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ENTIRE FUNCTIONS WITH TWO LINEARLY DISTRIBUTED VALUES

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In a series of papers [2, 3, 4] T. Kobayashi has given some interesting characterisations of the exponential function by its property of having its a-points collinear on a line $\lambda(a)$ for several values of a. In [4] he proved

Theorem A. *Let G be a transcendental entire function of finite lower order Assume that the zero points of G lie on the line* $\operatorname{Re} z=0$ *and that the one-points lie on* $\operatorname{Re} z=1$. *Then*

$$G(z) = P(\exp Cz)$$

where P is a polynomial and C a non-zero real constant.

Of course there are restrictions on the polynomials which can occur in Theorem A and the possible forms are given in [4].

Kobayashi asks whether Theorem A still holds if the assumption that G has finite order be omitted from the hypotheses. We prove the

Theorem. *Let f be a transcendental entire function such that all the zeros of f lie on* $\operatorname{Re} z=0$ *and all the one-points on* $\operatorname{Re} z=1$. *Then f has finite order, so by Theorem* A *there exist a polynomial P and a non-zero real constant C such that*

$$f(z) = P(\exp Cz).$$

The proof depends on a recent result of J. Miles (Lemma 6) which states that if an entire function has both infinite order and real zeros then the zeros are in a certain sense scarce. Applying this to $f(-iz)$ and $f(1-iz)-1$ and using Nevanlinna's second fundamental theorem gives the result after a number of subsidiary points have been checked.

The proof follows in six lemmas and a concluding section.

Lemma 1. *Let g be analytic in H:* $\operatorname{Im} z>0$ *and omit the values* 0 *and* 1 *in H. Then there exists a constant $K=K(g)$ such that*

$$\log|g(re^{i\theta})| < Kr/(\sin\theta), \quad r>1,\ 0<\theta<\pi. \tag{1}$$

Proof. The map $z=\varphi(t)=i(1+t)/(1-t)$ maps the disc D: $|t|<1$ to the half-plane H. Applying Schottky's theorem to $g(\varphi(t))$ we obtain

$$|g(\varphi(t))| < \exp\{K/(1-|t|)\}.$$

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Using

$$1-|t|^2 = 4r \sin\theta/\{r^2+2r\sin\theta+1\}, \quad z = re^{i\theta}$$

we have

$$1-|t| > 2r\sin\theta/\{r^2+2r\sin\theta+1\}, \quad |t|<1$$

and

$$\log|g(re^{i\theta})| < K(1+r)^2/(2r\sin\theta) < Kr/(\sin\theta).$$

Lemma 2. *Let g be analytic in the strip S: $0<\operatorname{Im} z<1$ and omit the values 0 and 1 there. Then there exists a constant $K=K(g)$ such that*

$$\log|g(x+iy)| < Ke^{\pi|x|}/(\sin\pi y), \quad 0<y<1, \ -\infty<x<\infty.$$

Proof. Putting $w=e^{\pi z}$ which maps S onto the half-plane H: $\operatorname{Im} w>0$, the result follows from Lemma 1 for $x>0$. For $x<0$ the result follows by symmetry.

Lemma 3. *Suppose f is entire and that all the zeros of f are real and all the one-points have imaginary part one. Then there is a constant A such that*

(2) $$T(r) = T(r,f) < Ar^{-4}e^{4\pi r}$$

for all sufficiently large r.

Remark. We assume without explanation the standard notations of Nevanlinna theory.

Proof. The result of Lemma 1 shows that f is of order one and exponential type in any angle which is either strictly interior to $\operatorname{Im} z>0$ or to $\operatorname{Im} z<0$. If the lower order $\mu=\underline{\lim}_{r\to\infty}(\log T(r))/(\log r)<\infty$ it follows from the Phragmèn—Lindelöf principle that f has at most order one in the plane and the assertion of the lemma holds.

Thus we may assume that $\mu=\infty$ so that $T(r)\to\infty$ faster than any power of r. Put $\eta=\sin^{-1}(1/r)$ and $\delta=\{T(r)\}^{-1/2}$. Split the range of integration in

(3) $$T(r) = \frac{1}{2\pi}\int_0^{2\pi} \log^+|f(re^{i\theta})|\,d\theta$$

at $\pm\delta, \eta\pm\delta, \pi-\eta\pm\delta$ and $\pi\pm\delta$. In the intervals $[-\delta,\delta]$, $[\eta-\delta, \eta+\delta]$, $[\pi-\eta-\delta, \pi-\eta+\delta]$ and $[\pi-\delta,\pi+\delta]$ we put

$$\log^+|f(re^{i\theta})| \leqq \log M(r,f),$$

and in the remaining intervals use the estimates from Lemmas 1 and 2. It follows that there are constants K, K', P and Q such that for sufficiently large r

