Annales Academiæ Scientiarum Fennicæ
Series A. I. Mathematica
Volumen 5, 1980, 377—379

A NOTE ON LIPSCHITZ COMPACTIFICATIONS

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1. In [1] we introduced *Lipschitz compactifications* or LIP *compactifications* as compactifications in the category LIP of metric spaces and locally Lipschitz (LIP) maps. That is, if $f\colon X\to Y$ is a dense LIP embedding of a metric space X into a compact metric space Y, then Y or, more properly, the pair (Y, f) is called a LIP compactification of X. Two LIP compactifications of X are called LIP *equivalent* if they are equivalent through a lipeomorphism. We proved in [1, 1.14] that a metric space has a LIP compactification if and only if it is separable and locally totally bounded.

In this note we consider the problem whether a metrizable compactification Z of a separable locally totally bounded metric space X is equivalent to a LIP compactification of X, or equivalently, whether Z can be metrized in such a way that Z becomes a LIP compactification of X. Let $K_Z(X)$ denote the set of the LIP equivalence classes of the LIP compactifications of X that are equivalent to Z. In Theorem 1 we give characterizations for $K_Z(X)\neq\emptyset$. Our main result, Theorem 2, is that $K_Z(X)\neq\emptyset$ for every Z if and only if X is locally compact. In Theorem 3 we consider the cardinality of $K_Z(X)$. We now give an example where $K_Z(X)=\emptyset$. Let X be the subspace $[0,1]\setminus\{1/n\,|\,n\geqq 1\}$ of $\boldsymbol{R}$ and let $Y\subset\boldsymbol{R}$ be the union of $\{0\}$ and the intervals $(1/(2n+1), 1/2n)$, $n\geqq 1$. Then $\bar{X}$ and $Z=\bar{Y}$ are compact and there is a homeomorphism f of X onto Y with $f(0)=0$. However, no neighborhood of 0 in $\bar{X}$ is homeomorphic to any neighborhood of 0 in Z. Hence the condition (2) of Theorem 1 is not satisfied and thus $K_Z(X)=\emptyset$.

For the undefined LIP terms we refer to [1].

2. A bijection f between uniform spaces is called a *locally uniform homeomorphism* if both f and f^{-1} are locally uniformly continuous, i.e. uniformly continuous on some neighborhood of every point. We need the following modification of Lavrentiev's theorem [2, 24.9].

Lemma. *Let S and T be complete Hausdorff uniform spaces, let $A\subset S$ and $B\subset T$ be dense subsets, and let $f\colon A\to B$ be a locally uniform homeomorphism. Then there are open sets $U\supset A$ and $V\supset B$ and a locally uniform homeomorphism $F\colon U\to V$ extending f.*

doi:10.5186/aasfm.1980.0516

Proof. By [1, 2.9.5] f and f^{-1} have locally uniformly continuous extensions to open neighborhoods of A and B, respectively. The proof can now be completed as in [2, 24.9]. □

Theorem 1. *Let X be a separable locally totally bounded metric space, let $f\colon X\to Z$ be a metrizable compactification of X, and let $\tilde{X}$ be the completion of X. Then the following conditions are equivalent:*

(1) $K_Z(X)\neq\emptyset$.

(2) *There are neighborhoods U of X in $\tilde{X}$ and V of fX in Z and a homeomorphism $g\colon U\to V$ extending f.*

(3) *There are a neighborhood U of X in $\tilde{X}$ and an embedding $g\colon U\to Z$ extending f.*

(4) *There are a neighborhood V of fX in Z and an embedding $h\colon V\to\tilde{X}$ extending $f^{-1}\colon fX\to X$.*

(5) *Consider Z with its unique compatible uniformity, given by any compatible metric. Then f defines a locally uniform homeomorphism of X onto fX.*

Proof. (1)⇒(5): Trivial.

(5)⇒(3) and (5)⇒(4): This follows from the Lemma.

(3)⇒(2): By [1, 1.13] X has a locally compact neighborhood in $\tilde{X}$. Thus we may assume that U is locally compact. Then gU is locally compact and hence open in Z.

(4)⇒(2): This is proved as (3)⇒(2).

(2)⇒(1): Let e be the metric on V for which $g\colon U\to(V,e)$ is an isometry. We may assume that V is open. Then by [1, 6.4] there is a compatible metric r on Z which is LIP equivalent to e on V. Hence $f\colon X\to(Z,r)$ is a LIP embedding. □

Theorem 2. *Let X be a separable locally totally bounded metric space. Then $K_Z(X)\neq\emptyset$ for every metrizable compactification Z of X if and only if X is locally compact.*

Proof. Suppose that X is locally compact and that $f\colon X\to Z$ is a metrizable compactification of X. Then X is open in $\tilde{X}$ and fX in Z. Hence the condition (2) of Theorem 1 is satisfied. Thus $K_Z(X)\neq\emptyset$.

Suppose now that X is not locally compact. Then X is not open in $\tilde{X}$. Hence by [1, 6.5] there is a compatible totally bounded metric e on X having no extension to a compatible metric on a neighborhood of X in $\tilde{X}$. Then the completion Z of (X, e) is a compactification of X such that the condition (2) of Theorem 1 is not satisfied. Thus $K_Z(X)=\emptyset$. □

The sufficiency part of Theorem 2 generalizes the sufficiency part of a similar result [1, 1.6] on one-point compactifications and gives it a new proof.

In the next theorem we consider $K_Z(X)$ with its partial order which one gets through representatives setting $(Y,f)\leqq(Y',f')$ if there is a LIP map $g\colon Y'\to Y$ with $gf'=f$.

Theorem 3. *Let X be a noncompact metric space and let Z be a metrizable compactification of X with $K_Z(X)\neq\emptyset$. Then $K_Z(X)$ has the cardinality of the continuum. In fact, $K_Z(X)$ contains a subset which has the cardinality of the continuum and whose elements are not comparable.*

Proof. We may assume that Z is a compact metric space and that X is a subspace of Z. Since $X\neq Z$, the proof can now be completed just as for one-point compactifications in the proof of [1, 1.9]. □

This generalizes [1, 1.9 and 1.10] and improves [1, 1.15.2].

3. Finally we consider the case where we allow the metric of X to vary.

Theorem 4. *Let X be a metrizable space which is not locally compact, and let $f\colon X\to Z$ be a metrizable compactification of X. Then X can be metrized by a totally bounded metric such that $K_Z(X)=\emptyset$.*

Proof. Since fX is not open in Z, by [1, 6.5] there is a compatible totally bounded metric e on fX such that no compatible metric on Z is LIP equivalent to e on fX. □

References

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Received 24 January 1980