zeta) \leqq \lambda(R+\zeta^2) \leqq \frac{2B_2\log\log R}{(\log R)^2}$$

if $\zeta$ lies in

$$G_1 = G-\left\{\zeta\colon\ 0<\operatorname{Im}\zeta<\sqrt{R},\ |\operatorname{Re}\zeta|<\sqrt{R}(\log R)^{-6}\right\}.$$

We choose $\zeta_4$ by the equation

$$\zeta_4-i\sqrt{R} = (\zeta_3-i\sqrt{R})\left(1-\frac{1}{B_4^2\log R}\right).$$

We may assume that $\zeta_4\in G_1$,

$$(10)\qquad \omega(\zeta_4) \geqq \frac{1}{\log(1/B_3)\, B_4^2\log R}.$$

It follows from (3) that

$$(11)\qquad \log|g(\zeta)| \geqq K_k^2\log R_k\left(\omega(\zeta)-v(\zeta)\right)$$

on the boundary of $G_0$, and since $\log|g(\zeta)|$ is superharmonic in $G_0$, we conclude that (11) holds in $G_0$. Therefore we may deduce from (9) and (10) that $\log|g(\zeta_4)|\geqq\sqrt{K_k}$.

Since now $|g(\zeta_3)|\leqq 2$, $|g(\zeta_4)|\geqq 10$ and

$$|\zeta_3-\zeta_4| \leqq \frac{t}{B_4^2\log R},$$

we conclude from Lemma 2 that the disc

$$U\left(\zeta_3, \frac{Kt}{B_4^2 \log R}\right)$$

contains at least one zero, 1-point or $w_3$-point of $g$. This is not possible, since from (7) and the fact that $\zeta_3$ lies outside $A$ it follows that $f$ omits the values 0, 1 and $w_3$ in

$$U\left(\zeta_3, \frac{t}{2B_4 \log R}\right).$$

Therefore we conclude now that $|g(\zeta)|>2$ in

$$U\left(i\sqrt{R}, \sqrt{R}\left(1-\frac{1}{B_4 \log R}\right)\right)-A.$$

This implies that

$$|\zeta_1| < \frac{2\sqrt{R}}{B_4 \log R}$$

because $\zeta_1$ lies on the boundary of $A$ and $|g(\zeta_1)|=2$. If we choose

$$\zeta_5 = \frac{8(1+i)\sqrt{R}}{B_4 \log R},$$

then $\zeta_5$ lies outside $A$, and we see in the same manner as above, $\zeta_5$ taking the role of $\zeta_4$, that

(12) $$\log|g(\zeta_5)| \geqq \sqrt{K_k}.$$

We set $z_2=R+\zeta_5^2$. Then $|f(z_2)|>10$ and

$$|z_1-z_2| \leqq |\zeta_1|^2+|\zeta_5|^2 \leqq \frac{500R}{B_4^2(\log R)^2}.$$

Using Lemma 2 again, we conclude that the disc

$$C_0 = U\left(z_1, \frac{500KR}{B_4^2(\log R)^2}\right)$$

contains at least one point of $S$. From the choices of $B_4$ and $z_1$ it follows that if $k$ is large, then $C_0$ cannot contain any point of $S$, and we conclude that, if $k$ is large, then $|f(z)|\geqq 2$ in

$$\{z\colon R_k \leqq |z| \leqq R_k e^{37}\}-\bigcup_{n=1}^{\infty} U\left(a_n, \frac{\varepsilon a_n}{M_6(\log a_n)^\alpha}\right),$$

which implies together with (4) that

(13) $$|f(z)| \geqq 1$$

in

$$\{z\colon R_k \leqq |z| \leqq R_k e^{37}\}-\bigcup_{n=1}^{\infty} U(a_n, M_6 d_n).$$