(4) $$T(r) < \frac{Kr}{\sin\delta} + \frac{K'e^{\pi r}\eta}{\sin(\pi r\delta)} + \frac{4\delta}{\pi}\log M(r,f)$$

$$< \frac{Pr}{\delta} + \frac{Qe^{\pi r}}{r^2\delta} + \frac{4\delta}{\pi}\log M(r,f).$$

Taking $R=r+\varphi$ where $\varphi=r/\log T(r)$ we have

$$\log M(r,f) \leqq \frac{R+r}{R-r}\cdot T(R) > \frac{3r}{\varphi} T\left(r+\frac{r}{T(r)}\right). \tag{5}$$

A lemma of Borel [1] states that for any increasing function $V(r)$ which is continuous in $r>r_0$ and such that $V(r)\to\infty$ as $r\to\infty$ and for any $\varepsilon>0$

$$V\left(r+\frac{r}{\log V(r)}\right) \leqq V(r)^{1+\varepsilon}$$

holds outside a set of finite logarithmic measure. Taking $V=T$ and $1/2>\varepsilon>0$, (4) and (5) show that outside a set E of finite logarithmic measure in $r>1$

$$T(r) < Pr\delta^{-1}+Qr^{-2}\delta^{-1}e^{\pi r}+12\delta r T(r)^{1+\varepsilon}/(\pi\varphi),$$

or putting in the values of δ and φ

$$T(r) < PrT(r)^{1/2}+Qr^{-2}e^{\pi r}T(r)^{1/2}+(12/\pi)(\log T(r))T(r)^{1/2+\varepsilon}.$$

For $0<\varepsilon<\varepsilon'<1/2$ we have

$$\log T(r) = o((T(r))^{\varepsilon'-\varepsilon}) \quad (r\to\infty)$$

and so

$$(1-o(1))T(r)^{1/2} < 2Qr^{-2}e^{\pi r}$$

as $r\to\infty$ outside E. Hence for large r outside E

$$T(r) < 5Q^2r^{-4}e^{2\pi r}. \tag{6}$$

Since E has finite logarithmic measure there exists r_0 such that for any $r>r_0$ there are $s\notin E$, $t\notin E$, $s<r<t<2s$, while (6) holds for $r=s$ and $r=t$. Thus

$$T(r) < T(t) < 5Q^2t^{-4}e^{2\pi t} < 5Q^2r^{-4}e^{4\pi r},$$

so that (2) holds with $A=5Q^2$.

Lemma 4. *Suppose the increasing continuous function $V(r)$ satisfies $V(r)<e^{Ar}$ for some constant A, at least for all $r\geqq r_0>0$. Suppose also that $V(r)\to\infty$ as $r\to\infty$. Then*

$$V(r+r^{-1}) < 2V(r) \tag{7}$$

holds outside a set of finite logarithmic measure in $[r_0,\infty)$.

Proof. Suppose the assertion is not true. Then there is a first $r_1\geqq r_0$ where (7) fails. Define $r_1^*=r_1+r_1^{-1}$. Then

$$V(r_1^*) \geqq 2V(r_1).$$

Denote the interval $[r_1, r_1^*]$ by I_1.

Now proceed inductively. Assuming r_n, $r_n^*=r_n+r_n^{-1}$ and $I_n=[r_n, r_n^*]$ have been constructed, let r_{n+1} be the first $r\geqq r_n^*$ at which (7) fails, i.e. $V(r_{n+1}^*)\geqq 2V(r_{n+1})$.

Such an r_{n+1} exists since we are supposing that (7) fails for a set of infinite logarithmic measure. Thus there is an infinite sequence r_n, $n=1, 2, \ldots,$ and we have

$$r_n < r_n^* \leqq r_{n+1} < r_{n+1}^*.$$

From this it follows that $r_n \to \infty$ for otherwise we would have $\lim r_n = \lim r_{n+1} = \lim r_n^* =$ finite and thence $\lim r_n^{-1} = 0$ which gives a contradiction. Further, the inequality (7) holds in the complement of $F = \bigcup_{n=1}^{\infty} I_n$.