We begin with $t_1=R_k$, where $k$ is large, and choose $\varrho_2$, $R_k e^{36}<\varrho_2<R_k e^{37}$, such that there are no poles of $f$ on $|z|=\varrho_2$ and that

$$U\left(\varrho_2, \frac{\varepsilon\varrho_2}{4(\log\varrho_2)^\alpha}\right)\cap S=\emptyset.$$

If $\gamma_p$ is a boundary component of

$$G_2=\{z\colon R_k<|z|<\varrho_2\}-\bigcup_{n=1}^{\infty} U(a_n, M_6 d_n),$$

we denote by $v_p$ its harmonic measure with respect to $G_2$. Let $u$ be the function harmonic in $R_k<|z|<\varrho_2$ which has the boundary values $\log|f(z)|$ on $|z|=\varrho_2$ and 0 on $|z|=R_k$. We denote $\beta_p=\max\{u(z)\colon z\in\gamma_p\}$ if $a_p$ lies in $R_k<|z|<\varrho_2$.

On the boundary of $G_2$ we have

$$\log|f(z)|\geqq u(z)-\sum\beta_p v_p(z), \tag{14}$$

and from the superharmonicity of $\log|f(z)|$ it follows that (14) holds in $G_2$, especially on $|z|=t_2$ where $t_2$, $\varrho_2/4<t_2<\varrho_2/2$, is chosen such that $|t_2-a_n|\geqq\varrho_2(\log\varrho_2)^{-3}$ for all $n$.

Let $z$ lie on the circle $|z|=t_2$. If $(8/9)\varrho_2\leqq a_p<\varrho_2$, then we see from Lemma 6 that

$$\beta_p\leqq 16m(\varrho_2,\infty)\varepsilon^{-1}(\log\varrho_2)^\alpha.$$

Since

$$v_p(\zeta)\leqq\frac{\log(2\varrho_2/|\zeta-a_p|)}{\log(2\varrho_2/(M_6 d_p))}$$

in $G_2$, we conclude from (K) that

$$v_p(z)\leqq\frac{2\log 8}{H(\log\varrho_2)^{2+\alpha}}.$$

The number of the points $a_p$ satisfying $(8/9)\varrho_2\leqq a_p<\varrho_2$ is at most $\varepsilon^{-1}(\log\varrho_2)^\alpha$, and we see that the sum $\sum_1$ over these $a_p$ satisfies

$$\sum\nolimits_1\beta_p v_p(z)\leqq\frac{32\log 8}{H\varepsilon^2}m(\varrho_2,\infty)\leqq\frac{1}{24}m(\varrho_2,\infty).$$

If $R_k<a_p<(8/9)\varrho_2$, then it follows from Lemma 6 that $\beta_p=O(m(\varrho_2,\infty))$, and we conclude from (5) that the sum $\sum_2$ over these $a_p$ satisfies

$$\sum\nolimits_2\beta_p v_p(z)\leqq O(m(\varrho_2,\infty)\lambda(z))=o(m(\varrho_2,\infty))\leqq\frac{1}{24}m(\varrho_2,\infty)$$

if $k$ was chosen sufficiently large. Combining these estimates with (14), we deduce that

$$\log|f(z)|\geqq u(z)-\frac{1}{12}m(\varrho_2,\infty)$$

on $|z|=t_2$, and from Lemma 6 we see now that

$$\log|f(z)| \geqq \frac{1}{12} m(\varrho_2, \infty) \tag{15}$$

for all $z$ lying on $|z|=t_2$.

Since

$$\lim_{r\to\infty} \frac{m(r, \infty)}{\log r} = \infty,$$

taking $t_2$ instead of $R_k$, we get $|f(z)|\geqq 1$ for

$$z\in\{z: t_2 \leqq |z| \leqq t_2 e^{37}\} - \bigcup_{n=1}^{\infty} U(a_n, M_6 d_n),$$

and continuing this process inductively, we conclude that there exists $R_0>0$ such that $|f(z)|\geqq 1$ for all $z$ lying in

$$\{z: |z| > R_0\} - \bigcup_{n=1}^{\infty} U(a_n, M_6 d_n).$$

We choose a sequence $r_n$, $r_1>(4+R_0)^{100}$, such that $r_{n-1}^2<r_n<2r_{n-1}^2$, there exist no poles of $f$ on $|z|=r_n$, and

$$U\left(r_n, \frac{\varepsilon r_n}{4(\log r_n)^\alpha}\right) \cap S = \emptyset. \tag{16}$$

The sequence $x_n$ is chosen such that $r_n^{1/100}/2<x_n\leqq r_n^{1/100}$ and that (16) is satisfied if $r_n$ is replaced by $x_n$.