The logarithmic measure of I_n is $\log(1+r_n^{-2}) < r_n^{-2}$. Since the logarithmic measure of F is infinite $\sum r_n^{-2} = \infty$. Take any constant B such that $B \log 2 > A$. There must be arbitrarily large n such that $r_n^{-2} \geqq B^2 n^{-2}$, that is such that $n \geqq Br_n$. For any n

$$V(r_n) \geqq V(r_{n-1}^*) \geqq 2V(r_{n-1}) \geqq \lambda 2^n,$$

where $\lambda = V(r_1)/2$. For infinitely many n we have in addition that $n \geqq Br_n$ so that

$$V(r_n) > \lambda \exp(r_n B \log 2).$$

Since $r_n \to \infty$ and $B \log 2 > A$ this contradicts the hypothesis that $V(r) < e^{Ar}$, $r > r_0$. Thus the lemma is established.

Lemma 5. *Suppose f satisfies the assumptions of Lemma 3 and let g be defined by $g(z) = f(i+z)$.*

Then

$$T(r, g) < 8T(r, f)$$

outside a set $E_1 \subset [1, \infty)$ of finite logarithmic measure.

Proof. We can assume f transcendental. By Nevanlinna's second fundamental theorem

$$T(r, g) \leqq N(r, 0, g) + N(r, 1, g) + S(r) \tag{8}$$

where $S(r) = O\{\log r + \log T(r, g)\}$ as $r \to \infty$ outside a set H of finite measure. Now

$$n(r, 0, g) \leqq n(r, 0, f),$$

$$n(r, 1, g) \leqq n((1+r^2)^{1/2}, 1, f),$$

so that

$$N(r, 0, g) \leqq N(r, 0, f) + O(\log r) \tag{9}$$

and

$$N(r, 1, g) = O(\log r) + \int_1^r n(t, 1, g)\frac{dt}{t} \tag{10}$$

$$\leqq O(\log r) + \int_{\sqrt{2}}^{\sqrt{(1+r^2)}} \frac{n(u, 1, f)}{u^2-1}\, u\, du$$

$$\leqq O(\log r) + 2N((1+r^2)^{1/2}, 1, f).$$

Thus from (8), (9) and (10) we have outside H that

$$T(r, g) < 3T((1+r^2)^{1/2}, f)+O(\log r+\log T(r, g))$$

whence

(11) $$T(r, g) < 4T(r+r^{-1}, f)$$

outside a set H' of finite measure.

By Lemma 3 $T(r, f,) < \exp(4\pi r)$ for large r, so applying Lemma 4 with $V(r)=T(r, f)$ (11) gives

$$T(r, g) < 8T(r, f)$$

outside a set E_1 in $[1, \infty)$, of finite logarithmic measure.

Lemma 6. (Cf. [5].) *Suppose h is entire of infinite order with zeros restricted to a finite number of rays through the origin. Then there exists a set $G\subset[1, \infty)$ having logarithmic density zero and such that* $\lim N(r, 0)/T(r, h)=0$ *as* $r\to\infty$ *outside G.*

Final section of the proof of the theorem. Let f satisfy the assumptions of the theorem and consider $h(z)=f(-iz)$ which has the same characteristic as f, while the zeros and ones of h lie on the lines $\operatorname{Im} z=0$ and $\operatorname{Im} z=1$ respectively.

Suppose that the order of f (and hence of h) is infinite. Then by Lemma 6 there is a set $G_1\subset[1, \infty)$ of logarithmic density zero and such that

(12) $$\lim N(r, 0, h)/T(r, h) = 0 \quad \text{as} \quad r\to\infty, \quad r\notin G_1.$$

Now consider $g(z)=h(i+z)-1$ which is also of infinite order with real zeros. By Lemma 5 there is a set $G_2\subset[1, \infty)$ of finite logarithmic measure such that $T(r, g)<8T(r, h)$, $r\notin G_2$.

Since $n(r, 1, h)\leqq n(r, 0, g)$ holds we have

$$N(r, 1, h) \leqq N(r, 0, g)+O(\log r)$$

and for $r\notin G_2$

(13) $$\frac{N(r, 1, h)}{T(r, h)} \leqq \frac{8N(r, 0, g)}{T(r, g)}+o(1).$$

Applying Lemma 6 to g shows that there is a set G_3 of logarithmic density zero such that as $r\to\infty$ outside G_3 the right hand side of (13) tends to zero.

The second fundamental theorem shows that

(14) $$(1+o(1))T(r, h) \leqq N(r, 0, h)+N(r, 1, h)$$

outside a set G_4 of finite measure. Thus $G=G_1\cup G_2\cup G_3\cup G_4$ has zero logarithmic density and as $r\to\infty$ outside G we have by (12), (13) and (14) that

$$(1+o(1))T(r, h) = o(T(r, h))$$

which is a contradiction. The proof is now complete.

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