Let $u$ be the function harmonic in $x_n<|z|<r_n$ which has the boundary values $\log|f(z)|$ on $|z|=r_n$ and 0 on $|z|=x_n$. For those $a_p$ which lie in $x_n<|z|<r_n$, we set

$$w_p(z) = \frac{\log(2r_n/|z-a_p|)}{\log(2r_n/M_6 d_p))}.$$

Using Lemma 6 as in the proof of Theorem 10, we deduce that

$$\log|f(z)| \geqq u(z) - 2m(r_n, \infty) \sum \frac{r_n+a_p}{r_n-a_p} w_p(z) \tag{17}$$

on $|z|=r_{n-1}$.

Let $|z|=r_{n-1}$. If $r_n/2\leqq a_p<r_n$, then

$$\frac{r_n+a_p}{r_n-a_p} w_p(z) \leqq \frac{32(\log r_n)^\alpha \log 8}{H\varepsilon(\log r_n)^{2+\alpha}},$$

the number of these $a_p$ is at most $2\varepsilon^{-1}(\log r_n)^\alpha$, and the sum $\sum_1$ over these $a_p$ satisfies

$$\sum\nolimits_1 \frac{r_n+a_p}{r_n-a_p} w_p(z) \leqq \frac{64\log 8}{H\varepsilon^2} \leqq \frac{1}{100}. \tag{18}$$

If $x_n < a_p < r_n/2$, then

(19) $$\frac{r_n + a_p}{r_n - a_p} w_p(z) \leqq \frac{6 \log r_n}{H(100^{-1} \log r_n)^{2+\alpha}}.$$

Let $n(r)$ be the counting function of the sequence $a_n$. From (J) we get

$$n(e^k) - n(e^{k-1}) \leqq \frac{8}{\varepsilon} k^\alpha,$$

which implies that

$$n(e^k) \leqq \frac{8}{\varepsilon} \sum_{s=1}^{k} s^\alpha + O(1) = \left(\frac{8}{\varepsilon} + o(1)\right) \int_1^k x^\alpha \, dx.$$

From this it follows that $n(r) \leqq 8\varepsilon^{-1} (\log r)^{1+\alpha}$ for all large $r$. Therefore the number of $a_p$ satisfying $x_n < a_p < r_n/2$ is at most $8\varepsilon^{-1} (\log r_n)^{1+\alpha}$, and we deduce from (19) that the sum $\sum_2$ over these $a_p$ satisfies

(20) $$\sum\nolimits_2 \frac{r_n + a_p}{r_n - a_p} w_p(z) \leqq \frac{48(100)^{2+\alpha}}{H\varepsilon} \leqq \frac{1}{100}.$$

Combining the estimates (18) and (20) with (17) we conclude that

(21) $$\log |f(z)| \geqq u(z) - \frac{1}{25} m(r_n, \infty)$$

on $|z| = r_{n-1}$.

From Lemma 6 we get

$$u(z) \geqq \frac{98}{198} (1 + o(1)) m(r_n, \infty),$$

and this implies together with (21) that

(22) $$m(r_{n-1}, \infty) \geqq \frac{9}{20} m(r_n, \infty).$$

In the same manner as in the proof of Theorem 10, we see that (22) implies that

$$T(r, f) = O\big((\log r)^{3/2}\big),$$

and that this leads to the impossibility that $\delta(0, f) \geqq \delta(\infty, f)$. Theorem 11 is proved.

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